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Conditional Residual Lifetimes of $(n-k+1)$-out-of-n Systems with Mixed Erlang Components

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Abstract. We consider an $(n-k+1)$-out-of-n system with component lifetimes being correlated. The main objective of this paper is to study the conditional residual lifetime of an $(n-k+1)$-out-of-n system, given that at a fixed time a certain number of components have failed, assuming that the component lifetimes follow a multivariate Erlang mixture. Comparison studies of the stochastic ordering of the $(n-k+1)$-out-of-n system are presented. Several examples are presented to illustrate and confirm our results.

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Key words: Conditional mean residual lifetime, multivariate Erlang mixture, $(n-k+1)$-out-of-n system, dependence structure, exchangeable variables.

1 Introduction

An $(n-k+1)$-out-of-n system is a system such that it consists of n components and works if and only if at least $(n-k+1)$ out of the n components are operating $(k \le n)$. Thus, this system fails if k or more of its components fail. If $k=1$ the system is a series system, and if $k=n$ the system is a parallel system. The system is often considered in the industrial and survival analysis context. Denote the lifetimes of the individual components by $X_1, X_2, \cdots, X_n$ and let $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$ be the corresponding order statistics. Then the lifetime of the $(n-k+1)$-out-of-n system will be represented by the kth order statistic $X_{k:n}$. Let X denote the lifetime of a component of a system. Then $X_t = (X-t|X>t)$ may be interpreted as the residual lifetime of the system at time t, given that the system is alive at time t.

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In recent years, the residual lifetime of an $(n-k+1)$-out-of-n system has been studied extensively. [2] considered the residual lifetime of an $(n-k+1)$-out-of-n system under the condition that at most $(l-1)$ components have failed at time t, i.e., $(X_{k:n}-t|X_{l:n}>t), 1\le l\le k\le n$. [14] extended the concept to an $(n-k+1)$-out-of-n system under the condition that at least j components have failed but the lth failure has not occurred yet at time t, i.e., $(X_{k:n}-t|X_{j:n}\le t<X_{l:n}), 1\le j<l\le k\le n$. Similar research can be found in [1], [3], [13], and [22].

The research in this area in general focuses on the distribution function, the mean residual lifetime (MRL) and stochastic ordering properties of residual lifetimes under the assumption that the components of a system are independent. See [8], [9], [18], and references therein. In many real situations however, there may be a structural dependence among components of the system. As a result, there are several recent studies considering the dependence among the components. For example, [15] adopted Archimedean copula to reflect the dependence among the components. Others may be found in [7], [17], [18] and references therein.

In this paper, we study the conditional mean residual lifetime function and stochastic ordering properties of an $(n-k+1)$-out-of-n system with assumption that the lifetimes have a multivariate Erlang mixture. The multivariate Erlang mixture is a useful model as it can capture the dependence structure of a large number of multiple variables well. Compared with copula method that is a dominant choice to model multivariate data these days, a multivariate Erlang mixture is more flexible in terms of dependence structure and has a wide range of dependence. Furthermore, it is easy to deal with high dimensional data with a multivariate Erlang mixture, while a copula approach may become much more difficult to use for higher dimensional data. See [12], [19], [20] and references therein. Hence the results in this paper may be useful when the components of a system are of strong dependency and the number of components is high.

Each marginal of a multivariate Erlang mixtures can be viewed as a compound exponential distribution. In this paper, we show that if the counting random variables satisfy the multivariate totally positive of order 2 (MTP_2) property, then the conditional residual lifetime $X^t_{k,j,l,n}=(X_{k:n}-t|X_{j:n}\le t<X_{l:n}), 1\le j<l\le k\le n$ is stochastically non-decreasing with respect to j and k and non-increasing with respect to l. These properties are consistent with the results when the component lifetimes are independent.

This paper is organized as follows. In Section 2, we present some properties of exchangeable variables with the joint distribution being a multivariate Erlang mixture. The purpose of the section is to simplify the proofs in following sections. In Section 3, we study the conditional mean residual lifetime of an $(n-k+1)$-out-of-n system under the assumption that the lifetimes of the components follow an Erlang mixture. In Section 4, we stochastically compare the conditional residual lifetimes of the $(n-k+1)$-out-of-n system with respect to its various parameters. We conclude Section 5 with some remarks.

