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Complex Oscillation of Differential Polynomials Generated by Meromorphic Solutions of [p,q] Order to Complex Non-homogeneous Linear Differential Equations

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Abstract. In this article we study the complex oscillation of differential polynomials generated by meromorphic solutions of the non-homogeneous linear differential equation

$$f^{(k)}+A_{k-1}(z)f^{(k-1)}+\cdots+A_1(z)f'+A_0(z)f=F,$$

where $A_i(z)$ $(i=0,1,\cdots,k-1)$ and F are meromorphic functions of finite $[p,q]$-order in the complex plane.

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1 Introduction and preliminaries

In this paper, we assume that the reader knows the standard notations and the fundamental results of the Nevanlinna's value distribution theory of meromorphic functions (see [10] , [15], [22]). Throughout this paper, we assume that a meromorphic function is meromorphic in the whole complex plane $\mathbb{C}$. Let us define inductively for $r\in\mathbb{R}$, $\exp_1 r:=e^r$ and

$$\exp_{p+1} r:=\exp\left(\exp_p r\right),\ p\in\mathbb{N}.$$

We also define for all r sufficiently large $\log_1 r:=\log r$ and

$$\log_{p+1} r:=\log\left(\log_p r\right),\ p\in\mathbb{N}.$$

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Moreover, we denote by $\exp_0 r := r$, $\log_0 r := r$, $\log_{-1} r := \exp_1 r$ and $\exp_{-1} r := \log_1 r$. In [12], [13], Juneja-Kapoor-Bajpai investigated some properties of growth of entire functions of $[p,q]$-order. In [21], in order to keep accordance with the general definitions of entire function $f(z)$ of iterated $p-$order [14], [15], Liu-Tu-Shi gave a minor modification to the original definition of $[p,q]-$order given in [12], [13]. With this new concept of $[p,q]-$order, the $[p,q]-$order of solutions of complex linear differential equations was investigated (see e.g. [2-4], [6], [11], [20], [21], [23]).

Now we introduce the definitions of the $[p,q]-$order as follows.

Definition 1.1. ([20, 21]) *Let $p \geq q \geq 1$ be integers. If $f(z)$ is a transcendental meromorphic function, then the $[p,q]$-order of $f(z)$ is defined by*

$$\rho_{[p,q]}(f) = \limsup_{r \longrightarrow +\infty} \frac{\log_p T(r,f)}{\log_q r},$$

where $T(r,f)$ is the Nevanlinna characteristic function of f. For $p=1$, this notation is called order and for $p=2$ hyper-order. It is easy to see that $0 \leq \rho_{[p,q]}(f) \leq \infty$. By Definition 1.1, we have that $\rho_{[1,1]}(f) = \rho_1(f) = \rho(f)$ usual order, $\rho_{[2,1]}(f) = \rho_2(f)$ hyper-order and $\rho_{[p,1]}(f) = \rho_p(f)$ iterated $p-$order.

Definition 1.2. ([11]) *Let $p \geq q \geq 1$ be integers. If $f(z)$ is a transcendental meromorphic function, then the lower $[p,q]$-order of $f(z)$ is defined by*

$$\mu_{[p,q]}(f) = \liminf_{r \longrightarrow +\infty} \frac{\log_p T(r,f)}{\log_q r}.$$

Remark 1.1. ([20]) If $f(z)$ is a meromorphic function satisfying $0 < \rho_{[p,q]}(f) < \infty$, then

(i) $\rho_{[p-n,q]}(f) = \infty$ $(n<p)$, $\rho_{[p,q-n]}(f) = 0$ $(n<q)$, $\rho_{[p+n,q+n]}(f) = 1$ $(n<p)$ for $n = 1,2,\cdots$

(ii) If $[p_1,q_1]$ is any pair of integers satisfying $q_1 = p_1 + q - p$ and $p_1 < p$, then $\rho_{[p_1,q_1]}(f) = 0$ if $0 < \rho_{[p,q]}(f) < 1$ and $\rho_{[p_1,q_1]}(f) = \infty$ if $1 < \rho_{[p,q]}(f) < \infty$.

(iii) $\rho_{[p_1,q_1]}(f) = \infty$ for $q_1 - p_1 > q - p$ and $\rho_{[p_1,q_1]}(f) = 0$ for $q_1 - p_1 < q - p$.

Definition 1.3. ([20]) *A transcendental meromorphic function $f(z)$ is said to have index-pair $[p,q]$ if $0 < \rho_{[p,q]}(f) < \infty$ and $\rho_{[p-1,q-1]}(f)$ is not a nonzero finite number.*

Remark 1.2. ([20]) Suppose that f_1 is a meromorphic function of $[p,q]$-order ρ_1 and f_2 is a meromorphic function of $[p_1,q_1]$-order ρ_2, let $p \leq p_1$. We can easily deduce the result about their comparative growth:

(i) If $p_1 - p > q_1 - q$, then the growth of f_1 is slower than the growth of f_2.

(ii) If $p_1-p<q_1-q$, then f_1 grows faster than f_2.

(iii) If $p_1-p=q_1-q>0$, then the growth of f_1 is slower than the growth of f_2 if $\rho_2\geq 1$, and the growth of f_1 is faster than the growth of f_2 if $\rho_2<1$.

(iv) Especially, when $p_1=p$ and $q_1=q$ then f_1 and f_2 are of the same index-pair $[p,q]$. If $\rho_1>\rho_2$, then f_1 grows faster than f_2; and if $\rho_1<\rho_2$, then f_1 grows slower than f_2. If $\rho_1=\rho_2$, Definition 1.1 does not show any precise estimate about the relative growth of f_1 and f_2.

Definition 1.4. ([20]) *Let $p\geq q\geq 1$ be integers. Let f be a meromorphic function satisfying $0<\rho_{[p,q]}(f)=\rho<\infty$. Then the $[p,q]-$type of $f(z)$ is defined by*

$$\tau_{[p,q]}(f)=\limsup_{r\to+\infty}\frac{\log_{p-1}T(r,f)}{\left[\log_{q-1}r\right]^{\rho}}.$$

Similarly, the $[p,q]-$type of an entire function f of $[p,q]-$order $0<\rho_{[p,q]}(f)=\rho<\infty$ is defined as

$$\tau_{M,[p,q]}(f)=\limsup_{r\to+\infty}\frac{\log_{p}M(r,f)}{\left[\log_{q-1}r\right]^{\rho}},$$

where $M(r,f)=\max_{|z|=r}|f(z)|$.

Definition 1.5. ([20]) *Let $p\geq q\geq 1$ be integers. The $[p,q]$-exponent of convergence of the zero-sequence of a meromorphic function $f(z)$ is defined by*

$$\lambda_{[p,q]}(f)=\limsup_{r\to+\infty}\frac{\log_{p}N\left(r,\frac{1}{f}\right)}{\log_{q}r}.$$

Similarly, the $[p,q]$-exponent of convergence of the sequence of distinct zeros of $f(z)$ is defined by

$$\overline{\lambda}_{[p,q]}(f)=\limsup_{r\to+\infty}\frac{\log_{p}\overline{N}\left(r,\frac{1}{f}\right)}{\log_{q}r},$$

where $N\left(r,\frac{1}{f}\right)$ $\left(\overline{N}\left(r,\frac{1}{f}\right)\right)$ is the integrated counting function of zeros (distinct zeros) of $f(z)$ in $\{z:|z|\leq r\}$. By Definition 1.5, we have that $\lambda_{[p,1]}(f)=\lambda_p(f)$ the iterated exponent of convergence of the sequence of zeros of $f(z)$ and $\overline{\lambda}_{[p,1]}(f)=\overline{\lambda}_p(f)$ the iterated exponent of convergence of the sequence of distinct zeros of $f(z)$.

