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An Example of Nonexistence of $\kappa-$solutions to the Ricci Flow

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Abstract. In this paper, we construct an example of three-dimensional complete smooth $\kappa-$noncollapsed manifold, which admits no short time smooth complete and $\kappa-$noncollapsed solutions to the Ricci flow.

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1 Introduction

In 1982, R. Hamilton introduced the Ricci flow [3] to the world of mathematics. In this seminal work [3], among other things, R. Hamilton proved that when the manifold M is compact, the Ricci flow has a solution for a short time. In 1989, W.X. Shi [10] extended the short time existence result to complete noncompact manifolds with bounded curvature.

In this paper, we will revisit the problem of existence of the Ricci flow. We attempt to show that, on a general manifold, the bounded curvature condition in Shi's theorem can not be dropped. More precisely, one can show the following:

Theorem 1.1. *There is a smooth complete $\kappa-$noncollapsed three-dimensional Riemannian manifold (M,g_{ij}) such that it admits no complete and $\kappa-$noncollapsed smooth solution to the Ricci flow for a short time with (M,g_{ij}) as initial data.*

Here, we say a Riemannian manifold of dimension n is $\kappa-$noncollapsed if there is a positive constant $\kappa>0$ such that for any $x_0\in M$ with $|Rm|\le r^{-2}$ on $B(x_0,r)$, we have $vol(B(x_0,r))\ge\kappa r^n$. A solution to the Ricci flow is said to be $\kappa-$noncollapsed (see [7] section 8) if for any spacetime point (x_0,t_0) such that $|Rm|\le r^{-2}$ on $B_t(x_0,r)$ for any $t\in[t_0-r_0^2,t_0]$, we have $vol(B_{t_0}(x_0,r))\ge\kappa r^n$.

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We remark that in the above theorem, besides the completeness of the metric, a $\kappa-$noncollapsed condition was imposed on the solutions. We expect that this technical condition could be removed.

It is well-known that the Ricci flow on spheres of dimension n with initial metric of constant curvature K will shrink the spheres to a point at time $T=\frac{1}{2(n-1)K}$. A tiny sphere will shrink to a point in short time. Intuitively, we can imagine that the Ricci flow on a manifold with many tiny necks N_j whose cross spheres have radius $r_j \to 0$ can not move for any short time. The purpose of this paper is trying to provide such an example.

Note that it is a nontrivial issue to control the behavior of the individual spheres or necks on the manifold during the evolution. In this paper, instead of dealing with each individual spheres, our idea is to choose and investigate the minimal surfaces in the homotopy classes of those spheres. The point is that we do not need to care about the shape of the minimal surfaces, we only care about their existence. Because whenever these minimal surfaces exist, their areas can be used to estimate the life span of the solution. Now the difficulty of the problem is to prove the existence of minimal surfaces which must be confined in some specific regions.

The minimal surface argument in Ricci flow was first used by Hamilton [4, 5], who proved the incompressibility of certain boundary tori of hyperbolic pieces in the long time nonsingular solutions. In [1] and [9], the argument with min-max constructions of minimal surfaces was used to prove the finite time extinction for the Ricci flow on homotopy three-spheres.

In the present paper, we will use [6] to construct minimal surfaces in certain domains. The difficulty is to prove that the boundaries of these domains are convex or mean convex for the evolving metrics. Ultimately, this will amount to certain technical local a priori curvature estimate, which is the main difficulty of the paper to overcome.

The paper is organized as follows. In Section 2, we first give the construction of the manifolds, then we outline the proof of nonexistence of the Ricci flow, modulo key Lemma 2.2. In Section 3, we prove Lemma 2.2.

2 Examples

2.1 Construction

We will construct a warped product metric on cylinder $M=\mathbb{R}\times\mathbb{S}^{n-1}$ in the form $ds^2=dr^2+f^2(r)ds^2_{\mathbb{S}^{n-1}}$, where $f(r)$ is a positive even smooth function on $\mathbb{R}$.