2 Properties of exchangeable mixed Erlang variables

[12] introduces the class of multivariate Erlang mixtures and shows that the class has many desirable properties. A multivariate Erlang mixture is defined as a random vector $\mathbf{X}=(X_1,X_2,\cdots,X_n)$ with probability density function (pdf)

$$h(\mathbf{x})=h(x_1,\cdots,x_n)=\sum_{m_1=1}^{\infty}\cdots\sum_{m_n=1}^{\infty}\alpha_{\mathbf{m}}\left\{\prod_{j=1}^{n}\frac{\beta(\beta x_j)^{m_j-1}e^{-\beta x_j}}{(m_j-1)!}\right\}, \tag{2.1}$$

where $\mathbf{m}=(m_1,\cdots,m_n)$, $\alpha_{\mathbf{m}}$'s are the mixing weights satisfying $\alpha_{\mathbf{m}}\geq 0$ and

$$\sum_{m_1=1}^{\infty}\cdots\sum_{m_n=1}^{\infty}\alpha_{\mathbf{m}}=1$$

and β is the common rate parameter ($\theta=1/\beta$ is called scale parameter).

It is shown in [12] that each marginal random variable X_p of a multivariate Erlang mixture has a compound exponential distribution, i.e.,

$$X_p=\sum_{i=1}^{N_p}E_{ip},\quad p=1,\cdots n, \tag{2.2}$$

where $E_{ip},i=1,\cdots;p=1,\cdots,n$ are iid (independent identical distributed) exponential random variables with mean $1/\beta$ and the weights $\alpha_{\mathbf{m}}$ form a joint probability function of a multivariate counting random vector $\mathbf{N}=(N_1,N_2,\cdots,N_n)$, that is,

$$\alpha_{\mathbf{m}}=P(N_1=m_1,N_2=m_2,\cdots,N_n=m_n). \tag{2.3}$$

We denote the density of an Erlang distribution with shape parameter m and rate parameter β as

$$f(x|m,\beta)=\frac{\beta(\beta x)^{m-1}e^{-\beta x}}{(m-1)!}. \tag{2.4}$$

Thus, the survival function is given by

$$\overline{F}(x|m,\beta)=\sum_{r=0}^{m-1}\frac{(\beta x)^r e^{-\beta x}}{r!}. \tag{2.5}$$

The distribution function $F(x|m,\beta)=1-\overline{F}(x|m,\beta)$.

For any permutation $\{\pi_1,\cdots,\pi_n\}$ of $\{1,\cdots,n\}$, if

$$(X_{\pi_1},\cdots,X_{\pi_n})\stackrel{d}{=}(X_1,\cdots,X_n), \tag{2.6}$$

namely, $(X_{\pi_1},\cdots,X_{\pi_n})$ and $(X_1,\cdots,X_n)$ have the same distribution, then we call $X_1,\cdots,X_n$ exchangeable variables. The vector $(X_1,\cdots,X_n)$ or its distribution is said to be exchangeable as well.

We will study the properties of the conditional residual lifetime of an $(n-k+1)$-out-of-n system. The proofs of the results in next sections are presented under the assumption that $(X_1,\cdots,X_n)$ is exchangeable. We now present some properties of exchangeable variables with the joint distribution being a multivariate Erlang mixture.

First, by Corollary 2.1 of [12] and the property of exchangeable random vector we immediately have the following property.

Property 2.1. If $(X_1,\cdots,X_n)$ is exchangeable with joint distribution being a multivariate Erlang mixture, then the marginal distribution of $X_j, j=1,\cdots,n$ is a univariate Erlang mixture and $X_1,\cdots,X_n$ are identical.

In fact, any p-variate marginal $(X_{i_1},\cdots,X_{i_p})$, is a p-variate exchangeable Erlang mixture, $\{i_1,\cdots,i_p\}$ is any subset of $\{1,\cdots,n\}$ with cardinality $p(p\leq n)$.

Property 2.2. If the joint distribution of $X_1,\cdots,X_n$ is a multivariate Erlang mixture, then $(X_1,\cdots,X_n)$ is exchangeable if and only if $(N_1,\cdots,N_n)$ is exchangeable.