Definition 1.6. ([11]) *Let $p\geq q\geq 1$ be integers. The lower $[p,q]$-exponent of convergence of the zero-sequence of a meromorphic function $f(z)$ is defined by*

$$\underline{\lambda}_{[p,q]}(f)=\liminf_{r\to+\infty}\frac{\log_{p}N\left(r,\frac{1}{f}\right)}{\log_{q}r}.$$

Similarly, the lower $[p,q]$-exponent of convergence of the sequence of distinct zeros of $f(z)$ is defined by

$$\underline{\overline{\lambda}}_{[p,q]}(f)=\liminf_{r\to+\infty}\frac{\log_p \overline{N}\left(r,\frac{1}{f}\right)}{\log_q r}.$$

By Definition 1.6*, we have that $\underline{\lambda}_{[p,1]}(f)=\underline{\lambda}_p(f)$ the iterated lower exponent of convergence of the sequence of zeros and $\underline{\overline{\lambda}}_{[p,1]}(f)=\underline{\overline{\lambda}}_p(f)$ the iterated lower exponent of convergence of the sequence of distinct zeros of $f(z)$. Moreover, we may obtain the definitions of $\lambda_{[p,q]}(f-\varphi)$, $\overline{\lambda}_{[p,q]}(f-\varphi)$, $\underline{\lambda}_{[p,q]}(f-\varphi)$ and $\underline{\overline{\lambda}}_{[p,q]}(f-\varphi)$, when f is replaced by a meromorphic function $f-\varphi$.*

2 Main results

Consider the complex differential equation

$$f^{(k)}+A_{k-1}(z)f^{(k-1)}+\cdots+A_1(z)f'+A_0(z)f=0 \tag{2.1}$$

and the differential polynomial

$$g_k=d_kf^{(k)}+d_{k-1}f^{(k-1)}+\cdots+d_0f, \tag{2.2}$$

where $A_0(z),A_1(z),...,A_{k-1}(z)$ and $d_0(z),d_1(z),...,d_k(z)$ are meromorphic functions in the complex plane.

Recently, many authors have investigated the complex oscillation properties of solutions and differential polynomials generated by solutions of differential equations in the unit disc and in the complex plane $\mathbb{C}$, see [1]- [9], [11], [16]- [21], [23]. Recently, the first author investigated the growth and oscillation of the differential polynomial (2.2) generated by meromorphic solutions of equation (2.1). Before we state those results and the results of this paper, we need to define the following sequences of functions $\alpha_{i,j},\beta_j$ $(i=0,\cdots,k-1;j=0,\cdots,k-1)$ by

$$\alpha_{i,j}=\begin{cases}\alpha'_{i,j-1}+\alpha_{i-1,j-1}-A_i\alpha_{k-1,j-1}, & \text{for all } i=1,\cdots,k-1,\\ \alpha'_{0,j-1}-A_0\alpha_{k-1,j-1}, & \text{for } i=0,\end{cases} \tag{2.3}$$

$$\alpha_{i,0}=d_i-d_kA_i,\ \text{ for } i=0,\cdots,k-1 \tag{2.4}$$

$$\beta_j=\begin{cases}\beta'_{j-1}+\alpha_{k-1,j-1}F, & \text{for all } j=1,\cdots,k-1,\\ d_kF, & \text{for } j=0.\end{cases} \tag{2.5}$$

We define also h_k by

$$h_k=\begin{vmatrix}\alpha_{0,0} & \alpha_{1,0} & \cdot & \cdot & \alpha_{k-1,0}\\ \alpha_{0,1} & \alpha_{1,1} & \cdot & \cdot & \alpha_{k-1,1}\\ \cdot & \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdot\\ \alpha_{0,k-1} & \alpha_{1,k-1} & \cdot & \cdot & \alpha_{k-1,k-1}\end{vmatrix}$$

and $\psi(z)$, $\psi_k(z)$ by

$$\psi(z)=\frac{1}{h_k(z)}\begin{vmatrix} \varphi & \alpha_{1,0} & \cdot & \cdot & \alpha_{k-1,0} \\ \varphi' & \alpha_{1,1} & \cdot & \cdot & \alpha_{k-1,1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \varphi^{(k-1)} & \alpha_{1,k-1} & \cdot & \cdot & \alpha_{k-1,k-1} \end{vmatrix},$$

$$\psi_k(z)=\frac{1}{h_k(z)}\begin{vmatrix} \varphi-\beta_0 & \alpha_{1,0} & \cdot & \cdot & \alpha_{k-1,0} \\ \varphi'-\beta_1 & \alpha_{1,1} & \cdot & \cdot & \alpha_{k-1,1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \varphi^{(k-1)}-\beta_{k-1} & \alpha_{1,k-1} & \cdot & \cdot & \alpha_{k-1,k-1} \end{vmatrix},$$

where $h_k\not\equiv 0$ and $\alpha_{i,j}$ $(i=0,\cdots,k-1;j=0,\cdots,k-1)$ are defined in (2.3) and (2.4), and φ is a meromorphic function with $\rho_{[p,q]}(\varphi)<\infty$.

Theorem 2.1. ([6]) *Let $p\geq q\geq 1$ be integers, and let $A_i(z)$ $(i=0,1,\cdots,k-1)$ be meromorphic functions of finite $[p,q]-$order. Let $d_j(z)$ $(j=0,1,\cdots,k)$ be finite $[p,q]-$order meromorphic functions that are not all vanishing identically such that $h_k\not\equiv 0$. If $f(z)$ is an infinite $[p,q]-$order meromorphic solution of* (2.1) *with $\rho_{[p+1,q]}(f)=\rho$, then the differential polynomial* (2.2) *satisfies*

$$\rho_{[p,q]}(g_k)=\rho_{[p,q]}(f)=\infty,$$
$$\rho_{[p+1,q]}(g_k)=\rho_{[p+1,q]}(f)=\rho.$$

Furthermore, if f is a finite $[p,q]-$order meromorphic solution of (2.1) *such that*

$$\rho_{[p,q]}(f)>\max\left\{\rho_{[p,q]}(A_i)(i=0,1,\cdots,k-1),\rho_{[p,q]}(d_j)\ \ (j=0,1,\cdots,k)\right\},$$

then

$$\rho_{[p,q]}(g_k)=\rho_{[p,q]}(f).$$

Theorem 2.2. ([6]) *Under the hypotheses of Theorem* 2.1 *, let $\varphi(z)\not\equiv 0$ be a meromorphic function of finite $[p,q]-$order such that $\psi(z)$ is not a solution of* (2.1)*. If $f(z)$ is an infinite $[p,q]-$order meromorphic solution of* (2.1) *with $\rho_{[p+1,q]}(f)=\rho$, then the differential polynomial* (2.2) *satisfies*

$$\overline{\lambda}_{[p,q]}(g_k-\varphi)=\lambda_{[p,q]}(g_k-\varphi)=\rho_{[p,q]}(f)=\infty,$$
$$\overline{\lambda}_{[p+1,q]}(g_k-\varphi)=\lambda_{[p+1,q]}(g_k-\varphi)=\rho_{[p+1,q]}(f)=\rho.$$

Furthermore, if f is a finite $[p,q]-$order meromorphic solution of (2.1) such that

$$\rho_{[p,q]}(f)>\max\Big\{\rho_{[p,q]}(A_i)\ \ (i=0,1,\cdots,k-1),\rho_{[p,q]}(\varphi),$$
$$\rho_{[p,q]}(d_j)\ \ (j=0,1,\cdots,k)\Big\}$$

then

$$\overline{\lambda}_{[p,q]}(g_k-\varphi)=\lambda_{[p,q]}(g_k-\varphi)=\rho_{[p,q]}(f).$$

Theorems 2.1 and 2.2 investigated the growth and oscillation of higher order differential polynomial (2.2) generated by meromorphic solutions of homogeneous linear differential equation (2.1). A natural question is now that when the equation is non-homogeneous linear differential, can the similar results hold? We give an affirmative answer by studying the controllability of solutions of the non-homogeneous linear differential equation

$$f^{(k)}+A_{k-1}(z)f^{(k-1)}+\cdots+A_1(z)f'+A_0(z)f=F, \tag{2.6}$$

where, $A_0(z)$, $A_1(z)$,..., $A_{k-1}(z)$, $F(z)$ are meromorphic functions of finite $[p,q]-$order.

