NUMERICAL MATHEMATICS
A Journal of Chinese Universities (English Series)

Vol. **15** No. 1, pp. 67-73
February 2006

On The Maximal-Like Solution of Matrix Equation $X + A^*X^{-2}A = I^*$ †

Minghui Wang*and Musheng Wei

Department of Mathematics, East China Normal University, Shanghai 200062, China.

Received October 17, 2003; Accepted (in revised version) September 5, 2004

Abstract. In this paper, we study several iterative methods for finding the maximal-like solution of the matrix equation $X + A^*X^{-2}A = I$, and deduce some properties of the maximal-like solution with these methods.

Key words: Matrix equation; maximal-like solution.

AMS subject classifications: 65F10, 65F30

---

1 Introduction

In this paper we consider the matrix equation

$$X + A^*X^{-2}A = I \tag{1}$$

where I is the $n \times n$ identity matrix and A is an $n \times n$ complex matrix.

Throughout this paper we denote $\|\cdot\|$ the Euclidean vector norm, or corresponding subordinate matrix norm (simply 2-norm). $\lambda(M)$, $\rho(M)$ are respectively the spectrum and spectral radius of a square matrix M, A^* is conjugate transpose of a matrix A. For two positive definite (Hermitian) matrices P, Q of the same dimension, $P > Q$ ($P \geq Q$) means that $P - Q$ is positive definite (semi-definite). For any positive definite solution X of Eq. (1), we have $X_S \leq X \leq X_L$, where X_L and X_S are respectively the maximal solution and minimal solution, and X_l is the maximal-like solution whose inverse has the minimal 2-norm.

In the literature, matrix equations of type like Eq. (1) have been extensively studied. Articles [3, 4, 11] discuss the matrix equation $X + A^*X^{-1}A = I$ and obtained some properties of the equation, including the existence of maximal and minimal solutions. [1, 2, 10] generalize the results, and [7, 8] directly discuss nonlinear matrix equation of type in Eq. (1). [7, 8] mainly study the following algorithms:

$$\begin{cases} X_0 = \alpha I \\ X_k = I - A^*X_{k-1}^{-2}A \end{cases}, \quad \begin{cases} X_0 = \alpha I \\ X_{k+1} = \sqrt{A(I - X_k)^{-1}A^*} \end{cases}, \tag{2}$$

---

*Correspondence to: Minghui Wang, Department of Mathematics, East China Normal University, Shanghai 200062, China. Email: wangmh2000@eyou.com

†This work was supported by the National Natural Science Foundation of China under grant 10371044, and the Science and Technology Commission of Shanghai Municipality under grant 04JC14031.

and provide some convergence properties under different conditions. However, they do not show the existence of the maximal and minimal solutions and the properties of solutions. [5] proves the existence of the minimal solutions. [9] studies more general matrix equations of the type $X^s \pm A^T X^{-t} A = I$.

In this paper, we discuss the maximal-like solution X_l, which is the maximal solution X_L when X_L exists. In Sections 2 and 3 we propose two algorithms for finding X_l, and study properties of these algorithms; in Section 4 we provide some numerical experiments.

2 An algorithm for computing X_l

In this section, we propose an iterative algorithm for computing X_l. We will prove that the algorithm is linearly convergent, and derive some properties of X_l. Unlike the commonly used algorithms given in Eq. (2) which involve computing the inverse, our algorithm only requires matrix multiplications.

We first give a necessary condition for existence of a solution of Eq. (1)

Theorem 2.1 ([5]). *If Eq. (1) has a positive definite solution X, then*

$$\rho(A) \leq \frac{2\sqrt{3}}{9}.$$

Corollary 2.1. *Suppose that A is normal. If Eq. (1) has a positive definite solution, then*

$$\|A\| \leq \frac{2\sqrt{3}}{9}.$$

Lemma 2.1. *Define*

$$f(\eta) = \frac{\eta}{(1+\eta)^3}, \ \eta \geq 0.$$

Then f is increasing for $0 \leq \eta \leq \frac{1}{2}$, decreasing for $\frac{1}{2} \leq \eta \leq +\infty$, and

$$f_{\max} = f(\frac{1}{2}) = \frac{4}{27}.$$

Proof: From

$$f'(\eta) = \frac{1}{(1+\eta)^4}(1-2\eta),$$

we know that $f(\eta)$ is increasing in $[0, \frac{1}{2}]$, and decreasing in $[\frac{1}{2}, +\infty]$. When $\eta = \frac{1}{2}$, $f\max = f(\frac{1}{2}) = \frac{4}{27}$. ■

We now present the main result of this section.

Theorem 2.2. *If $\|A\| < \frac{2\sqrt{3}}{9}$, then there exists a* unique *solution X_l of Eq. (1) satisfying*

$$\|X_l^{-1}\| < \frac{3}{2}.$$

Moreover, for any other positive definite solution X we have

$$\|X^{-1}\| \geq \frac{3}{2}.$$