Proof. (1) If $(N_1,\cdots,N_n)$ is exchangeable, that is, for any permutation $\{\pi_1,\cdots,\pi_n\}$ of $\{1,\cdots,n\}$, $\alpha_{\mathbf{m}_\pi}=\alpha_{\mathbf{m}}$, $\mathbf{m}_\pi=(m_{\pi_1},\cdots,m_{\pi_n})$, then

$$\begin{aligned} h(x_{\pi_1},\cdots,x_{\pi_n}) &= \sum_{m_1=1}^{\infty}\cdots\sum_{m_n=1}^{\infty}\alpha_{\mathbf{m}}\prod_{j=1}^{n}f(x_{\pi_j}|m_j,\beta) \\ &= \sum_{m_{\pi_1}=1}^{\infty}\cdots\sum_{m_{\pi_n}=1}^{\infty}\alpha_{\mathbf{m}_\pi}\prod_{j=1}^{n}f(x_{\pi_j}|m_{\pi_j},\beta) = \sum_{m_1=1}^{\infty}\cdots\sum_{m_n=1}^{\infty}\alpha_{\mathbf{m}_\pi}\prod_{j=1}^{n}f(x_j|m_j,\beta) \\ &= \sum_{m_1=1}^{\infty}\cdots\sum_{m_n=1}^{\infty}\alpha_{\mathbf{m}}\prod_{j=1}^{n}f(x_j|m_j,\beta)=h(x_1,\cdots,x_n). \end{aligned} \tag{2.7}$$

Hence we conclude that $(X_1,\cdots,X_n)$ is exchangeable.

(2) If $(X_1,\cdots,X_n)$ is exchangeable, that is, for any permutation $\{\pi_1,\cdots,\pi_n\}$ of $\{1,\cdots,n\}$, $h(x_{\pi_1},\cdots,x_{\pi_n})=h(x_1,\cdots,x_n)$, then similar to the above procedure, we have

$$\sum_{m_1=1}^{\infty}\cdots\sum_{m_n=1}^{\infty}\alpha_{\mathbf{m}_\pi}\prod_{j=1}^{n}f(x_j|m_j,\beta)=\sum_{m_1=1}^{\infty}\cdots\sum_{m_n=1}^{\infty}\alpha_{\mathbf{m}}\prod_{j=1}^{n}f(x_j|m_j,\beta). \tag{2.8}$$

According to the identifiability property of the multivariate Erlang mixture, we have $\alpha_{\mathbf{m}_\pi}=\alpha_{\mathbf{m}}$, that is, $(N_1,\cdots,N_n)$ is exchangeable. □

Property 2 provides us an idea to construct a set of exchangeable Erlang mixture variables. The dependence of $X_1,\cdots,X_n$ relies on the dependence of $N_1,\cdots,N_n$.

Example 2.1. Let $\mathbf{U}=\mathbf{N}-\mathbf{1}$ and $(U_1,\cdots,U_n)\sim Mult_n(M-n;\frac{1}{n},\cdots,\frac{1}{n})$, that is $(U_1,\cdots,U_n)$ is from a multinomial distribution with number of trials $M-n$ and event probabilities

$(\frac{1}{n},\cdots,\frac{1}{n})$, then

$$\begin{aligned}\alpha_{\mathbf{m}} &= P(N_1=m_1,\cdots,N_n=m_n)\\ &= P(U_1=m_1-1,\cdots,U_n=m_n-1)\\ &= \left(\begin{array}{c} M-n \\ m_1-1,\cdots,m_n-1 \end{array}\right)(\frac{1}{n})^{M-n}, if\ m_1+\cdots+m_n=M,\end{aligned}$$

and the rest mixing weights are set to be 0. It is obvious that $(N_1,\cdots,N_n)$ is a random vector with exchangeable distribution and hence $(X_1,\cdots,X_n)$ is exchangeable.

The next property provides a method to obtain more discrete exchangeable variables.

Property 2.3. If variables $Y_1,\cdots,Y_n$ are exchangeable and another variable Y is independent with $Y_1,\cdots,Y_n$, then $N_i=Y_i+Y, i=1,\cdots,n$ are exchangeable variables.