Theorem 2.3. *Let $p\geq q\geq 1$ be integers, and let $A_i(z)$ $(i=0,1,\cdots,k-1)$, $F(z)$ be meromorphic functions of finite $[p,q]-$order. Let $d_j(z)$ $(j=0,1,\cdots,k)$ be finite $[p,q]-$order meromorphic functions that are not all vanishing identically such that $h_k\not\equiv 0$. If $f(z)$ is an infinite $[p,q]-$order meromorphic solution of* (2.6) *with $\rho_{[p+1,q]}(f)=\rho$, then the differential polynomial* (2.2) *satisfies*

$$\rho_{[p,q]}(g_k)=\rho_{[p,q]}(f)=\infty,$$
$$\rho_{[p+1,q]}(g_k)=\rho_{[p+1,q]}(f)=\rho.$$

Furthermore, if f is a finite $[p,q]-$order meromorphic solution of (2.6) *such that*

$$\rho_{[p,q]}(f)>\max\Big\{\rho_{[p,q]}(A_i)(i=0,1,\cdots,k-1),\rho_{[p,q]}(F),$$
$$\rho_{[p,q]}(d_j)\ \ (j=0,1,\cdots,k)\Big\}, \tag{2.7}$$

then

$$\rho_{[p,q]}(g_k)=\rho_{[p,q]}(f).$$

Remark 2.1. In Theorem 2.3, if we do not have the condition $h_k\not\equiv 0$, then the conclusions of Theorem 2.3 cannot hold. For example, if we take $d_i=d_kA_i$ $(i=0,\cdots,k-1)$, then $h_k\equiv 0$. It follows that $g_k\equiv d_kF$. So, if $f(z)$ is an infinite $[p,q]-$order meromorphic solution of (2.6), then $\rho_{[p,q]}(g_k)=\rho_{[p,q]}(d_kF)<\rho_{[p,q]}(f)=\infty$, and if f is a finite $[p,q]-$order meromorphic solution of (2.6) such that (2.7) holds, then

$$\rho_{[p,q]}(g_k)=\rho_{[p,q]}(d_kF)\leq\max\Big\{\rho_{[p,q]}(F),\rho_{[p,q]}(d_k)\Big\}<\rho_{[p,q]}(f).$$

Theorem 2.4. *Under the hypotheses of Theorem* 2.3*, let $\varphi(z)\not\equiv 0$ be a meromorphic function of finite $[p,q]-$order such that $\psi_k(z)$ is not a solution of* (2.6)*. If $f(z)$ is an infinite $[p,q]-$order meromorphic solution of* (2.6) *with $\rho_{[p+1,q]}(f)=\rho$, then the differential polynomial* (2.2) *satisfies*

$$\overline{\lambda}_{[p,q]}(g_k-\varphi)=\lambda_{[p,q]}(g_k-\varphi)=\rho_{[p,q]}(f)=\infty,$$
$$\overline{\lambda}_{[p+1,q]}(g_k-\varphi)=\lambda_{[p+1,q]}(g_k-\varphi)=\rho_{[p+1,q]}(f)=\rho.$$

Furthermore, if f is a finite $[p,q]-$order meromorphic solution of (2.6) *such that*

$$\rho_{[p,q]}(f) > \max\Big\{\rho_{[p,q]}(A_i)\ (i=0,1,\cdots,k-1), \rho_{[p,q]}(F),\\ \rho_{[p,q]}(\varphi), \rho_{[p,q]}(d_j)\ (j=0,1,\cdots,k)\Big\}, \tag{2.8}$$

then

$$\overline{\lambda}_{[p,q]}(g_k-\varphi)=\lambda_{[p,q]}(g_k-\varphi)=\rho_{[p,q]}(f).$$

Corollary 2.1. *Let $p\geq q\geq 1$ be integers, and let $A_0(z),\cdots, A_{k-1}(z), F(z)\not\equiv 0$ be finite $[p,q]-$order meromorphic functions such that*

$$\max\left\{\rho_{[p,q]}(A_i), \lambda_{[p,q]}\left(\frac{1}{A_0}\right), \rho_{[p+1,q]}(F) : i=1,\cdots,k-1\right\} < \rho_{[p,q]}(A_0).$$

Let $d_j(z)$ $(j=0,1,\cdots,k)$ be finite $[p,q]-$order meromorphic functions that are not all vanishing identically such that $h_k\not\equiv 0$, and let $\varphi(z)$ be a meromorphic function of finite $[p,q]-$order such that $\psi_k(z)$ is not a solution of (2.6). *Then the differential polynomial* (2.2) *satisfying for all meromorphic solutions f whose poles are of uniformly bounded multiplicities of equation* (2.6)

$$\overline{\lambda}_{[p,q]}(g_k-\varphi)=\lambda_{[p,q]}(g_k-\varphi)=\rho_{[p,q]}(f)=\infty,$$
$$\overline{\lambda}_{[p+1,q]}(g_k-\varphi)=\lambda_{[p+1,q]}(g_k-\varphi)=\rho_{[p+1,q]}(g_k)=\rho_{[p+1,q]}(f)=\rho_{[p,q]}(A_0),$$

with at most one exceptional solution f_0 satisfying $\rho_{[p+1,q]}(f_0)<\rho_{[p,q]}(A_0)$.

We consider now the differential equation

$$f''+A(z)f=F, \tag{2.9}$$

where $A(z)$ and $F(z)$ are meromorphic functions of finite $[p,q]-$order. In the following, we will give sufficient conditions on A and F which satisfied the results of Theorem 2.3 and Theorem 2.4 without the conditions " $h_k\not\equiv 0$ " and " $\psi_k(z)$ is not a solution of (2.6) " where $k=2$.

Corollary 2.2. *Let $p\geq q\geq 1$ be integers, and let $A(z)$, $F(z)\not\equiv 0$ be meromorphic functions satisfying*

$$\rho_{[p,q]}(F)<\mu_{[p,q]}(A)\leq\rho_{[p,q]}(A)<\infty, \quad 0<\tau_{[p,q]}(A)<\infty.$$

Suppose that $A(z)=\sum_{n=0}^{\infty}c_{\lambda_n}z^{\lambda_n}$ is also an entire function such that the sequence of exponents $\{\lambda_n\}$ satisfies the gap condition $\lambda_n/n>(\log n)^{2+\eta}$ $(\eta>0, n\in\mathbb{N})$. Let d_0, d_1, d_2 be meromorphic functions that are not all vanishing identically such that

$$\max\Big\{\rho_{[p,q]}(d_j)\ (j=0,1,2)\Big\}<\mu_{[p,q]}(A)\leq\rho_{[p,q]}(A)<\infty.$$

If $f(z)$ is a transcendental meromorphic solution to (2.9) *satisfying*

$$\lambda_{[p,q]}(1/f) < \mu_{[p,q]}(A),$$

then the differential polynomial $g_2 = d_2 f'' + d_1 f' + d_0 f$ satisfies

$$\mu_{[p,q]}(g_2) = \rho_{[p,q]}(g_2) = \mu_{[p,q]}(f) = \rho_{[p,q]}(f) = \infty,$$
$$\mu_{[p+1,q]}(g_2) = \mu_{[p+1,q]}(f) = \mu_{[p,q]}(A) \le \rho_{[p+1,q]}(g_2) = \rho_{[p+1,q]}(f) = \rho_{[p,q]}(A).$$

Corollary 2.3. *Under the hypotheses of Corollary* 2.2*, let φ be a meromorphic function such that $\rho_{[p,q]}(\varphi) < \infty$. If $f(z)$ is a transcendental meromorphic solution to* (2.9) *satisfying $\lambda_{[p,q]}(1/f) < \mu_{[p,q]}(A)$, then the differential polynomial $g_2 = d_2 f'' + d_1 f' + d_0 f$ with $d_2 \not\equiv 0$ satisfies*

$$\underline{\overline{\lambda}}_{[p+1,q]}(g_2 - \varphi) = \underline{\lambda}_{[p+1,q]}(g_2 - \varphi) = \mu_{[p,q]}(A)$$
$$\le \overline{\lambda}_{[p+1,q]}(g_2 - \varphi) = \lambda_{[p+1,q]}(g_2 - \varphi) = \rho_{[p,q]}(A).$$

Remark 2.2. The present article may be understood as an extension and improvement of the recent article of the first author [6] from equation (2.1) to equation (2.6).

