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Dynamical System Method for Solving Ill-Posed Operator Equations†

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Abstract. Two dynamical system methods are studied for solving linear ill-posed problems with both operator and right-hand nonexact. The methods solve a Cauchy problem for a linear operator equation which possesses a global solution. The limit of the global solution at infinity solves the original linear equation. Moreover, we also present a convergent iterative process for solving the Cauchy problem.

Key words: Ill-posed problems; dynamical system method; operator equations.

AMS subject classifications: 65J20, 65R30

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1 Introduction

Dynamical systems method (DSM) is a general method for solving operator equations, especially for non-linear, ill-posed as well as well-posed operator equations [1-6]. In [5, 6], Ramm proposed a DSM for linear ill-posed problem with right hand nonexact. However, in practice, not only the right-hand side of equations but also the operators are approximately given. This paper is to provide a DSM for linear operator equation with not only noisy data but also perturbed operators.

We first briefly describe the dynamical systems method for solving operator equations. Consider an operator equation

$$\mathcal{A}u = f, \quad f \in H \tag{1}$$

Let us denote by $(\mathbf{\Sigma})$ the following assumption:

$(\mathbf{\Sigma})$: $\mathcal{A}$ is a linear, bounded operator in H, defined on all of H; the range $R(\mathcal{A})$ is not closed, so (1) is ill-posed problem. There is a y such that $\mathcal{A}y = f, \quad y \in N(\mathcal{A})^{\perp}$, where $N(\mathcal{A})$ is the null-space of $\mathcal{A}$.

Let $\dot{u}$ denote the derivative of u with respect to time. Consider the following dynamical system (the Cauchy problem):

$$\dot{u} = \Phi(t, u), \quad t > 0, \quad u(0) = u_0 \tag{2}$$

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where $\Phi(t, u)$ is globally *Lipchitz* with respect to $u \in H$ and continuous with respect to $t \geq 0$:

$$\sup_{u,v \in H, t \in [0,\infty)} \|\Phi(t,u) - \Phi(t,v)\| \leq c\|u - v\|, \quad c = const > 0. \tag{3}$$

Problem (2) has a global solution if (3) holds. The DSM for solving (1) consists of solving (2), where Φ is so chosen that the following three conditions hold:

$$\exists u(t)\ \forall t > 0; \quad \exists y := u(\infty) := \lim_{t\to\infty} u(t); \quad \mathcal{A}y = f. \tag{4}$$

For real number $h > 0$, let $\mathcal{A}_h$ be a bounded linear operator in a real Hilbert space H such that

$$\|\mathcal{A} - \mathcal{A}_h\| \leq h. \tag{5}$$

Problem (1) with noisy data $f^\delta, \|f - f^\delta\| \leq \delta$ and perturbed operator $\mathcal{A}_h$, satisfying (5), given in place of f and $\mathcal{A}$, respectively, generates the problem:

$$\dot{u}_{\delta,h} = \Phi_{\delta,h}(t, u), \quad t > 0, \quad u_{\delta,h}(0) = u_0. \tag{6}$$

The solution $u_{\delta,h}$ to (6) at $t = t_{\delta,h}$, will have the property

$$\lim_{r\to 0} \|u_{\delta,h}(t_{\delta,h}) - y\| = 0, \tag{7}$$

where $r = \sqrt{\delta^2 + h^2}$. The choice of $t_{\delta,h}$ with this property is called the stopping rule. One has usually $\lim_{r\to 0} t_{\delta,h} = \infty$.

We organize this paper into four sections. In Section 2, we describe one DSM for solving linear problem. In Section 3, we present another version of DSM. In Section 4, we propose two convergent iterative processes to solve the two Cauchy problems.

2 DSM I for solving the linear problem

Consider the Cauchy problem

$$\dot{u}_{\delta,h}(t) = \Phi_{\delta,h}(t, u_{\delta,h}(t)), \quad t > 0, \quad u_{\delta,h}(0) = u_0 \tag{8}$$

where $\Phi_{\delta,h}(t, u_{\delta,h}(t)) = -[\mathcal{B}_h u_{\delta,h}(t) + \varepsilon(t) u_{\delta,h}(t) - \mathcal{F}_{\delta,h}]$, $\mathcal{B}_h := \mathcal{A}_h^* \mathcal{A}_h$, $\quad \mathcal{F}_{\delta,h} = \mathcal{A}_h^* f^\delta$ and

$$\varepsilon(t) \in C^1[0,\infty), \quad \varepsilon(t) > 0, \quad \varepsilon(t) \searrow 0\ (t \to \infty), \tag{9}$$

$$\frac{|\dot{\varepsilon}(t)|}{\varepsilon(t)^{\frac{5}{2}}} \to 0\ (t \to \infty). \tag{10}$$

Lemma 2.1. [5] *Let $\mathcal{A}$ and $\mathcal{A}_h$ are linear operator in a real Hilbert space H, $\mathcal{B} = \mathcal{A}^*\mathcal{A}$, $\quad \mathcal{B}_h = \mathcal{A}_h^*\mathcal{A}_h$, $\varepsilon(t) \in C[0,\infty)$ and $\varepsilon(t) > 0$. Then the following inequalities hold*

(i). $\|(\varepsilon(t) + \mathcal{B})^{-1}\mathcal{A}^*\| \leq \frac{1}{2\sqrt{\varepsilon(t)}}$,

(ii). $\|(\varepsilon(t) + \mathcal{B})^{-1}\mathcal{A}^*\mathcal{A}\| \leq 1$,

(iii). $\|\varepsilon(t)(\varepsilon(t) + \mathcal{B})^{-1}\| \leq 1$.

If $\mathcal{A}, \mathcal{B}$ are replaced by $\mathcal{A}_h, \mathcal{B}_h$, respectively, the above conclusions are still correct.