Proof. For any permutation $\{\pi_1,\cdots,\pi_n\}$ of $\{1,\cdots,n\}$,

$$\begin{aligned}&P(N_{\pi_1}=m_1,\cdots,N_{\pi_n}=m_n)=E[P(Y_{\pi_1}+Y=m_1,\cdots,Y_{\pi_n}+Y=m_n)|Y]\\ =&\sum_{k=0}^{\infty}P(Y_{\pi_1}+k=m_1,\cdots,Y_{\pi_n}+k=m_n)P(Y=k)\\ =&\sum_{k=0}^{\infty}P(Y_1+k=m_1,\cdots,Y_n+k=m_n)P(Y=k)=P(N_1=m_1,\cdots,N_n=m_n). \qquad (2.9)\end{aligned}$$

Hence, the result holds. □

Remark 2.1. In practice, we often extend a certain univariate distribution to multivariate case as described above. If we set variables $Y_1,\cdots,Y_n$ to be identical independent, then from Property 2 we know the corresponding multivariate distribution is exchangeable because it is easily to know that identical independent variables are also exchangeable.

3 Conditional residual lifetime of $(n-k+1)$-out-of-n system

We will study the conditional residual lifetime of an $(n-k+1)$-out-of-n system, given that at least j components have failed but the lth failure has not occurred yet at time t:

$$X^t_{k,j,l,n}=(X_{k:n}-t|X_{j:n}\le t<X_{l:n}),\ 1\le j<l\le k\le n.$$

Let $\mathbf{X}=(X_1,\cdots,X_n)$ be the vector of the components' lifetimes having joint density function $h(\mathbf{x})$. In order to simplify the notation, we denote

$$\sum_{\mathbf{m}=1}^{\infty}\triangleq\sum_{m_1=1}^{\infty}\cdots\sum_{m_n=1}^{\infty}$$

in next sections and we suppress the common rate parameter β.

For the conditional residual lifetime $X^t_{k,j,l,n}=(X_{k:n}-t|X_{j:n}\le t<X_{l:n})$ of an $(n-k+1)$-out-of-n system, the distribution function is denoted by

$$F^t_{k,j,l,n}(x)=P(X_{k:n}-t\le x|X_{j:n}\le t<X_{l:n}) \text{ for } x>0$$

and the survival function $\overline{F}^t_{k,j,l,n}(x)=1-F^t_{k,j,l,n}(x)$.

First we derive the survival function $\overline{F}^t_{k,j,l,n}(x)$ and then calculate the conditional mean residual lifetime which will be defined later. To simplify the calculation, the results in this section are under the assumption that the lifetimes of the components are exchangeable. The corresponding results when the joint distribution is an arbitrary multivariate Erlang mixture are presented in appendix.

Theorem 3.1. *If the lifetimes of $\boldsymbol{X}=(X_1,\cdots,X_n)$ have joint density function $h(\boldsymbol{x})$. For $1\le j<l\le k\le n$, the survival function of the conditional residual lifetime is given by*

$$\overline{F}^t_{k,j,l,n}(x)=\frac{\sum\limits_{\boldsymbol{m}=1}^{\infty}\alpha_{\boldsymbol{m}}w_{\boldsymbol{m}}\sum\limits_{i=j}^{l-1}\sum\limits_{C_i}\phi_i(t)\overline{F}^{(C_i^c)}_{k-i,n-i}(x,t|\boldsymbol{m})}{\sum\limits_{\boldsymbol{m}=1}^{\infty}\alpha_{\boldsymbol{m}}w_{\boldsymbol{m}}\sum\limits_{i=j}^{l-1}\sum\limits_{C_i}\phi_i(t)}, \tag{3.1}$$

where

$$w_{\boldsymbol{m}}=\prod_{i=1}^{n}\overline{F}(t|m_i),\quad \phi_i(t)=\prod_{s\in C_i}\frac{F(t|m_s)}{\overline{F}(t|m_s)},$$

C_i is any subset of $\{1,\cdots,n\}$ with cardinality i, $\varphi(x,t|m)=\dfrac{\overline{F}(x+t|m)}{\overline{F}(t|m)}$ and

$$\overline{F}^{(C_i^c)}_{k-i,n-i}(x,t|\boldsymbol{m})=\sum_{p=0}^{k-i-1}\sum_{C_{i(p)}}\prod_{s\in C_{i(p)}}(1-\varphi(x,t|m_s))\prod_{s\in C'_{i(p)}}\varphi(x,t|m_s), \tag{3.2}$$

where C_i^c is the complementary of C_i, $C_{i(p)}$ is any subset of C_i^c with cardinality p and $C'_{i(p)}=C_i^c-C_{i(p)}$.