3 Some lemmas

Lemma 3.1. ([20]) *Let $p \ge q \ge 1$ be integers, and let $A_0, A_1, \cdots, A_{k-1}, F \not\equiv 0$ be meromorphic functions. If $f(z)$ is a meromorphic solution of equation* (2.6) *such that*

$$\max\Big\{\rho_{[p,q]}(F), \rho_{[p,q]}(A_i) \ \ (i = 0, \cdots, k-1)\Big\} < \rho_{[p,q]}(f) < +\infty,$$

then $\overline{\lambda}_{[p,q]}(f) = \lambda_{[p,q]}(f) = \rho_{[p,q]}(f)$.

Lemma 3.2. ([6]) *Let $p \ge q \ge 1$ be integers, and let $A_0, A_1, \cdots, A_{k-1}, F \not\equiv 0$ be finite $[p,q]-$order meromorphic functions. If $f(z)$ is a meromorphic solution of equation* (2.6) *with $\rho_{[p,q]}(f) = +\infty$ and $\rho_{[p+1,q]}(f) = \rho < +\infty$, then*

$$\overline{\lambda}_{[p,q]}(f) = \lambda_{[p,q]}(f) = \rho_{[p,q]}(f) = \infty, \quad \overline{\lambda}_{[p+1,q]}(f) = \lambda_{[p+1,q]}(f) = \rho_{[p+1,q]}(f) = \rho.$$

Lemma 3.3. ([6]) *Let $p \ge q \ge 1$ be integers, and let f, g be non-constant meromorphic functions of $[p,q]$-order. Then we have*

$$\rho_{[p,q]}(f+g) \le \max\Big\{\rho_{[p,q]}(f), \rho_{[p,q]}(g)\Big\},$$
$$\rho_{[p,q]}(fg) \le \max\Big\{\rho_{[p,q]}(f), \rho_{[p,q]}(g)\Big\}.$$

Furthermore, if $\rho_{[p,q]}(f) > \rho_{[p,q]}(g)$, then we obtain $\rho_{[p,q]}(f+g) = \rho_{[p,q]}(fg) = \rho_{[p,q]}(f)$.

Lemma 3.4. *Let $p \geq q \geq 1$ be integers, and let f, g be non-constant meromorphic functions with $\rho_{[p,q]}(f)$ as $[p,q]$-order of f and $\mu_{[p,q]}(g)$ as lower $[p,q]$-order of g. Then we have*

$$\mu_{[p,q]}(f+g) \leq \max\left\{\rho_{[p,q]}(f), \mu_{[p,q]}(g)\right\},$$
$$\mu_{[p,q]}(fg) \leq \max\left\{\rho_{[p,q]}(f), \mu_{[p,q]}(g)\right\}.$$

Furthermore, if $\mu_{[p,q]}(g) > \rho_{[p,q]}(f)$, then we obtain

$$\mu_{[p,q]}(f+g) = \mu_{[p,q]}(fg) = \mu_{[p,q]}(g).$$

Proof. Without loss of generality, we assume that $\rho_{[p,q]}(f) < +\infty$ and $\mu_{[p,q]}(g) < +\infty$. From the definition of the lower $[p,q]-$order, there exists a sequence $r_n \to +\infty$ $(n \to +\infty)$ such that

$$\lim_{n\to+\infty} \frac{\log_p T(r_n,g)}{\log_q r_n} = \mu_{[p,q]}(g).$$

Then, for any given $\varepsilon > 0$, there exists a positive integer N_1 such that

$$T(r_n,g) \leq \exp_p\left\{(\mu_{[p,q]}(g)+\varepsilon)\log_q r_n\right\}$$

holds for $n > N_1$. From the definition of the $[p,q]-$order, for any given $\varepsilon > 0$, there exists a positive number R such that

$$T(r,f) \leq \exp_p\left\{(\rho_{[p,q]}(f)+\varepsilon)\log_q r\right\}$$

holds for $r \geq R$. Since $r_n \to +\infty$ $(n \to +\infty)$, there exists a positive integer N_2 such that $r_n > R$, and thus

$$T(r_n,f) \leq \exp_p\left\{(\rho_{[p,q]}(f)+\varepsilon)\log_q r_n\right\}$$

holds for $n > N_2$. Note that

$$T(r,f+g) \leq T(r,f) + T(r,g) + \ln 2,$$
$$T(r,fg) \leq T(r,f) + T(r,g).$$

Then, for any given $\varepsilon > 0$, we have for $n > \max\{N_1, N_2\}$

$$\begin{aligned} &T(r_n,f+g) \leq T(r_n,f) + T(r_n,g) + \ln 2 \\ &\leq \exp_p\{(\rho_{[p,q]}(f)+\varepsilon)\log_q r_n\} + \exp_p\{(\mu_{[p,q]}(g)+\varepsilon)\log_q r_n\} + \ln 2 \\ &\leq 2\exp_p\left\{\left(\max\left\{\rho_{[p,q]}(f), \mu_{[p,q]}(g)\right\}+\varepsilon\right)\log_q r_n\right\} + \ln 2 \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} &T(r_n,fg) \leq T(r_n,f) + T(r_n,g) \\ &\leq 2\exp_p\left\{\left(\max\left\{\rho_{[p,q]}(f), \mu_{[p,q]}(g)\right\}+\varepsilon\right)\log_q r_n\right\}. \end{aligned} \tag{3.2}$$

Since $\varepsilon>0$ is arbitrary, from (3.1) and (3.2), we easily obtain

$$\mu_{[p,q]}(f+g)\leq \max\left\{\rho_{[p,q]}(f),\mu_{[p,q]}(g)\right\}, \tag{3.3}$$

$$\mu_{[p,q]}(fg)\leq \max\left\{\rho_{[p,q]}(f),\mu_{[p,q]}(g)\right\}. \tag{3.4}$$

Suppose now that $\mu_{[p,q]}(g)>\rho_{[p,q]}(f)$. Considering that

$$T(r,g)=T(r,f+g-f)\leq T(r,f+g)+T(r,f)+\ln 2, \tag{3.5}$$

$$\begin{aligned} T(r,g)&=T\left(r,\frac{fg}{f}\right)\leq T(r,fg)+T\left(r,\frac{1}{f}\right)\\ &=T(r,fg)+T(r,f)+O(1). \end{aligned} \tag{3.6}$$

By (3.5), (3.6) and the same method as above we obtain that

$$\mu_{[p,q]}(g)\leq \max\left\{\mu_{[p,q]}(f+g),\rho_{[p,q]}(f)\right\}=\mu_{[p,q]}(f+g), \tag{3.7}$$

$$\mu_{[p,q]}(g)\leq \max\left\{\mu_{[p,q]}(fg),\rho_{[p,q]}(f)\right\}=\mu_{[p,q]}(fg). \tag{3.8}$$

By using (3.3) and (3.7) we obtain $\mu_{[p,q]}(f+g)=\mu_{[p,q]}(g)$ and by (3.4) and (3.8), we get $\mu_{[p,q]}(fg)=\mu_{[p,q]}(g)$. □

