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Composite Implicit Iteration Process for Asymptotically Hemi-Pseudocontractive Mappings

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Abstract. In Banach space, the composite implicit iterative process for uniformly L-Lipschitzian asymptotically hemi-pesudocontractive mappings are studied, and the sufficient and necessary conditions of strong convergence for the composite implicit iterative process are obtained.

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1 Introduction and preliminaries

Throughout this work, we assume that E is a real Banach space. E^* is the dual space of E and $J:E\to 2^{E^*}$ is the normalized duality mapping defined by

$$J(x)=\{f\in E^*:<x,f>=\|x\|\|f\|,\|f\|=\|x\|\},\quad \forall x\in E,$$

where $\langle\cdot,\cdot\rangle$ denotes duality pairing between E and E^*. A single-valued normalized duality mapping is denoted by j.

Let C be a nonempty subset of E and $T:C\to C$ a mapping, we denote the set of fixed points of T by $F(T)=\{x\in C;Tx=x\}$.

Definition 1.1. ([1]) T is said to be asymptotically nonexpansive, if there exists a sequence $\{k_n\}\subset[1,\infty)$ with $\lim_{n\to\infty}k_n=1$ such that

$$\|T^nx-T^ny\|\le k_n\|x-y\|,\quad \forall x,y\in C \text{ and } n\ge 1.$$

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(2) ([2]) T is said to be uniformly L-Lipschitzian, if there exists $L>0$ such that

$$\|T^n x - T^n y\| \le L\|x-y\|, \quad \forall x,y\in C \text{ and } n\ge 1.$$

(3) ([3]) T is said to be asymptotically pseudocontractive, if there exists a sequence $\{k_n\}\subset[1,\infty)$ with $\lim_{n\to\infty}k_n=1$, for any $x,y\in C$, there exists $j(x-y)\in J(x-y)$ such that

$$\langle T^n x - T^n y,\, j(x-y)\rangle \le k_n\|x-y\|^2,\ \ n\ge 1.$$

(4) ([4]) T is said to be asymptotically hemi-pseudocontractive, if $F(T)\neq\emptyset$ and there exists a sequence $\{k_n\}\subset[1,\infty)$ with $\lim\limits_{n\to\infty}k_n=1$ such that, for any $x\in C$ and $p\in F(T)$, there exists $j(x-p)\in J(x-p)$ such that

$$\langle T^n x - p,\, j(x-p)\rangle \le k_n\|x-p\|^2,\ \ n\ge 1.$$

Remark 1.1. It is easy to see that if T is an asymptotically nonexpansive mapping, then T is a uniformly L-Lipschitzian and asymptotically pseudocontractive mapping, where $L=sup_{n\ge1}\{k_n\}$; if T is an asymptotically pseudocontractive mapping with $F(T)\neq\emptyset$, then T is an asymptotically hemi-pseudocontractive mapping.

Let C be a nonempty closed convex subset of E and $T:C\to C$ be a uniformly L-Lipschitzian asymptotically hemi-pseudocontractive mapping, for any given $x_1\in C$, we introduce a composite implicit iteration process $\{x_n\}$ as follows:

$$\begin{cases} x_{n+1}=(1-\alpha_n)x_n+\alpha_n T^n y_n, \\ y_n=(1-\beta_n)x_n+\beta_n T^n x_{n+1}, \quad \forall n\ge 1, \end{cases} \tag{1.1}$$

where $\{\alpha_n\},\{\beta_n\}$ are two real sequences in $[0,1]$.

As $\beta_n=0$ for all $n\ge1$, then (1.1) reduces to

$$x_{n+1}=(1-\alpha_n)x_n+\alpha_n T^n x_n. \tag{1.2}$$

Remark 1.2. For any given $x_n\in C$, define the mapping $A_n:C\to C$, such as:

$$A_n x=(1-\alpha_n)x_n+\alpha_n T^n[(1-\beta_n)x_n+\beta_n T^n x],\quad \forall x\in C,$$

where C is a nonempty closed convex subset of E and T:$C\to C$ is a uniformly L-Lipschitzian. Then

$$\begin{aligned}\|A_n x - A_n y\| &=\|\alpha_n(T^n[(1-\beta_n)x_n+\beta_n T^n x]-T^n[(1-\beta_n)x_n+\beta_n T^n y])\|\\ &\le \alpha_n\beta_n L\|T^n x - T^n y\|\\ &\le \alpha_n\beta_n L^2\|x-y\|\end{aligned}$$

for all $x,y\in C$. Thus A_n is a contraction mapping if $\alpha_n\beta_n L^2<1$ for all $n\ge1$, and so there exists a unique fixed point $x_{n+1}\in C$ of A_n, such that $x_{n+1}=(1-\alpha_n)x_n+\alpha_n T^n[(1-\beta_n)x_n+\beta_n T^n x_{n+1}]$. This shows that the composite implicit iteration process (1.1) is well defined.