Proof. When the multivariate Erlang mixture is exchangeable, we have

$$\overline{F}^t_{k,j,l,n}(x)=P(X_{k:n}-t>x|X_{l:n}>t\ge X_{j:n})=\frac{P(X_{k:n}>t+x,X_{l:n}>t\ge X_{j:n})}{P(X_{l:n}>t\ge X_{j:n})},$$

and

$$
\begin{aligned}
&P(X_{k:n}>t+x,X_{l:n}>t\geq X_{j:n})\\
=&\sum_{i=j}^{l-1}\sum_{p=0}^{k-i-1}P(\text{exactly } i \text{ of } X's \text{ are } <t, t< \text{ exactly } p \text{ of } X's \text{ are } \leq t+x)\\
=&\sum_{i=j}^{l-1}\sum_{p=0}^{k-i-1}\frac{n!}{i!p!(n-i-p)!}\sum_{\mathbf{m}=1}^{\infty}\alpha_{\mathbf{m}}\prod_{s=1}^{i}F(t|m_s)\prod_{s=i+1}^{i+p}(F(t+x|m_s)-F(t|m_s))\prod_{s=i+p+1}^{n}\overline{F}(t+x|m_s)\\
=&\sum_{i=j}^{l-1}\sum_{p=0}^{k-i-1}\frac{n!}{i!p!(n-i-p)!}\sum_{\mathbf{m}=1}^{\infty}\alpha_{\mathbf{m}}\prod_{s=1}^{n}\overline{F}(t|m_s)\phi_i(t)\prod_{s=i+1}^{i+p}(1-\varphi(x,t|m_s))\prod_{s=i+p+1}^{n}\varphi(x,t|m_s)\\
=&\sum_{\mathbf{m}=1}^{\infty}\alpha_{\mathbf{m}}w_{\mathbf{m}}\sum_{i=j}^{l-1}\binom{n}{i}\phi_i(t)\psi_{i,k}(x,t),
\end{aligned}
\tag{3.3}
$$

where $C_i=\{1,\cdots,i\}$ and

$$
\psi_{i,k}(x,t)=\sum_{p=0}^{k-i-1}\binom{n-i}{p}\prod_{s=i+1}^{i+p}(1-\varphi(x,t|m_s))\prod_{s=i+p+1}^{n}\varphi(x,t|m_s).
$$

Similarly, we can obtain that

$$
P(X_{j:n}\leq t<X_{l:n})=\sum_{m=1}^{\infty}\alpha_{\mathbf{m}}w_{\mathbf{m}}\sum_{i=j}^{l-1}\binom{n}{i}\phi_i(t). \tag{3.4}
$$

the result follows immediately and it is obvious a special case of (3.1). □

Remark 3.1. For exchangeable variables with a joint distribution being a multivariate Erlang mixture, the survival function of the residual lifetime can also be written as (see [18])

$$
\overline{F}^{t}_{k,j,l,n}(x)=\frac{\sum_{i=j}^{l-1}\binom{n}{i}G_{i,n}(t)\overline{G}^{(C_i^c)}_{k-i,n-i}(x,t)}{\sum_{i=j}^{l-1}\binom{n}{i}G_{i,n}(t)}, \tag{3.5}
$$

where $G_{i,n}(t)=P\{A^t_{i,n}\}=P(X_1\leq t,\cdots,X_i\leq t,X_{i+1}>t,\cdots,X_n>t)$ and $\overline{G}^{(C_i^c)}_{k-i,n-i}(x,t)$ is the survival function of the $(k-i)$th order statistic of conditional random vector $(X_{i+1}-t,\cdots,X_n-t|A^t_{i,n})$, $A^t_{i,n}$ denotes event $[X_1\leq t,\cdots,X_i\leq t,X_{i+1}>t,\cdots,X_n>t]$.