Lemma 3.5. ([6]) *Let $p\geq q\geq 1$ be integers, and let f,g be meromorphic functions with $[p,q]-$order $0<\rho_{[p,q]}(f),\rho_{[p,q]}(g)<\infty$ and $[p,q]-$type $0<\tau_{[p,q]}(f),\tau_{[p,q]}(g)<\infty$. Then the following statements hold:*
(i) If $\rho_{[p,q]}(g)<\rho_{[p,q]}(f)$, then

$$\tau_{[p,q]}(f+g)=\tau_{[p,q]}(fg)=\tau_{[p,q]}(f).$$

(ii) If $\rho_{[p,q]}(f)=\rho_{[p,q]}(g)$ and $\tau_{[p,q]}(g)\neq\tau_{[p,q]}(f)$, then

$$\rho_{[p,q]}(f+g)=\rho_{[p,q]}(fg)=\rho_{[p,q]}(f).$$

Lemma 3.6. *(see Theorem 1.6 in [20]) Let $p\geq q\geq 1$ be integers, and let $A_0(z),\cdots, A_{k-1}(z)$, $F(z)\not\equiv 0$ be meromorphic functions in the plane satisfying*

$$\max\left\{\rho_{[p,q]}(A_i),\lambda_{[p,q]}\left(\frac{1}{A_0}\right),\rho_{[p+1,q]}(F):i=1,\cdots,k-1\right\}<\rho_{[p,q]}(A_0).$$

Then all meromorphic solutions f whose poles are of uniformly bounded multiplicities of (2.6) satisfy $\rho_{[p+1,q]}(f)=\rho_{[p,q]}(A_0)$ with at most one exceptional solution f_0 satisfying $\rho_{[p+1,q]}(f_0)<\rho_{[p,q]}(A_0)$.

Lemma 3.7. *(see Theorem* 2.1 *in* [23]*) Let $p\geq q\geq 1$ be integers, and let $A_0(z),\cdots, A_{k-1}(z)$, $F(z)$ be meromorphic functions satisfying that there exists some $d\in\{0,...,k-1\}$ such that*

$$\rho_1=\max\{\rho_{[p,q]}(F),\rho_{[p,q]}(A_i),i\neq d: i=0,1,\cdots,k-1\}<\mu_{[p,q]}(A_d)\leq\rho_{[p,q]}(A_d)<\infty.$$

Suppose that $A_d(z)=\sum_{n=0}^{\infty}c_{\lambda_n}z^{\lambda_n}$ is also an entire function such that the sequence of exponents $\{\lambda_n\}$ satisfies the gap condition

$$\lambda_n/n>(\log n)^{2+\eta}\ (\eta>0,n\in\mathbb{N}).$$

If $f(z)$ is a transcendental meromorphic solution to (2.6) *satisfying $\lambda_{[p,q]}(1/f)<\mu_{[p,q]}(A_d)$, then we have*

$$\mu_{[p+1,q]}(f)=\mu_{[p,q]}(A_d)\leq\rho_{[p+1,q]}(f)=\rho_{[p,q]}(A_d).$$

Lemma 3.8. ([11]) *Let $p\geq q\geq 1$ be integers, and let A_0, $A_1,\cdots$, A_{k-1}, $F\not\equiv 0$ be meromorphic functions. If $f(z)$ is a meromorphic solution of equation* (2.6) *satisfying*

$$\max\Big\{\rho_{[p+1,q]}(F),\rho_{[p+1,q]}(A_i):i=0,1,\cdots,k-1\Big\}<\mu_{[p+1,q]}(f), \tag{3.9}$$

then $\overline{\lambda}_{[p+1,q]}(f)=\underline{\lambda}_{[p+1,q]}(f)=\mu_{[p+1,q]}(f)$.

Lemma 3.9. *Assume that $f(z)$ is a solution of equation* (2.6)*. Then the differential polynomial g_k defined in* (2.2) *satisfies the system of equations*

$$\begin{cases} g_k-\beta_0=\alpha_{0,0}f+\alpha_{1,0}f'+\cdots+\alpha_{k-1,0}f^{(k-1)},\\ g_k'-\beta_1=\alpha_{0,1}f+\alpha_{1,1}f'+\cdots+\alpha_{k-1,1}f^{(k-1)},\\ g_k''-\beta_2=\alpha_{0,2}f+\alpha_{1,2}f'+\cdots+\alpha_{k-1,2}f^{(k-1)},\\ \cdots\\ g_k^{(k-1)}-\beta_{k-1}=\alpha_{0,k-1}f+\alpha_{1,k-1}f'+\cdots+\alpha_{k-1,k-1}f^{(k-1)}, \end{cases} \tag{3.10}$$

where

$$\alpha_{i,j}=\begin{cases} \alpha'_{i,j-1}+\alpha_{i-1,j-1}-A_i\alpha_{k-1,j-1}, & \text{for all } i=1,\cdots,k-1,\\ \alpha'_{0,j-1}-A_0\alpha_{k-1,j-1}, & \text{for } i=0, \end{cases} \tag{3.11a}$$

$$\alpha_{i,0}=d_i-d_kA_i,\quad \text{for } i=0,\cdots,k-1 \tag{3.11b}$$

$$\beta_j=\begin{cases} \beta'_{j-1}+\alpha_{k-1,j-1}F, & \text{for all } j=1,\cdots,k-1,\\ d_kF, & \text{for } j=0. \end{cases} \tag{3.11c}$$

Proof. Suppose that f is a solution of (2.6). We can rewrite (2.6) as

$$f^{(k)}=F-\sum_{i=0}^{k-1}A_if^{(i)} \tag{3.12}$$

which implies

$$g_k = d_k f^{(k)} + d_{k-1} f^{(k-1)} + \cdots + d_1 f' + d_0 f$$
$$= \sum_{i=0}^{k-1} (d_i - d_k A_i) f^{(i)} + d_k F. \tag{3.13}$$

We can rewrite (3.13) as

$$g_k - \beta_0 = \sum_{i=0}^{k-1} \alpha_{i,0} f^{(i)}, \tag{3.14}$$

where $\alpha_{i,0}$ are defined in (2.4) and $\beta_0 = d_k F$. Differentiating both sides of Eq. (3.14) and replacing $f^{(k)}$ with (3.12), we obtain

$$g_k' - \beta_0' = \sum_{i=0}^{k-1} \alpha_{i,0}' f^{(i)} + \sum_{i=0}^{k-1} \alpha_{i,0} f^{(i+1)} = \sum_{i=0}^{k-1} \alpha_{i,0}' f^{(i)} + \sum_{i=1}^{k} \alpha_{i-1,0} f^{(i)}$$
$$= \alpha_{0,0}' f + \sum_{i=1}^{k-1} \alpha_{i,0}' f^{(i)} + \sum_{i=1}^{k-1} \alpha_{i-1,0} f^{(i)} + \alpha_{k-1,0} f^{(k)}$$
$$= \alpha_{0,0}' f + \sum_{i=1}^{k-1} \alpha_{i,0}' f^{(i)} + \sum_{i=1}^{k-1} \alpha_{i-1,0} f^{(i)} - \sum_{i=0}^{k-1} \alpha_{k-1,0} A_i f^{(i)} + \alpha_{k-1,0} F$$
$$= \left(\alpha_{0,0}' - \alpha_{k-1,0} A_0\right) f + \sum_{i=1}^{k-1} \left(\alpha_{i,0}' + \alpha_{i-1,0} - \alpha_{k-1,0} A_i\right) f^{(i)} + \alpha_{k-1,0} F. \tag{3.15}$$

We can rewrite (3.15) as

$$g_k' - \beta_1 = \sum_{i=0}^{k-1} \alpha_{i,1} f^{(i)}, \tag{3.16}$$

where

$$\alpha_{i,1} = \begin{cases} \alpha_{i,0}' + \alpha_{i-1,0} - \alpha_{k-1,0} A_i, & \text{for all } i = 1, \cdots, k-1, \\ \alpha_{0,0}' - A_0 \alpha_{k-1,0}, & \text{for } i = 0, \end{cases} \tag{3.17}$$
$$\beta_1 = \beta_0' + \alpha_{k-1,0} F.$$