For any a point z and a set K in E, we denote the distance between z and K by $d(z,K)=\inf_{x\in K}\|z-x\|$.

Recently, Kan Xuzhou and Guo Weiping [5] proved the sufficient and necessary condition for the strong convergence of the composite implicit iterative process for a Lipschitzian pseudocontractive mapping in Banach space.

In this work, we obtained sufficient and necessary conditions of the strong convergence of the iterations sequences (1.1) and (1.2) for uniformly L-Lipschitzian asymptotically hemi-pseudocontractive mappings in Banach spaces.

Lemma 1.1. [6] *Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be sequences of nonnegative real numbers satisfying the ineqality*

$$a_{n+1}\le(1+c_n)a_n+b_n,\quad \forall n\ge n_0,$$

where n_0 is some positive integer, $\sum_{n=1}^{\infty}c_n<\infty$ and $\sum_{n=1}^{\infty}b_n<\infty$. Then $\lim_{n\to\infty}a_n$ exists.

Lemma 1.2. [7] *Let C be a nonempty subset of a Banach space E and $T:C\to C$ be an asymptotically hemi-pseudocontractive mapping with the sequence $\{k_n\}\subset[1,\infty)$,$\lim_{n\to\infty}k_n=1$, then*

$$\|x-p\|\le\|x-p+r[(k_nI-T^n)x-(k_nI-T^n)p]\|$$

for all $x\in C,p\in F(T),r>0$ and $n\ge1$, where I is a identity mapping.

2 Main results

Lemma 2.1. *Let E be a real Banach space and C be a nonempty closed convex subset of E. Let $T:C\to C$ be a uniformly L-Lipschitzian asymptotically hemi-pseudocontractive mapping with the sequence $\{k_n\}\subset[1,\infty)$, $\lim_{n\to\infty}k_n=1$ and Lipschitz constant $L>1$. Suppose that the sequence $\{x_n\}$ is defined by* (1.1) *satisfying the following conditions:*

(i) $\sum_{n=1}^{\infty}\alpha_n\beta_n<\infty$ *and* $\sum_{n=1}^{\infty}\alpha_n^2<\infty$;

(ii) $\sum_{n=1}^{\infty}\alpha_n(k_n-1)<\infty$;

(iii) $\alpha_n\beta_nL^2<1$ *for all* $n\ge1$.

Then

(1) there exists a sequence $\{\gamma_n\}\subseteq[0,\infty)$ and some positive integer n_0, such that $\sum_{n=1}^{\infty}\gamma_n<\infty$ and

$$\|x_{n+1}-p\|\le(1+\gamma_n)\|x_n-p\|$$

for all $p\in F(T)$ and $n\ge n_0$.

(2) The limit $\lim_{n\to\infty}d(x_n,F(T))$ exists.

Proof. It follows from the condition (iii) and Remark 1.2 that the sequence (1.1) is well defined. By (1.1), we have

$$\begin{aligned}x_n &= x_{n+1}+\alpha_n x_n-\alpha_n T^n y_n\\ &=(1+\alpha_n)x_{n+1}+\alpha_n(k_nI-T^n)x_{n+1}-(1+k_n)\alpha_n x_{n+1}+\alpha_n x_n+\alpha_n(T^nx_{n+1}-T^ny_n)\\ &=(1+\alpha_n)x_{n+1}+\alpha_n(k_nI-T^n)x_{n+1}-(1+k_n)\alpha_n[x_n+\alpha_n(T^ny_n-x_n)]\\ &\quad+\alpha_n x_n+\alpha_n(T^nx_{n+1}-T^ny_n)\\ &=(1+\alpha_n)x_{n+1}+\alpha_n(k_nI-T^n)x_{n+1}-k_n\alpha_n x_n+(1+k_n)\alpha_n^2(x_n-T^ny_n)\\ &\quad+\alpha_n(T^nx_{n+1}-T^ny_n)\end{aligned}\tag{2.1}$$