Let $X_1,X_2,\cdots,X_{n-1}$ be an orthogonal basis of M tangent to the sphere $\mathbb{S}^{n-1}$ and $T=\frac{\partial}{\partial r}$. It is well-known that

$$
\begin{aligned}
&R(T,X_i,T,X_j) = -\frac{f''}{f}\langle X_i,X_j\rangle,\\
&R(X_i,X_j,X_k,X_l) = \frac{1-f'^2}{f^2}(\langle X_i,X_k\rangle\langle X_j,X_l\rangle - \langle X_i,X_l\rangle\langle X_j,X_k\rangle),\\
&R(T,X_i,X_j,X_k) = 0.
\end{aligned}
\tag{2.1}
$$

In order that the constructed metric has required curvature growth property, let us arbitrarily fix two increasing sequences of positive numbers $\{a_i\}_{i=1}^{\infty}$ and $\{b_i\}_{i=1}^{\infty}$ such that $\lim\limits_{i\to\infty} a_i = +\infty, \lim\limits_{i\to\infty} b_i = +\infty$. For these sequences, we construct a sequence of smooth positive functions $0<\varphi_i \le b_i$ on $[0,b_i]$ for $i\ge 1$, which is increasing on $[0,\frac{b_i}{2}]$, decreasing on $[\frac{b_i}{2},b_i]$, such that

$$
\varphi_i(r) = \begin{cases}
\dfrac{1}{a_i+100}, & r\in\left[0,\dfrac{1}{2}\right],\\
r, & r\in\left[\dfrac{3}{4},\dfrac{b_i}{2}-1\right],\\
b_i - r, & r\in\left[\dfrac{b_i}{2}+1, b_i-\dfrac{3}{4}\right],\\
\dfrac{1}{a_{i+1}+100}, & r\in\left[b_i-\dfrac{1}{2}, b_i\right].
\end{cases}
\tag{2.2}
$$

Clearly, we may choose φ_i such that $|\varphi_i'|+|\varphi_i''|\le C$, where the constant C is independent of i. Now we define a positive even function $f(r)$ on $\mathbb{R}$ in the following way. Define $f(r)=\varphi_1(r)$ on $[0,b_1]$ and inductively

$$
f(r) = \varphi_i\left(r - \sum_{j=1}^{i-1} b_j\right), \quad r\in\left[\sum_{j=1}^{i-1} b_j, \sum_{j=1}^{i} b_j\right]
\tag{2.3}
$$

for positive integer $i\ge 2$.

By (2.1), we see that the curvature satisfies $|Rm|\le \frac{C}{f^2}$, hence for some point $x_0\in\{0\}\times \mathbb{S}^{n-1}$, we have

$$
\sup_{B(x_0,\sum\limits_{j=1}^{i-1} b_j)} |Rm|(x) \le C(a_i+100)^2, \quad \text{for all } 0<i<+\infty.
\tag{2.4}
$$

By direct computations, we find that the scalar curvature satisfies

$$
R = (n-1)\left[(n-2)\frac{1-f'^2}{f^2} - 2\frac{f''}{f}\right].
\tag{2.5}
$$

Note that we may choose f such that if for some point r_0 satisfying $|f'(r_0)|\ge \frac{1}{2}$, then we have $f(r_0)\ge \frac{1}{C'}$, where C' is a uniform positive constant.

Combining this observation and (2.5), we have

Lemma 2.1. *When $n\geq 3$, for the metric constructed above, the scalar curvature is bounded from below, i.e.*

$$R\geq -C. \tag{2.6}$$

In the next section, we will prove that when $n=3$, the manifold constructed above admits no short time smooth complete and κ-noncollapsed solution to the Ricci flow.

2.2 Proof of the nonexistence theorem

The proof of Theorem 1.1 is an argument by contradiction. Let (M,g) be the Riemannian manifold constructed in Section 2.1 of dimension $n=3$. Suppose there is a smooth complete and κ-noncollapsed solution to the Ricci flow for a short time $[0,T_0]$ $(T_0>0)$ with (M,g) as initial data.

Let I be an interval in $\mathbb{R}$, here and in the followings, we denote the corresponding region on the manifold by M^I. For example, $M^{[a,b]}=[a,b]\times\mathbb{S}^2\subset M$ and $M^{\{r_0\}}=\{r_0\}\times\mathbb{S}^2\subset M$.