Differentiating both sides of Eq. (3.16) and replacing $f^{(k)}$ with (3.12), we obtain

$$g_k'' - \beta_1' = \sum_{i=0}^{k-1} \alpha_{i,1}' f^{(i)} + \sum_{i=0}^{k-1} \alpha_{i,1} f^{(i+1)} = \sum_{i=0}^{k-1} \alpha_{i,1}' f^{(i)} + \sum_{i=1}^{k} \alpha_{i-1,1} f^{(i)}$$
$$= \alpha_{0,1}' f + \sum_{i=1}^{k-1} \alpha_{i,1}' f^{(i)} + \sum_{i=1}^{k-1} \alpha_{i-1,1} f^{(i)} + \alpha_{k-1,1} f^{(k)}$$
$$= \alpha_{0,1}' f + \sum_{i=1}^{k-1} \alpha_{i,1}' f^{(i)} + \sum_{i=1}^{k-1} \alpha_{i-1,1} f^{(i)} - \sum_{i=0}^{k-1} A_i \alpha_{k-1,1} f^{(i)} + \alpha_{k-1,1} F$$
$$= \left(\alpha_{0,1}' - \alpha_{k-1,1} A_0\right) f + \sum_{i=1}^{k-1} \left(\alpha_{i,1}' + \alpha_{i-1,1} - A_i \alpha_{k-1,1}\right) f^{(i)} + \alpha_{k-1,1} F \tag{3.18}$$

which implies that

$$g_k'' - \beta_2 = \sum_{i=0}^{k-1} \alpha_{i,2} f^{(i)}, \tag{3.19}$$

where

$$\alpha_{i,2} = \begin{cases} \alpha_{i,1}' + \alpha_{i-1,1} - A_i \alpha_{k-1,1}, & \text{for all } i = 1, \cdots, k-1, \\ \alpha_{0,1}' - A_0 \alpha_{k-1,1}, & \text{for } i = 0 \end{cases} \tag{3.20}$$
$$\beta_2 = \beta_1' + \alpha_{k-1,1} F.$$

By using the same method as above we can easily deduce that

$$g_k^{(j)} - \beta_j = \sum_{i=0}^{k-1} \alpha_{i,j} f^{(i)}, \quad j = 0, 1, \cdots, k-1, \tag{3.21}$$

where $\alpha_{i,j}$ and β_j are given by (3.11). This completes the proof of Lemma 3.9. □

Lemma 3.10. ([20]) *If $f(z)$ is a meromorphic function, then $\rho_{(p,q)}(f') = \rho_{(p,q)}(f)$.*

Lemma 3.11. ([23]) *If $f(z)$ is a meromorphic function, then $\mu_{(p,q)}(f') = \mu_{(p,q)}(f)$.*

4 Proof of the Theorems and the Corollaries

Proof of Theorem 2.3. Suppose that $f(z)$ is an infinite $[p,q]-$order meromorphic solution of (2.6) with $\rho_{[p+1,q]}(f) = \rho$. By Lemma 3.9, g_k satisfies the system of equations

$$\begin{cases} g_k - \beta_0 = \alpha_{0,0} f + \alpha_{1,0} f' + \cdots + \alpha_{k-1,0} f^{(k-1)}, \\ g_k' - \beta_1 = \alpha_{0,1} f + \alpha_{1,1} f' + \cdots + \alpha_{k-1,1} f^{(k-1)}, \\ g_k'' - \beta_2 = \alpha_{0,2} f + \alpha_{1,2} f' + \cdots + \alpha_{k-1,2} f^{(k-1)}, \\ \cdots \\ g_k^{(k-1)} - \beta_{k-1} = \alpha_{0,k-1} f + \alpha_{1,k-1} f' + \cdots + \alpha_{k-1,k-1} f^{(k-1)}, \end{cases} \tag{4.1}$$

where

$$\alpha_{i,j} = \begin{cases} \alpha_{i,j-1}' + \alpha_{i-1,j-1} - A_i \alpha_{k-1,j-1}, & \text{for all } i = 1, \cdots, k-1, \\ \alpha_{0,j-1}' - A_0 \alpha_{k-1,j-1}, & \text{for } i = 0, \end{cases} \tag{4.2}$$

$$\alpha_{i,0} = d_i - d_k A_i, \quad \text{for all } i = 0, 1, \cdots, k-1 \tag{4.3}$$

$$\beta_j = \begin{cases} \beta_{j-1}' + \alpha_{k-1,j-1} F, & \text{for all } j = 1, \cdots, k-1, \\ d_k F, & \text{for } j = 0. \end{cases} \tag{4.4}$$

By Cramer's rule, and since $h_k \not\equiv 0$, then we have

$$f=\frac{1}{h_k}\begin{vmatrix} g_k-\beta_0 & \alpha_{1,0} & . & . & \alpha_{k-1,0} \\ g_k'-\beta_1 & \alpha_{1,1} & . & . & \alpha_{k-1,1} \\ . & . & . & . & . \\ . & . & . & . & . \\ g_k^{(k-1)}-\beta_{k-1} & \alpha_{1,k-1} & . & . & \alpha_{k-1,k-1} \end{vmatrix}.$$

It follows that

$$\begin{aligned} f=&C_0(g_k-\beta_0)+C_1(g_k'-\beta_1)+\cdots+C_{k-1}\left(g_k^{(k-1)}-\beta_{k-1}\right) \\ =&C_0g_k+C_1g_k'+\cdots+C_{k-1}g_k^{(k-1)}-\sum_{j=0}^{k-1}C_j\beta_j, \end{aligned} \tag{4.5}$$

where C_j are finite $[p,q]-$order meromorphic functions depending on $\alpha_{i,j}$, where $\alpha_{i,j}$ are defined in (4.2) ,(4.3) and β_j are defined in (4.4).

If $\rho_{[p,q]}(g_k)<+\infty$, then by (4.5) we obtain $\rho_{[p,q]}(f)<+\infty$, which is a contradiction. Hence $\rho_{[p,q]}(g_k)=\rho_{[p,q]}(f)=+\infty$.

Now, we prove that $\rho_{[p+1,q]}(g_k)=\rho_{[p+1,q]}(f)=\rho$. By (2.2), Lemma 3.3 and Lemma 3.10, we get $\rho_{[p+1,q]}(g_k)\le\rho_{[p+1,q]}(f)$ and by (4.5) we have $\rho_{[p+1,q]}(f)\le\rho_{[p+1,q]}(g_k)$. This yield $\rho_{[p+1,q]}(g_k)=\rho_{[p+1,q]}(f)=\rho$.