and

$$p=(1+\alpha_n)p+\alpha_n(k_nI-T^n)p-k_n\alpha_n p \tag{2.2}$$

for all $p\in F(T)$. Together with (2.1) and (2.2), we can obtain

$$\begin{aligned}x_n-p=&(1+\alpha_n)(x_{n+1}-p)+\alpha_n[(k_nI-T^n)x_{n+1}-(k_nI-T^n)p]-k_n\alpha_n(x_n-p)\\ &+(1+k_n)\alpha_n^2(x_n-T^ny_n)+\alpha_n(T^nx_{n+1}-T^ny_n).\end{aligned}\tag{2.3}$$

Notice that

$$\begin{aligned}&(1+\alpha_n)(x_{n+1}-p)+\alpha_n[(k_nI-T^n)x_{n+1}-(k_nI-T^n)p]\\ =&(1+\alpha_n)[(x_{n+1}-p)+\frac{\alpha_n}{1+\alpha_n}((k_nI-T^n)x_{n+1}-(k_nI-T^n)p)].\end{aligned}$$

Using Lemma 1.2, we obtain that

$$\|(1+\alpha_n)(x_{n+1}-p)+\alpha_n(k_nI-T^n)(x_{n+1}-p)\|\geq(1+\alpha_n)\|x_{n+1}-p\|. \tag{2.4}$$

It follows from (2.3) and (2.4) that

$$\|x_n-p\|\geq(1+\alpha_n)\|x_{n+1}-p\|-k_n\alpha_n\|x_n-p\|-(1+k_n)\alpha_n^2\|x_n-T^ny_n\|-\alpha_n\|T^nx_{n+1}-T^ny_n\|.$$

This implies that

$$\begin{aligned}(1+\alpha_n)\|x_{n+1}-p\|\leq&(1+k_n\alpha_n)\|x_n-p\|+(1+k_n)\alpha_n^2\|x_n-T^ny_n\|\\ &+\alpha_n\|T^nx_{n+1}-T^ny_n\|.\end{aligned}\tag{2.5}$$

Next, we make the following estimations:

$$\begin{aligned}\|y_n-p\|&=\|(1-\beta_n)(x_n-p)+\beta_n(T^nx_{n+1}-p)\|\\ &\leq(1-\beta_n)\|x_n-p\|+\beta_nL\|x_{n+1}-p\|\end{aligned}$$

and

$$\begin{aligned}\|x_n - T^n y_n\| &\le \|x_n - p\| + \|T^n y_n - p\| \\ &\le \|x_n - p\| + L\|y_n - p\| \\ &\le [1 + L(1-\beta_n)]\|x_n - p\| + \beta_n L^2 \|x_{n+1} - p\|.\end{aligned} \tag{2.6}$$

Furthermore,

$$\begin{aligned}\|x_n - y_n\| &= \beta_n \|x_n - T^n x_{n+1}\| \\ &\le \beta_n (\|x_n - p\| + \|T^n x_{n+1} - p\|) \\ &\le \beta_n \|x_n - p\| + L\beta_n \|x_{n+1} - p\|\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}\|T^n x_{n+1} - T^n y_n\| &\le L\|x_{n+1} - y_n\| \\ &= L\|x_n - y_n + \alpha_n (T^n y_n - x_n)\| \\ &\le L\|x_n - y_n\| + \alpha_n L \|T^n y_n - x_n\|.\end{aligned} \tag{2.8}$$

Substituting (2.6) and (2.7) into (2.8), we have

$$\|T^n x_{n+1} - T^n y_n\| \le [\alpha_n L + \beta_n L + \alpha_n L^2 (1-\beta_n)]\|x_n - p\| + \beta_n L^2 (1 + \alpha_n L)\|x_{n+1} - p\|. \tag{2.9}$$