Since for $j\in\mathbb{N}$, the regions $M^{[\sum_{i=1}^{j-1}b_i+\frac{b_j}{8},\sum_{i=1}^{j-1}b_i+\frac{3b_j}{8}]}$ and $M^{[-\sum_{i=1}^{j-1}b_i-\frac{3b_j}{8},-\sum_{i=1}^{j-1}b_i-\frac{b_j}{8}]}$ are flat annuluses at time $t=0$, they have a short time so that the curvature is bounded. Actually we claim the following lemma holds:

Lemma 2.2. *There are constants $C=C(\kappa)$ depending only on κ, and $j_0>0$ such that as $j>j_0$, we have*

$$|Rm|(x,t)\leq 1, \tag{2.7}$$

for $(x,t)\in M^{[\sum_{i=1}^{j-1}b_i+\frac{b_j}{8},\sum_{i=1}^{j-1}b_i+\frac{3b_j}{8}]}\cup M^{[-\sum_{i=1}^{j-1}b_i-\frac{3b_j}{8},-\sum_{i=1}^{j-1}b_i-\frac{b_j}{8}]}\times[0,\min\{T_0,\frac{1}{C(\kappa)}\}]$.

The proof of the above lemma will be given in the next section. We proceed to derive the convexity estimate of the relevant regions.

By local gradient estimate of Shi [10], we have

$$|\nabla Rm|(x,t)\leq\frac{C(n)}{t^{\frac{1}{2}}}, \tag{2.8}$$

for $(x,t)\in M^{[\sum_{i=1}^{j-1}b_i+\frac{b_j}{8}+1,\sum_{i=1}^{j-1}b_i+\frac{3b_j}{8}-1]}\cup M^{[-\sum_{i=1}^{j-1}b_i-\frac{3b_j}{8}+1,-\sum_{i=1}^{j-1}b_i-\frac{b_j}{8}-1]}\times[0,\min\{T_0,\frac{1}{C(\kappa)}\}]$.

Now we only consider the piece $M^{[\sum_{i=1}^{j-1}b_i+\frac{b_j}{8}+1,\sum_{i=1}^{j-1}b_i+\frac{3b_j}{8}-1]}$. Since the initial metric on the above region is flat and it has the form $dr^2+(r-c)^2ds^2_{\mathbb{S}^2}$, where $c=\sum_{i=1}^{j-1}b_i$. It is obvious that at $t=0$

$$\nabla_{0\alpha}\nabla_{0\beta}(r-c)^2=2g_{\alpha\beta}(\cdot,0)$$

on $M^{[\sum_{i=1}^{j-1} b_i+\frac{b_j}{8}+1,\sum_{i=1}^{j-1} b_i+\frac{3b_j}{8}-1]}$.

By combining with (2.7) and (2.8), we have

$$\nabla_\alpha\nabla_\beta(r-c)^2\geq 2g_{\alpha\beta}(\cdot,0)-C(n)\sqrt{t}g_{\alpha\beta},$$

for any $t\in[0,\min\{T_0,\frac{1}{C(\kappa)}\}]$ and $c+\frac{b_j}{8}+1\leq r\leq c+\frac{3b_j}{8}-1$.

Without loss of generality, we may assume that $T_0<\frac{1}{C(\kappa)}$ and T_0 is small. This implies

$$\nabla_\alpha\nabla_\beta(r-c)^2\geq g_{\alpha\beta}, \tag{2.9}$$

for any $t\in[0,T_0]$ and $c+\frac{b_j}{8}+1\leq r\leq c+\frac{3b_j}{8}-1$. Obviously, the similar result also holds for the other piece where $-(c+\frac{3b_j}{8}-1)\leq r\leq -(c+\frac{b_j}{8}+1)$. We have proved the following lemma.

Lemma 2.3. *The regions* $M^{[-\sum_{i=1}^{j-1} b_i-\frac{b_j}{8}-1,\sum_{i=1}^{j-1} b_i+\frac{b_j}{8}+1]}$ *have convex boundaries for any* $t\in[0,T_0]$ *and* $j>j_0$*, where* j_0 *is the constant in Lemma* 2.2.