Furthermore, if $f(z)$ is a finite $[p,q]-$order meromorphic solution of Eq. (2.6) such that

$$\rho_{[p,q]}(f)>\max\left\{\rho_{[p,q]}(F),\rho_{[p,q]}(A_i),\quad \rho_{[p,q]}(d_j):i=0,\cdots,k-1,j=0,1,\cdots,k-1\right\}, \tag{4.6}$$

then

$$\rho_{[p,q]}(f)>\max\left\{\rho_{[p,q]}(\alpha_{i,j}),\rho_{[p,q]}(C_j\beta_j):i=0,\cdots,k-1,j=0,\cdots,k-1\right\}. \tag{4.7}$$

By (2.2) and (4.6) we have $\rho_{[p,q]}(g_k)\le\rho_{[p,q]}(f)$. Now, we prove $\rho_{[p,q]}(g_k)=\rho_{[p,q]}(f)$. If $\rho_{[p,q]}(g_k)<\rho_{[p,q]}(f)$, then by (4.5) and (4.7) we get

$$\rho_{[p,q]}(f)\le\max\left\{\rho_{[p,q]}(C_j\beta_j)\ \ (j=0,\cdots,k-1),\rho_{[p,q]}(g_k)\right\}<\rho_{[p,q]}(f)$$

which is a contradiction. Hence $\rho_{[p,q]}(g_k)=\rho_{[p,q]}(f)$. □

Proof of Theorem 2.4. Suppose that $f(z)$ is an infinite $[p,q]-$order meromorphic solution of equation (2.6) with $\rho_{[p+1,q]}(f)=\rho$. Set $w(z)=g_k-\varphi$. Since $\rho_{[p,q]}(\varphi)<\infty$, then by Lemma 3.3 and Theorem 2.3 we have $\rho_{[p,q]}(w)=\rho_{[p,q]}(g_k)=\infty$ and $\rho_{[p+1,q]}(w)=\rho_{[p+1,q]}(g_k)=\rho$. To prove

$$\overline{\lambda}_{[p,q]}(g_k-\varphi)=\lambda_{[p,q]}(g_k-\varphi)=\infty,\quad \overline{\lambda}_{[p+1,q]}(g_k-\varphi)=\lambda_{[p+1,q]}(g_k-\varphi)=\rho,$$

we need to prove $\overline{\lambda}_{[p,q]}(w)=\lambda_{[p,q]}(w)=\infty$ and $\overline{\lambda}_{[p+1,q]}(w)=\lambda_{[p+1,q]}(w)=\rho$. By $g_k=w+\varphi$, and using (4.5), we get

$$f=C_0w+C_1w'+\cdots+C_{k-1}w^{(k-1)}+\psi_k(z), \tag{4.8}$$

where

$$\psi_k(z)=C_0(\varphi-\beta_0)+C_1(\varphi'-\beta_1)+\cdots+C_{k-1}\left(\varphi^{(k-1)}-\beta_{k-1}\right).$$

Substituting (4.8) into (2.6), we obtain

$$C_{k-1}w^{(2k-1)}+\sum_{j=0}^{2k-2}\phi_jw^{(j)}=F-\left(\psi_k^{(k)}+A_{k-1}(z)\psi_k^{(k-1)}+\cdots+A_0(z)\psi_k\right)=H,$$

where ϕ_j $(j=0,\cdots,2k-2)$ are meromorphic functions of finite $[p,q]$-order. Since $\psi_k(z)$ is not a solution of (2.6), it follows that $H\not\equiv 0$. Then, by Lemma 3.2, we obtain $\overline{\lambda}_{[p,q]}(w)=\lambda_{[p,q]}(w)=\infty$ and $\overline{\lambda}_{[p+1,q]}(w)=\lambda_{[p+1,q]}(w)=\rho$, i. e.,

$$\overline{\lambda}_{[p,q]}(g_k-\varphi)=\lambda_{[p,q]}(g_k-\varphi)=\infty,$$
$$\overline{\lambda}_{[p+1,q]}(g_k-\varphi)=\lambda_{[p+1,q]}(g_k-\varphi)=\rho.$$

Suppose that $f(z)$ is a finite $[p,q]-$order meromorphic solution of equation (2.6) such that (2.8) holds. Set $w(z)=g_k-\varphi$. Since $\rho_{[p,q]}(\varphi)<\rho_{[p,q]}(f)$, then by Lemma 3.3 and Theorem 2.3 we have $\rho_{[p,q]}(w)=\rho_{[p,q]}(g_k)=\rho_{[p,q]}(f)$. To prove $\overline{\lambda}_{[p,q]}(g_k-\varphi)=\lambda_{[p,q]}(g_k-\varphi)=\rho_{[p,q]}(f)$ we need to prove $\overline{\lambda}_{[p,q]}(w)=\lambda_{[p,q]}(w)=\rho_{[p,q]}(f)$. Using the same reasoning as above, we get

$$C_{k-1}w^{(2k-1)}+\sum_{j=0}^{2k-2}\phi_jw^{(j)}=F-\left(\psi_k^{(k)}+A_{k-1}(z)\psi_k^{(k-1)}+\cdots+A_0(z)\psi_k\right)=H,$$

where ϕ_j $(j=0,\cdots,2k-2)$ are meromorphic functions with $[p,q]-$order satisfying $\rho_{[p,q]}(\phi_j)<\rho_{[p,q]}(f)$ $(j=0,\cdots,2k-2)$ and

$$\psi_k(z)=C_0(\varphi-\beta_0)+C_1(\varphi'-\beta_1)+\cdots+C_{k-1}\left(\varphi^{(k-1)}-\beta_{k-1}\right),$$
$$\rho_{[p,q]}(H)<\rho_{[p,q]}(f).$$

Since $\psi_k(z)$ is not a solution of (2.6), it follows that $H\not\equiv 0$. Then by Lemma 3.1, we obtain $\overline{\lambda}_{[p,q]}(w)=\lambda_{[p,q]}(w)=\rho_{[p,q]}(f)$, i. e., $\overline{\lambda}_{[p,q]}(g_k-\varphi)=\lambda_{[p,q]}(g_k-\varphi)=\rho_{[p,q]}(f)$. □

Proof of Corollary 2.1. Suppose that $f(z)$ is a meromorphic solution of equation (2.6). Then, by Lemma 3.6, we have all meromorphic solutions f whose poles are of uniformly bounded multiplicities of (2.6) satisfy $\rho_{[p+1,q]}(f)=\rho_{[p,q]}(A_0)$ with at most one exceptional solution

f_0 satisfying $\rho_{[p+1,q]}(f_0)<\rho_{[p,q]}(A_0)$. Since $\psi_k(z)$ is not a solution of equation (2.6), then by Theorem 2.4 we obtain

$$\overline{\lambda}_{[p,q]}(g_k-\varphi)=\lambda_{[p,q]}(g_k-\varphi)=\rho_{[p,q]}(g_k)=\rho_{[p,q]}(f)=\infty,$$
$$\overline{\lambda}_{[p+1,q]}(g_k-\varphi)=\lambda_{[p+1,q]}(g_k-\varphi)=\rho_{[p+1,q]}(g_k)=\rho_{[p+1,q]}(f)=\rho_{[p,q]}(A_0)$$

hold for all meromorphic solutions f whose poles are of uniformly bounded multiplicities of (2.6) with at most one exceptional solution f_0 satisfying $\rho_{[p+1,q]}(f_0)<\rho_{[p,q]}(A_0)$. □

Proof of Corollary 2.2. Suppose that f is a transcendental meromorphic solution to (2.9). Then by Lemma 3.7, we have $\mu_{[p,q]}(f)=\rho_{[p,q]}(f)=\infty$ and

$$\mu_{[p+1,q]}(f)=\mu_{[p,q]}(A)\leq\rho_{[p+1,q]}(f)=\rho_{[p,q]}(A).$$

On the other hand, we have

$$g_2=d_2f''+d_1f'+d_0f. \tag{4.9}$$

It follows that

$$\begin{cases} g_2-\beta_0=\alpha_{0,0}f+\alpha_{1,0}f', \\ g_2'-\beta_1=\alpha_{0,1}f+\alpha_{1,1}f'. \end{cases} \tag{4.10}$$

By (2.3) and (2.4), we obtain

$$\alpha_{i,0}=\begin{cases} d_1, & \text{for } i=1, \\ d_0-d_2A, & \text{for } i=0 \end{cases} \tag{4.11a}$$

$$\alpha_{i,1}=\begin{cases} \alpha_{1,0}'+\alpha_{0,0}, & \text{for } i=1, \\ \alpha_{0,0}'-A\alpha_{1,0}, & \text{for } i=0, \end{cases} \tag{4.11b}$$

and by (2.5) we get

$$\beta_0=d_2F,\quad \beta_1=\beta_0'+\alpha_{1,0}F=(d_2F)'+d_1F.$$

Hence

$$\begin{cases} \alpha_{0,0}=d_0-d_2A, \quad \alpha_{1,0}=d_1, \\ \alpha_{0,1}=-(d_2A)'-d_1A+d_0', \\ \alpha_{1,1}=-d_2A+d_0+d_1' \end{cases} \tag{4.12}$$

and

$$\begin{aligned} h_2&=\begin{vmatrix} \alpha_{0,0} & \alpha_{1,0} \\ \alpha_{0,1} & \alpha_{1,1} \end{vmatrix} \\ &=d_2^2A^2+\left(d_2'd_1-d_1'd_2-2d_0d_2+d_1^2\right)A+d_1d_2A'-d_0'd_1+d_0d_1'+d_0^2. \end{aligned}$$