Example 3.1. (Example 2.1 continued)

We set $M=4$ and $n=3$ in Example 2.1, then the joint distribution of the components is given by (with the common rate parameter $\beta=1$)

$$
h(x_1,x_2,x_3)=\frac{1}{3}\Big[f(x_1|1)f(x_2|1)f(x_3|2)+f(x_1|1)f(x_2|2)f(x_3|1)+f(x_1|2)f(x_2|1)f(x_3|1)\Big].
$$

Figure 1 shows the survival curves for different time points for $j=1,l=2,k=3$.

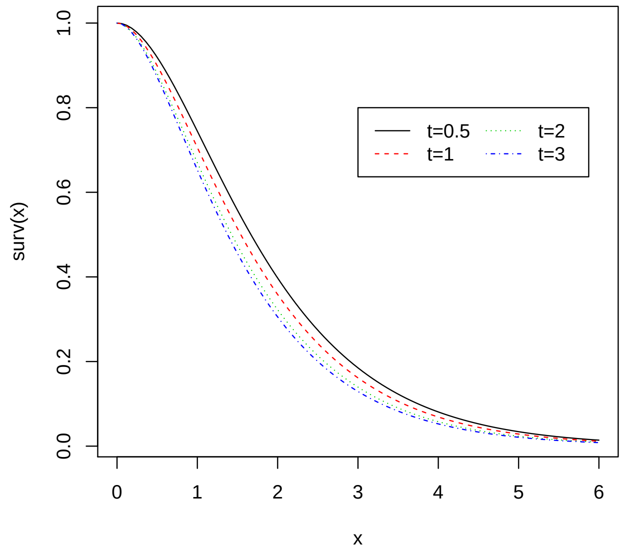

Figure 1: Survival curves for different times.

[5] investigated the MRL functions of parallel systems and [8] considered systems with independent but nonidentical components. Here we consider a system consisting of dependent components with lifetimes having a multivariate Erlang mixture. The conditional mean residual life function is defined as the expectation of the conditional residual lifetime,

$$\mu^t_{k,j,l,n} = E[X^t_{k,j,l,n}] = E[X_{k:n} - t | X_{j:n} \le t < X_{l:n}]. \tag{3.6}$$

Generally speaking, it is difficult to obtain an explicit expression for an MRL. It is an advantage to use a multivariate Erlang mixture in this situation as an explicit expression for an MRL is available.

The following result from [10] will be used to derive the MRL in this section.

Lemma 3.1. *Let $X_1,\cdots,X_n$ be independent random variables where $X_p, p=1,\cdots,n$ has pdf*

$$f_p(x) = \sum_{m=1}^{\infty} q_m^{(p)} f(x|m,\beta).$$

Then the pdf of rth order statistic is given by

$$f_{r:n}(x) = \sum_{m=1}^{\infty} q_m f(x|m,n\beta), \tag{3.7}$$

where

$$q_m=\frac{1}{n^m}\sum_{A_{m,n}}\binom{m-1}{m_1-1,\cdots,m_n-1}\frac{1}{(r-1)!(n-r)!}\sum_P q_{m_r}^{(p_r)}\prod_{j=1}^{r-1}Q_{m_j}^{(p_j)}\prod_{j=r+1}^{n}\overline{Q}_{m_j}^{(p_j)},$$

$$A_{m,n}=\left\{(m_1,\cdots,m_n)|\sum_{i=1}^{n}m_i=m+n-1,m_i,i=1,\cdots,n \text{ are positive integers}\right\},$$

$\sum P$ *denotes the sum over all* $n!$ *permutations* $\{p_1,p_2,\cdots,p_n\}$ *of* $\{1,2,\cdots,n\}$,

$Q_{m_j}^{(p_j)}=\sum_{i=1}^{m_j}q_i^{(p_j)}$ *and* $\overline{Q}_{m_j}^{(p_j)}=1-Q_{m_j}^{(p_j)}$.