In the following, we will use the minimal surface argument. First of all, we need the existence of the minimal surfaces. Clearly, for each $j\geq 2$, at time $t=0$, there is a smooth minimal surface Σ_j in $M^{[-\sum_{i=1}^{j-1} b_i-\frac{b_j}{8}-1,\sum_{i=1}^{j-1} b_i+\frac{b_j}{8}+1]}$, with area $\leq\frac{C}{{a_j}^2}$. Actually, by the construction of the metric, we may take $\Sigma_j=M^{\{\sum_{i=1}^{j-1} b_i\}}$. We denote by $A_j(t)$ the minimum of the areas of smooth surfaces homotopic to Σ_j in $M^{[-\sum_{i=1}^{j-1} b_i-\frac{b_j}{8}-1,\sum_{i=1}^{j-1} b_i+\frac{b_j}{8}+1]}$ at time t.

Lemma 2.4. *For any fixed* $j>j_0$*, and time* $t\in[0,T_0]$*, there exists a smooth minimal surface* $\Sigma_j(t)$ *in the interior of* $M^{[-\sum_{i=1}^{j-1} b_i-\frac{b_j}{8}-1,\sum_{i=1}^{j-1} b_i+\frac{b_j}{8}+1]}$ *achieves* $A_j(t)$ *in the homotopy class of* Σ_j.

Proof. By Lemma 2.3, the domain $M^{[-\sum_{i=1}^{j-1} b_i-\frac{b_j}{8}-1,\sum_{i=1}^{j-1} b_i+\frac{b_j}{8}+1]}$ has convex boundaries $M^{\{-\sum_{i=1}^{j-1} b_i-\frac{b_j}{8}-1\}}$ and $M^{\{\sum_{i=1}^{j-1} b_i+\frac{b_j}{8}+1\}}$ for any $t\in[0,T_0]$. Then we know that a smooth minimal surfaces $\Sigma_j(t)$ exists from [6]. □

Lemma 2.5. *There is a constant* $C>0$ *such that*

$$R(x,t)\geq -C,\quad \text{for all}\ \ (x,t)\in M^n\times[0,T_0]. \tag{2.10}$$

Proof. This follows from Corollary 2.3 in [2]. □

For any fixed $t_0 \in [0,T_0]$, we may compute the rates of the area changes of the fixed $\Sigma_j(t_0)$ under the Ricci flow (see [5], section 11):

$$\begin{aligned}\frac{d}{dt}\text{area}_t(\Sigma_j(t_0))\,|_{t_0} &= -\int_{\Sigma_j(t_0)} \frac{1}{2}(R+|A|^2)+G \\ &\le -4\pi - \frac{1}{2}\int_{\Sigma_j(t_0)} R \\ &\le -4\pi + \frac{C}{2}\text{area}\Sigma_j(t_0)\end{aligned} \tag{2.11}$$

where we have used Gauss-Bonnet theorem and (2.10). Here A and G denote the second fundamental form and the Gaussian curvature of $\Sigma_j(t_0)$ respectively.

Then for any $t_0 \in [0,T_0]$, we have

$$\begin{aligned}\frac{d^+}{dt}A_j(t)\,|_{t=t_0} &:= \limsup_{\triangle t\to 0}\frac{A_j(t_0+\triangle t)-A_j(t_0)}{\triangle t} \\ &\le \frac{d}{dt}\text{area}_t(\Sigma_j(t_0))\,|_{t_0} \\ &\le -4\pi + \frac{C}{2}\text{area}\Sigma_j(t_0) \\ &= -4\pi + \frac{C}{2}A_j(t_0).\end{aligned} \tag{2.12}$$

By combining $A_j(0) \le \frac{C}{a_j^2}$ and (2.12), we see that there is $j_1 > 0$ such that as $j > j_1$, for any $t \in [0,T_0]$ we have

$$0 < A_j(t) \le \frac{C}{a_j^2} - 2\pi t.$$

This implies

$$T_0 \le \frac{C}{2\pi a_j^2}, \quad \text{for any} \quad j > j_1.$$

Since $a_j \to \infty$, we conclude that $T_0 = 0$. This is a contradiction. We complete the proof of the Theorem 1.1 modulo Lemma 2.2.

3 Proof of Lemma 2.2

Now we recall the useful local curvature pinching estimate proved in [2].