Substituting (2.6) and (2.9) into (2.5), we have

$$\begin{aligned}(1+\alpha_n)\|x_{n+1} - p\| \le\ & (1 + k_n \alpha_n)\|x_n - p\| + (1+k_n)\alpha_n^2 [1 + L(1-\beta_n)]\|x_n - p\| \\ & + (1+k_n)\alpha_n^2 \beta_n L^2 \|x_{n+1} - p\| \\ & + \alpha_n [\alpha_n L + \beta_n L + \alpha_n L^2 (1-\beta_n)]\|x_n - p\| \\ & + \alpha_n \beta_n L^2 (1 + \alpha_n L)\|x_{n+1} - p\|.\end{aligned}$$

Since $1+\alpha_n \ge 1$, this implies that

$$\begin{aligned}\|x_{n+1} - p\| \le\ & (1 + (k_n - 1)\alpha_n)\|x_n - p\| + (1+k_n)\alpha_n^2 [1 + L(1-\beta_n)]\|x_n - p\| \\ & + \alpha_n [\alpha_n L + \beta_n L + \alpha_n L^2 (1-\beta_n)]\|x_n - p\| \\ & + (1+k_n)\alpha_n^2 \beta_n L^2 \|x_{n+1} - p\| + \alpha_n \beta_n L^2 (1 + \alpha_n L)\|x_{n+1} - p\| \\ \le\ & (1 + (k_n - 1)\alpha_n)\|x_n - p\| + [(1+k_n)(\alpha_n^2 + L\alpha_n^2 - L\alpha_n^2 \beta_n) \\ & + L\alpha_n^2 + L\alpha_n \beta_n + L^2 \alpha_n^2 - L^2 \alpha_n^2 \beta_n)]\|x_n - p\| \\ & + (1+k_n)\alpha_n^2 \beta_n L^2 \|x_{n+1} - p\| + \alpha_n \beta_n L^2 (1 + \alpha_n L)\|x_{n+1} - p\| \\ \le\ & (1 + (k_n - 1)\alpha_n)\|x_n - p\| + [(1 + k_n + L)(1 + L)]\alpha_n^2 \|x_n - p\| \\ & + (L - L\alpha_n - k_n L\alpha_n - L^2 \alpha_n)\alpha_n \beta_n \|x_n - p\| \\ & + (1+k_n)\alpha_n^2 \beta_n L^2 \|x_{n+1} - p\| + \alpha_n \beta_n L^2 (1 + \alpha_n L)\|x_{n+1} - p\|\end{aligned}$$

and so

$$
\begin{aligned}
&[1-(1+k_n)\alpha_n^2\beta_n L^2-\alpha_n\beta_n L^2-\alpha_n^2\beta_n L^3]\|x_{n+1}-p\| \\
\leq &[1+(k_n-1)\alpha_n+(1+k_n+L)(1+L)\alpha_n^2+(L-L\alpha_n-Lk_n\alpha_n-L^2\alpha_n)\alpha_n\beta_n]\|x_n-p\|.
\end{aligned}
$$

Since $\lim_{n\to\infty}\alpha_n\beta_n=0$ and $\lim_{n\to\infty}k_n=1$, there exists some positive integer n_0, such that $\alpha_n\beta_n\leq\frac{1}{8L^3}$ and $k_n<L$ for all $n\geq n_0$. Thus

$$
\begin{aligned}
1-(1+k_n)\alpha_n^2\beta_n L^2-\alpha_n\beta_n L^2-\alpha_n^2\beta_n L^3 &\geq 1-\frac{1+k_n}{8L^3}L^2-\frac{1}{8L^3}L^2-\frac{1}{8L^3}L^3 \\
&=\frac{8L-2-L-k_n}{8L} \\
&\geq\frac{8L-4L}{8L}=\frac{1}{2}.
\end{aligned}
$$

After finishing deformation, we have

$$
\begin{aligned}
\|x_{n+1}-p\| &\leq\{1+2[(k_n-1)\alpha_n+(1+k_n+L)(1+L)\alpha_n^2 \\
&\quad+(L+(2+k_n)L^2+L^3)\alpha_n\beta_n]\}\|x_n-p\| \\
&=(1+\gamma_n)\|x_n-p\|,\quad n\geq n_0,
\end{aligned}
\tag{2.10}
$$

where $\gamma_n=2[(k_n-1)\alpha_n+(1+k_n+L)(1+L)\alpha_n^2+(L+(2+k_n)L^2+L^3)\alpha_n\beta_n]$, and $\sum_{n=1}^{\infty}\gamma_n<\infty$ by conditions (i) and (ii), Thus (1) is proved.