Theorem 3.2. *For* $1\le j<l\le k\le n$ *and* $t>0$, *the conditional mean residual lifetime is given by*

$$\mu_{k,j,l,n}(t)=\frac{\sum_{\boldsymbol{m}=1}^{\infty}\alpha_{\boldsymbol{m}}w_{\boldsymbol{m}}\sum_{i=j}^{l-1}\sum_{C_i}\phi_i(t)\mu_{k-i,n-i}^{(C_i^c)}(t|\boldsymbol{m})}{\sum_{\boldsymbol{m}=1}^{\infty}\alpha_{\boldsymbol{m}}w_{\boldsymbol{m}}\sum_{i=j}^{l-1}\sum_{C_i}\phi_i(t)},\tag{3.8}$$

where $\mu_{k-i,n-i}^{(C_i^c)}(t|\boldsymbol{m})$ *is the mean function of the* $(k-i)$*th order statistic from independent variables* $(Z_p)_t=(Z_p-t|Z_p>t),p\in C_i^c$, *where* Z_p *has an Erlang distribution with shape parameter* m_p.

Proof: Note that

$$\mu_{k,j,l,n}(t)=E[X_{k,j,l,n}^t]=\int_0^\infty \overline{F}_{k,j,l,n}^t(x)dx.\tag{3.9}$$

We notice that we only need to calculate the term

$$\mu_{k-i,n-i}^{(C_i^c)}(t)=\int_0^\infty \overline{F}_{k-i,n-i}^{(C_i^c)}(x,t|\mathbf{m})dx.\tag{3.10}$$

Let random variables $Z_p,p\in C_i^c$ be independent variables with X_p has an Erlang distribution with shape parameter $m_p,p\in C_i^c$. Then the residual lifetime of Z_p at time t denoted by $(Z_p)_t$ has an Erlang mixture distribution with pdf

$$h_p(x)=\frac{e^{-t\beta}}{\overline{F}(t|m_p)}\sum_{m=1}^{m_p}\frac{(t\beta)^{m_p-m}}{(m_p-m)!}f(x|m)=\sum_{m=1}^{\infty}q_m^{(p)}f(x|m),\tag{3.11}$$

where

$$q_m^{(p)}=\begin{cases}\dfrac{e^{-t\beta}}{\overline{F}(t|m_p)}\dfrac{(t\beta)^{m_p-m}}{(m_p-m)!}, & m\le m_p,\\ 0, & m>m_p,\end{cases}$$

Hence, $(X_p)_t,p\in C_i^c$ are independent but nonidentical Erlang mixture variables.

According to (3.2), $\overline{F}_{k-i,n-i,t}^{(C_i^c)}$ is exactly the survival function of the $(k-i)$th order statistic from independent variables $(Z_p)_t,p\in C_i^c$ and thus $\mu_{k-i,n-i}^{(C_i^c)}(t)$ is exactly the mean function of the $(k-i)$th order statistic.

From Lemma 3.1, the pdf of $(k-i)$th order statistic of $(Z_p)_t, p\in C_i^c$ is given by

$$f_{k-i,n-i}(x)=\sum_{m=1}^{\infty} q_m f(x|m,(n-i)\beta), \tag{3.12}$$

where the mixing weights $q_m, m=1,2,\cdots$ can be seen in Lemma 3.1 with n replaced by $n-i$ and k replaced by $k-i$. Then we have

$$\mu_{k-i,n-i}^{(C_i^c)}(t)=\frac{1}{(n-i)\beta}\sum_{m=1}^{\infty} m q_m. \tag{3.13}$$

This completes the proof. □

Remark 3.2. The mean function of the $(k-i)$th order statistic of $(Z_p)_t, p\in C_i^c$ can be obtained by calculating the following integral expression directly:

$$\mu_{k-i,n-i}^{(C_i^c)}(t)=\sum_{p=0}^{k-i-1}\sum_{C_{i(p)}}\int_0^{\infty}\prod_{s\in C_{i(p)}}(1-\varphi(x,t|m_s))\prod_{s\in C'_{i(p)}}\varphi(x,t|m_s)dx, \tag{3.14}$$

where the notations $C_{i(p)}$ and $C'_{i(p)}$ are the same as what presented in Theorem 3.2. We can also obtain an analytic expression, but it is tedious and is omitted here.

Example 3.2. (Example 3.1 continued)

The conditional mean residual lifetime of the system given in Example 3.1 can be easily computed through expression (3.8). Here we set the time $t=10$, then it follows that

$$\mu_{3,1,2,3}(10)=1.5652.$$

4 Stochastic orders of $(n-k+1)$-out-of-n system

In this section, we examine some stochastic orders of the conditional residual lifetime of an $(n-k+1)$-out-of-n system.