In dimension 3, let $\lambda \ge \mu \ge \nu$ be the eigenvalues of the curvature operator $M_{ij} = Rg_{ij} - 2R_{ij}$.

Proposition 3.1. For any $k\in\mathbb{Z}_+$, $0<\delta<1$, there is a constant $C_{k,\delta}>0$ depending only on k and δ satisfying the following property. Suppose we have a smooth solution $g_{ij}(x,t)$ to the three dimensional Ricci flow, such that for any $t\in[0,T]$, $B_t(x_0,Ar_0)$ are compactly contained in the manifold and assume that $Ric(x,t)\leq(n-1)r_0^{-2}$ for $x\in B_t(x_0,r_0)$, $t\in[0,T]$. Then we have

$$\lambda+\mu+k\nu\geq\min\left\{-\frac{C_{k,\delta}}{t+\frac{1}{K_k}},-\frac{C_{k,\delta}}{Ar_0^2}\right\}, \tag{3.1}$$

whenever $x\in B_t(x_0,(1-\delta)Ar_0)$, $t\in[0,T]$, where $\lambda+\mu+k\nu\geq-K_k$ on $B(x_0,Ar_0)$ at time 0.

By choosing $x_0=\tilde{x}$, $r_0=1$, $A=\frac{3}{16}b_j$, and $\delta=\frac{1}{100}$ in the above proposition, we have

Proposition 3.2. For any $k\in\mathbb{Z}_+$, there is a constant $C_k>0$ depending only on k such that

$$(\lambda+\mu+k\nu)(x,t)\geq-\frac{C_k}{b_j}, \tag{3.2}$$

for $x\in B_t(\tilde{x},\frac{5}{32}b_j)$, $t\in[0,T]$, and $\tilde{x}\in M^{\{\sum_{i=1}^{j-1}b_i+\frac{b_j}{4}\}}$.

To state the following lemma, we need the notion of canonical neighborhood (see [8]) which is important in the singularity analysis of Ricci flow on 3-d manifolds. We say a spacetime point (x,t) has a canonical neighborhood (with accuracy ε) if the parabolic neighborhood $\cup_{s\in[t-\frac{\varepsilon^{-2}}{R(x,t)},t]}B_s(x,\frac{\varepsilon^{-1}}{\sqrt{R(x,t)}})\times\{s\}$ (after normalizing the solution with the factor $R(x,t)$ and shifting the time t to 0) is $\varepsilon-$ close (in $C^{[\frac{1}{\varepsilon}]}$ topology) to the corresponding subset of some ancient $\kappa-$ solution. Geometrically, for fixed time, these canonical neighborhoods are long necks or caps with long neck ends. It is known (see [8]) that there is a positive constant $C(\varepsilon)$ depending only on ε such that $\lim_{\varepsilon\to0}C(\varepsilon)=0$ and

$$vol_t(B_t(x,r))\leq C(\varepsilon)r^n \tag{3.3}$$

where $r=\frac{\varepsilon^{-1}}{\sqrt{R(x,t)}}$.

Lemma 3.1. *For any given $\varepsilon>0$, there is a constant $K>0$ depending only on κ and ε such that for any $(y,t)\in M\times[0,T_0]$, one of the followings holds:*

(1) $|Rm|(y,t)\leq\frac{K}{t}$;

(2) the point (y,t) has a canonical neighborhood with accuracy ε.

Proof. We argue by contradiction.

Suppose there exist $\varepsilon>0$ and a sequence of points $y_j\in M$ with $|Rm|(y_j,t_j)\geq\frac{K_j}{t_j}$ and $K_j\to\infty$, but the points (y_j,t_j) have no canonical neighborhoods with accuracy ε.

From each point (y_j,t_j) with $|Rm|(y_j,t_j)\geq\frac{K_j}{t_j}$, we use Perelman's point picking technique (see section 10 in [7]) to find another point $(\bar{y}_j,\bar{t}_j)$ with $Q_j=|Rm|(\bar{y}_j,\bar{t}_j)\geq\frac{K_j}{\bar{t}_j}$, such that the conclusion of the proposition fails at $(\bar{y}_j,\bar{t}_j)$, but holds for any point (y,t) with $|Rm|(y,t)\geq 2Q_j$ and $d_t(y,\bar{y}_j)\leq d_{\bar{t}_j}(y_j,\bar{y}_j)+K_j^{\frac{1}{2}}Q_j^{-\frac{1}{2}}$, $\bar{t}_j-\frac{1}{4}K_jQ_j^{-1}\leq t\leq\bar{t}_j$.