First we suppose that $d_2\not\equiv0$. By $d_2\not\equiv0$, $A\not\equiv0$ and Lemma 3.5 we have $\rho_{[p,q]}(h_2)=\rho_{[p,q]}(A)>0$. Now suppose $d_2\equiv0$, $d_1\not\equiv0$. Then, by Lemma 3.3, we get

$$\rho_{[p,q]}(h_2)=\rho_{[p,q]}\left(d_1^2A-d_0'd_1+d_0d_1'+d_0^2\right)=\rho_{[p,q]}(A)>0.$$

Finally, if $d_2\equiv 0$, $d_1\equiv 0$ and $d_0\not\equiv 0$, then $h_2=d_0^2\not\equiv 0$. Hence $h_2\not\equiv 0$. By $h_2\not\equiv 0$ and (4.10), we obtain

$$f=\frac{\alpha_{1,1}(g_2-\beta_0)-\alpha_{1,0}(g_2'-\beta_1)}{h_2}. \tag{4.13}$$

By (4.9), Lemma 3.3, Lemma 3.4 and Lemma 3.11, we have

$$\begin{aligned}
&\rho_{[p,q]}(g_2)\le\rho_{[p,q]}(f)\quad(\rho_{[p+1,q]}(g_2)\le\rho_{[p+1,q]}(f)),\\
&\mu_{[p,q]}(g_2)\le\max\left\{\mu_{[p,q]}(f),\rho_{[p,q]}(d_j)\ \ (j=0,1,2)\right\}=\mu_{[p,q]}(f),\\
&\mu_{[p+1,q]}(g_2)\le\max\left\{\mu_{[p+1,q]}(f),\rho_{[p+1,q]}(d_j)\ \ (j=0,1,2)\right\}=\mu_{[p+1,q]}(f),
\end{aligned}$$

and by (4.13) we have

$$\begin{aligned}
&\rho_{[p,q]}(f)\le\rho_{[p,q]}(g_2)\ \ (\rho_{[p+1,q]}(f)\le\rho_{[p+1,q]}(g_2)),\\
&\mu_{[p,q]}(f)\le\mu_{[p,q]}(g_2)\ \ (\mu_{[p+1,q]}(f)\le\mu_{[p+1,q]}(g_2)).
\end{aligned}$$

Therefore, $\rho_{[p,q]}(g_2)=\rho_{[p,q]}(f)=\mu_{[p,q]}(g_2)=\mu_{[p,q]}(f)=\infty$ and

$$\mu_{[p+1,q]}(g_2)=\mu_{[p+1,q]}(f)=\mu_{[p,q]}(A)\le\rho_{[p+1,q]}(g_2)=\rho_{[p+1,q]}(f)=\rho_{[p,q]}(A).$$

This completes the proof. □

Proof of Corollary 2.3. Suppose that $f(z)$ is a transcendental meromorphic solution of (2.9). Then, by Corollary 2.2, we have $\rho_{[p,q]}(g_2)=\rho_{[p,q]}(f)=\mu_{[p,q]}(g_2)=\mu_{[p,q]}(f)=\infty$ and

$$\mu_{[p+1,q]}(g_2)=\mu_{[p+1,q]}(f)=\mu_{[p,q]}(A)\le\rho_{[p+1,q]}(g_2)=\rho_{[p+1,q]}(f)=\rho_{[p,q]}(A).$$

Set $w(z)=d_2f''+d_1f'+d_0f-\varphi$. Then, by $\rho_{[p,q]}(\varphi)<\infty$, Lemma 3.3 and Lemma 3.4, we have $\mu_{[p,q]}(w)=\mu_{[p,q]}(g_2)=\mu_{[p,q]}(f)=\rho_{[p,q]}(f)=\infty$ and

$$\begin{aligned}
&\mu_{[p+1,q]}(w)=\mu_{[p+1,q]}(g_2)=\mu_{[p+1,q]}(f)=\mu_{[p,q]}(A)\\
\le&\rho_{[p+1,q]}(w)=\rho_{[p+1,q]}(g_2)=\rho_{[p+1,q]}(f)=\rho_{[p,q]}(A).
\end{aligned}$$

To prove

$$\begin{aligned}
&\overline{\underline{\lambda}}_{[p+1,q]}(g_2-\varphi)=\underline{\lambda}_{[p+1,q]}(g_2-\varphi)=\mu_{[p,q]}(A)\\
\le&\overline{\lambda}_{[p+1,q]}(g_2-\varphi)=\lambda_{[p+1,q]}(g_2-\varphi)=\rho_{[p,q]}(A),
\end{aligned}$$

we need to prove

$$\overline{\underline{\lambda}}_{[p+1,q]}(w)=\underline{\lambda}_{[p+1,q]}(w)=\mu_{[p,q]}(A)\le\overline{\lambda}_{[p+1,q]}(w)=\lambda_{[p+1,q]}(w)=\rho_{[p,q]}(A).$$

Using $g_2=w+\varphi$, we get from (4.13)

$$f=\frac{-\alpha_{1,0}w'+\alpha_{1,1}w}{h_2}+\psi_2, \tag{4.14}$$

where

$$\psi_2(z)=\frac{-\alpha_{1,0}(\varphi'-\beta_1)+\alpha_{1,1}(\varphi-\beta_0)}{h_2}$$
$$=\frac{-d_1\left(\varphi'-\left((d_2F)'+d_1F\right)\right)+(-d_2A+d_0+d_1')(\varphi-d_2F)}{d_2^2A^2+\left(d_2'd_1-d_1'd_2-2d_0d_2+d_1^2\right)A+d_1d_2A'-d_0'd_1+d_0d_1'+d_0^2}. \tag{4.15}$$

Substituting (4.14) into equation (2.9), we obtain

$$\frac{-\alpha_{1,0}}{h_2}w'''+\phi_2w''+\phi_1w'+\phi_0w=F-\left(\psi_2''+A(z)\psi_2\right)=G, \tag{4.16}$$

where ϕ_j $(j=0,1,2)$ are meromorphic functions with $\rho_{[p,q]}(\phi_j)<\infty$ $(j=0,1,2)$. First, we suppose that $\psi_2\equiv 0$. Then, $G=F\not\equiv 0$. Now, if $\psi_2(z)\not\equiv 0$, then by Lemma 3.7 it follows that $\psi_2(z)$ is not a solution of equation (2.9) because ψ_2 is a transcendental meromorphic function with $\rho_{[p,q]}(\psi_2)<\infty$. Hence $G\not\equiv 0$. By Lemma 3.2 and Lemma 3.8, we obtain

$$\underline{\overline{\lambda}}_{[p+1,q]}(w)=\underline{\lambda}_{[p+1,q]}(w)=\mu_{[p,q]}(A)\le\overline{\lambda}_{[p+1,q]}(w)=\lambda_{[p+1,q]}(w)=\rho_{[p,q]}(A),$$

that is,

$$\underline{\overline{\lambda}}_{[p+1,q]}(g_2-\varphi)=\underline{\lambda}_{[p+1,q]}(g_2-\varphi)=\mu_{[p,q]}(A)$$
$$\le\overline{\lambda}_{[p+1,q]}(g_2-\varphi)=\lambda_{[p+1,q]}(g_2-\varphi)=\rho_{[p,q]}(A).$$

This completes the proof. □

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