(2) Taking the infimun over all $p\in F(T)$ on both sides in (2.10), we get

$$
d(x_{n+1},F(T))\leq(1+\gamma_n)d(x_n,F(T)),\ n\geq n_0.
$$

It follows from Lemma 1.1 that the limit $\lim_{n\to\infty}d(x_n,F(T))$ exists. This completes the proof. □

Theorem 2.1. *Let E be a real Banach space and C be a nonempty closed convex subset of E. Let $T:C\to C$ be a uniformly L-Lipschitzian asymptotically hemi-pseudocontractive mapping with the sequence $\{k_n\}\subset[1,\infty)$, $\lim_{n\to\infty}k_n=1$ and Lipschitz constant $L>1$. Suppose that the sequence $\{x_n\}$ is defined by* (1.1) *satisfying the following conditions:*

(i) $\sum_{n=1}^{\infty}\alpha_n\beta_n<\infty$ *and* $\sum_{n=1}^{\infty}\alpha_n^2<\infty$;

(ii) $\sum_{n=1}^{\infty}\alpha_n(k_n-1)<\infty$;

(iii) $\alpha_n\beta_n L^2<1$ *for all* $n\geq1$.

Then $\{x_n\}$ converges strongly to some fixed point of T if and only if $\liminf_{n\to\infty}d(x_n,F(T))=0$.

Proof. The necessary of Theorem 2.1 is obvious. we just need to prove the sufficiency.

From Lemma 2.1 (2) and the condition $\liminf_{n\to\infty} d(x_n,F(T))=0$, we have

$$\lim_{n\to\infty} d(x_n,F(T))=0.$$

Next, we show that $\{x_n\}$ is a Cauchy sequence. In fact, using Lemma 2.1(1), for any $p\in F(T)$ and any positive integers $m,n,m>n\geq n_0$, we have

$$\begin{aligned}\|x_m-p\| &\leq (1+\gamma_{m-1})\|x_{m-1}-p\| \\ &\leq e^{\gamma_{m-1}}\|x_{m-1}-p\| \\ &\leq e^{\sum_{j=n}^{m-1}\gamma_j}\|x_n-p\| \\ &\leq M\|x_n-p\|,\end{aligned}$$

where $M=e^{\sum_{j=1}^{\infty}\gamma_j}$. Thus, we have

$$\|x_n-x_m\|\leq\|x_n-p\|+\|x_m-p\|\leq(1+M)\|x_n-p\|.$$

Taking the infimum over all $p\in F(T)$, we have

$$\|x_n-x_m\|\leq(1+M)d(x_n,F(T)).$$

It follows from $\lim_{n\to\infty} d(x_n,F(T))=0$ that $\{x_n\}$ is a Cauchy sequence. Since C is closed subset of E, so there exists a $p_0\in C$ such that $x_n\to p_0$ as $n\to\infty$. Further, since T is uniformly L-Lipschitzian, it is easy to prove that $F(T)$ is closed. Again since $\lim_{n\to\infty} d(x_n,F(T))=0$ and $p_0\in F(T)$, this shows that $\{x_n\}$ converges strongly to a fixed point of T, this completes the proof. □

Using Theorem 2.1, we have the following:

Theorem 2.2. *Let E be a real Banach space and C be a nonempty closed convex subset of E. Let $T:C\to C$ be a uniformly L-Lipschitzian asymptotically hemi-pseudocontractive mapping with the sequence $\{k_n\}\subset[1,\infty)$, $\lim_{n\to\infty}k_n=1$ and Lipschitz constant $L>1$. Suppose that the sequence $\{x_n\}$ is defined by* (1.2) *satisfying the following conditions:*

(i) $\sum_{n=1}^{\infty}\alpha_n^2<\infty$;

(ii) $\sum_{n=1}^{\infty}\alpha_n(k_n-1)<\infty$.

Then $\{x_n\}$ converges strongly to some fixed point of T if and only if $\liminf_{n\to\infty} d(x_n,F(T))=0$.

Remark 2.1. By Remark 1.1, clearly, Theorem 2.1 and Theorem 2.2 hold for uniformly L-Lipschitzian and asymptotically pseudocontractive mappings with $F(T)\neq\emptyset$ and for asymptotically nonexpansive mappings with $F(T)\neq\emptyset$.

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