We rescale the solution around $(\bar{y}_j,\bar{t}_j)$ with factor Q_j and translate the time $\bar{t}_j$ to zero so that it is still a solution to the Ricci flow. We will show that after passing to a subsequence, the rescaled solutions will converge to a smooth complete ancient $\kappa-$solution. This will give a contradiction.

Recall that, we assume our solution is $\kappa-$noncollapsed. By using Hamilton's compactness, we may extract a subsequence which converges on the spacetime regions with uniformally (independent of j) bounded curvature. In order to adapt the whole argument of [7] (section 12) to the present case, we observe that it suffices to show that the convergent limit (on the regions where it exists) always has nonnegative sectional curvature. To prove this, we use our local pinching estimate (3.1). Putting $A=\infty$ and $K_k=\infty$ in Proposition 3.1, we know

$$(\lambda+\mu+k\nu)(t)\geq-\frac{C_k}{t}$$

holds whenever $t\in[0,\bar{t}_j]$ for the unrescaled solution. For the rescaled solution, we have

$$(\lambda+\mu+k\nu)(s)\geq-\frac{C_k}{Q_j\bar{t}_j+|s|}$$

whenever $s\in[-Q_j\bar{t}_j,0]$. Since $Q_j\bar{t}_j\geq K_j\to\infty$, we know the convergent limit satisfies

$$(\lambda+\mu+k\nu)\geq 0.$$

The arbitrariness of k implies $\nu\geq 0$, i.e. the limit has nonnegative sectional curvature. □

The purpose of this section is to prove Lemma 2.2. For simplicity, we only show the estimate for the pieces with $r>0$. For any $j\in\mathbb{N}$, let $M_j=M^{[\sum_{i=1}^{j-1}b_i+\frac{b_j}{8},\sum_{i=1}^{j-1}b_i+\frac{3b_j}{8}]}$, $S_j=M^{\{\sum_{i=1}^{j-1}b_i+\frac{b_j}{4}\}}$.

Now we fix a very small $\varepsilon>0$ and a constant $K=K(\varepsilon)$ in Lemma 3.1. Set

$$t_j=\min_{\tilde{x}\in S_j}\max\left\{t\in[0,T_0]:s|Rm|(x,s)\leq 2K,\quad\forall d_s(x,\tilde{x})\leq\frac{b_j}{7},\quad\forall s\in[0,t]\right\}.\tag{3.4}$$

By smoothness of the solution, we know $t_j>0$ for each j. Recall Theorem 3.1 in [2] says the following:

Theorem 3.1. ([2]) *There is a constant $C=C(n)$ with the following property. Suppose we have a smooth solution to the Ricci flow $(g_{ij})_t=-2R_{ij}$, $0\leq t\leq T$, on an n-manifold M such that $B_t(x_0,r_0)$, for $0\leq t\leq T$, is compactly contained in M and* *(i)* $|Rm| \le r_0^{-2}$ *on* $B_0(x_0,r_0)$ *at* $t=0$*;*

(ii)

$$|Rm|(x,t) \le \frac{D}{t}$$

where $D \ge 1$, *whenever* $d_t(x_0,x) < r_0$ *and* $0 \le t \le T$.

Then we have

$$|Rm|(x,t) \le e^{CD}(r_0 - d_t(x_0,x))^{-2}$$

whenever $d_t(x_0,x) = dist_t(x_0,x) < r_0$ *and* $0 \le t \le T$.

By Theorem 3.1, we have

$$|Rm|(x,t) \le e^{C(n)K}\left(\frac{b_j}{7} - d_t(x,\tilde{x})\right)^{-2} \tag{3.5}$$

whenever $\tilde{x} \in S_j$, $d_t(x,\tilde{x}) \le \frac{b_j}{7}$ and $t \in [0,t_j]$.

From (3.5), Lemma 2.2 can be proved immediately if one can show that t_j is bounded from below by a positive constant independent of j.

Actually, we will show that $t_j = T_0$ for all large j. Suppose the contrary holds, we will find a contradiction in the rest of the paper.

Let $(\bar{x},t_j)$ be a point achieving the minmax in the definition (3.4), i.e., $|Rm|(\bar{x},t_j) = \frac{2K}{t_j}$ and there is a point $\tilde{x} \in S_j$ such that $d_{t_j}(\bar{x},\tilde{x}) \le \frac{b_j}{7}$, moreover we have $|Rm|(x,s) \le \frac{2K}{s}$ whenever $d_s(\tilde{x},x) < \frac{b_j}{7}$ and $0 \le t \le t_j$.

By (3.5), we have

$$\frac{K}{t_j} \le e^{CK}\left(\frac{b_j}{7} - d_{t_j}(\bar{x},\tilde{x})\right)^{-2}.$$

This implies

$$0 \le \frac{b_j}{7} - d_{t_j}(\bar{x},\tilde{x}) \le \frac{e^{\frac{CK}{2}}}{\sqrt{K}}\sqrt{t_j}.$$

Let $\gamma(s)$ be a minimal geodesic at time t_j connecting $\bar{x}$ and $\tilde{x}$ of length $d_{t_j}(\bar{x},\tilde{x})$, $\gamma(0) = \bar{x}$. Then (3.5) implies

$$|Rm|(x,t_j) \le 4e^{CK}s^{-2}, \tag{3.6}$$

for all $x \in B_{t_j}(\gamma(s),\frac{s}{2})$.

For $x \in B_{t_j}(\gamma(s),\frac{s}{2})$, $t \in [0,t_j]$, we have (see Lemma 8.3 in [7])

$$\frac{d}{dt}d_t(\tilde{x},x) \ge -C(n)\sqrt{\frac{K}{t}},$$

as long as $d_t(\tilde{x},x)<\frac{b_j}{7}$, where $C(n)$ is a constant depending only on the dimension $n=3$.

This implies

$$d_b(\tilde{x},x)\leq \frac{b_j}{7}-\frac{1}{2}s+2C(n)\sqrt{Kt_j}\,,$$

as long as $d_t(\tilde{x},x)<\frac{b_j}{7}$ for any $t\in[b,t_j]$.

From above estimate, we know

$$d_t(\tilde{x},x)\leq \frac{b_j}{7}-\frac{1}{2}s+2C(n)\sqrt{Kt_j}<\frac{b_j}{7}$$

for all $(x,t)\in B_{t_j}(\gamma(s),\frac{s}{2})\times[0,t_j]$ if $4C(n)\sqrt{Kt_j}<s<d_{t_j}(\tilde{x},\bar{x})$.

It follows from (3.5) that

$$|Rm|(x,t)\leq e^{CK}\left(\frac{1}{2}s-2C(n)\sqrt{Kt_j}\right)^{-2} \tag{3.7}$$

on $B_{t_j}(\gamma(s),\frac{s}{2})\times[0,t_j]$ whenever $4C(n)\sqrt{K}\sqrt{t_j}<s<d_{t_j}(\tilde{x},\bar{x})$.

By the Ricci flow equation and (3.7), we know

$$e^{-Ce^{CK}s^{-2}t_j}g_{ij}(x,0)\leq g_{ij}(x,t)\leq e^{Ce^{CK}s^{-2}t_j}g_{ij}(x,0) \tag{3.8}$$

on $B_{t_j}(\gamma(s),\frac{s}{2})\times[0,t_j]$, for $8C(n)\sqrt{Kt_j}<s<\frac{b_j}{16}$. Note that $B_{t_j}(\gamma(s),\frac{s}{2})\subset B_0(\tilde{x},\frac{b_j}{7})$ and $g_{ij}(x,0)$ is Euclidean on $B_0(\tilde{x},\frac{b_j}{7})$. This implies

$$vol_{t_j}\left(B_{t_j}\left(\gamma(s),\frac{s}{2}\right)\right)\geq e^{-Ce^{CK}s^{-2}t_j}\alpha(n)(\frac{s}{2})^n e^{-Ce^{CK}s^{-2}t_j} \tag{3.9}$$

where $\alpha(n)$ is the volume of unit ball in the Euclidean space. Hence,

$$vol_{t_j}\left(B_{t_j}\left(\bar{x},\frac{3s}{2}\right)\right)\geq \alpha(n)(\frac{s}{2})^n e^{-Ce^{CK}s^{-2}t_j}, \tag{3.10}$$

whenever $8C(n)\sqrt{Kt_j}<s<\frac{b_j}{16}$.

Scaling the solution around $(\bar{x},t_j)$ by the factor t_j, so that we get a family of solutions to the Ricci flow $\tilde{g}_j(\tau)=\frac{1}{t_j}g(t_j\tau)$ for $\tau\in[0,1]$. We claim:

Lemma 3.2. *As $j\to\infty$, the pointed family of the solutions $\tilde{g}(\tau)$ will converge to a smooth complete solution $\tilde{g}(\tau)$ to the Ricci flow on some pointed manifold $\tilde{M}$ with nonnegative sectional curvature and maximal volume growth, for $\tau\in(0,1]$.*

Proof. By using Lemma 3.1, we know that any point with curvature greater than $\frac{K}{t}$ for the unrescaled solution will have a canonical neighborhood structure. Because we scale the solution with the factor t_j, one can prove (see section 12 in [7]) that the rescaled solutions at time $\tau=1$ will have bounded curvature on any fixed bounded distance. The curvature can also be shown to be uniformally bounded on $\tau=1$. The solution can be extended backward to whole interval (0,1]. The reason is that any point (x,τ) in the limit with curvature $|Rm|(x,\tau)\tau>K$ is a limit of points which have canonical neighborhoods with suitable curvature control. Here the key point is that the limit has nonnegative sectional curvature on the region where the limit exists.

To prove that $\tilde{M}$ has nonnegative sectional curvature, we note that for any fixed $m>1$, and each time $t\in[0,t_j]$, $B_t(\bar{x},m\sqrt{t_j})\subset B_t(\tilde{x},\frac{3b_j}{16})\subset B_t(\tilde{x},\frac{7b_j}{32})$ for large j. Since $B_0(\tilde{x},\frac{7b_j}{32})$ is flat at $t=0$, by pinching estimate (3.2), we know

$$\lambda+\mu+k\nu\geq-\frac{C_k}{b_j}$$

for the unrecaled solution. For the rescaled solution, we have

$$\lambda+\mu+k\nu\geq-\frac{C_k t_j}{b_j}.$$

Let $j\to\infty$, we find $\nu\geq 0$ on the limit. That is to say, the limit $\tilde{M}$ has nonnegative sectional curvature for any $\tau\in(0,1]$.

For any large and fixed $m>1$, let $s=m\sqrt{t_j}$ in (3.10), and let $j\to\infty$, we know that at time $\tau=1$,

$$vol_1\left(B_{\tilde{M}}\left(O,\frac{3m}{2}\right)\right)\geq\alpha(n)\left(\frac{m}{2}\right)^n e^{-Ce^{CK}m^{-2}}, \tag{3.11}$$

where O is the origin of $\tilde{M}$. This implies

$$\lim_{m\to\infty}\frac{vol_1(B_{\tilde{M}}(O,m))}{m^n}\geq\frac{1}{3^n}\alpha(n). \tag{3.12}$$

Since the curvature of $\tilde{M}$ is nonnegative, by the volume comparison theorem, we have

$$\frac{vol_1(B_{\tilde{M}}(O,m))}{m^n}\geq\frac{1}{3^n}\alpha(n) \tag{3.13}$$

for any $m\geq 0$. □

Proof of Lemma 2.2. Note that the origin O has curvature $2K$ at time $\tau=1$, hence it has a canonical neighborhood of accuracy ε. Combining (3.13) and (3.3), we get

$$C(\varepsilon)\geq\frac{1}{3^n}\alpha(n),$$

which is a contradiction for small ε since $\lim_{\varepsilon\to 0}C(\varepsilon)=0$. This contradiction shows that the curvature bound $\frac{2K(\varepsilon)}{t}$ can never be achieved in (3.4) for large j. That means $t_j=T_0$ for large j. Then Theorem 3.1 gives us the required curvature estimate. □

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