Definition 4.1. *Let X and Y be two random variables with distribution functions F and G and survival functions $\overline{F}=1-F$ and $\overline{G}=1-G$, respectively. Then random variable X is said to be smaller than random variable Y in the usual stochastic order (denoted by $X\le_{st}Y$) if $\overline{F}(x)\le\overline{G}(x)$ for all x.*

Definition 4.2. *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two n-dimensional random vectors with joint density function f and g. Then $\boldsymbol{X}$ is said to be smaller than $\boldsymbol{Y}$ in the multivariate likelihood ratio order (denoted by $\boldsymbol{X}\le_{lr}\boldsymbol{Y}$) if for all $(x_1,\cdots,x_n)$ and $(y_1,\cdots,y_n)$ in $\mathbb{R}^n$,*

$$f(x_1,\cdots,x_n)g(y_1,\cdots,y_n)\le f(x_1\wedge y_1,\cdots,x_n\wedge y_n)g(x_1\vee y_1,\cdots,x_n\vee y_n), \tag{4.1}$$

where $x\wedge y=\min\{x,y\}$ and $x\vee y=\max\{x,y\}$. If $\boldsymbol{X}\le_{lr}\boldsymbol{X}$, then $\boldsymbol{X}$ is said to satisfy the multivariate totally positive of order 2 (MTP_2) property.

Some common distributions have the MTP_2 property such as multivariate gamma distribution, multivariate logistic distribution and negative multinomial distribution. Other examples and properties about multivariate totally positive distributions can be found in [6].

The following results will be used in our derivation.

Lemma 4.1. *If* $\boldsymbol{X}=(X_1,\cdots,X_n)$ *has a multivariate Erlang mixture and* $\boldsymbol{N}=(N_1,\cdots,N_n)$ *is the corresponding counting random vector as defined in Section 2. Then*

$$\boldsymbol{N}\leq_{lr}\boldsymbol{N}\Rightarrow\boldsymbol{X}\leq_{lr}\boldsymbol{X}. \tag{4.2}$$

Proof. It is known that each marginal of $\mathbf{X}=(X_1,\cdots,X_n)$ has a compound exponential distribution as in (2.2) and it is obvious that the density function of $E_{ip}, p=1,\cdots,n$ is logconcave. Under the assumption that $\mathbf{N}$ satisfies the MTP_2 property, it follows from Theorem 6.E.5 in [16] that we have

$$\left(\sum_{i=1}^{N_1}E_{i1},\cdots,\sum_{i=1}^{N_n}E_{in}\right)\leq_{lr}\left(\sum_{i=1}^{N_1}E_{i1},\cdots,\sum_{i=1}^{N_n}E_{in}\right), \tag{4.3}$$

namely,

$$(X_1,\cdots,X_n)\leq_{lr}(X_1,\cdots,X_n), \tag{4.4}$$

which means $\mathbf{X}$ also satisfies MTP_2 property. □

The following lemma from [16] will also be used in our derivation.

Lemma 4.2. *Let* $X_1,X_2,\cdots,X_m$ *and* $Y_1,Y_2,\cdots,Y_n$ *be two positively correlated sets of continuous random variables. Then*

$$X_i\leq_{st}Y_i \text{ for all } i\Rightarrow X_{i:m}\leq_{st}Y_{j:n},\quad i\leq j,\ m-i\geq n-j. \tag{4.5}$$

Next we study the stochastic orders of the conditional residual lifetime with respect to k,j,l,n. The proofs in this section are under the assumption that the lifetimes of the components are exchangeable. The corresponding results when the joint distribution is an arbitrary multivariate Erlang mixture are presented in appendix.

First, according to the definition of order statistics, we can immediately obtain the result that $X^t_{k,j,l,n}$ is stochastically non-decreasing with respect to k.

Theorem 4.1. *Under the assumptions of Theorem* 3.1, *for* $1\leq j<l\leq k\leq n$ *and* $t>0$, *we have*

$$X^t_{k,j,l,n}\leq_{st}X^t_{k+1,j,l,n}. \tag{4.6}$$

Now we consider $X^t_{k,j,l,n}$ with respect to j and l. The following theorem shows that if $\mathbf{N}$ satisfies the MTP_2 property, then $X^t_{k,j,l,n}$ is non-increasing in l and non-decreasing in j.

Theorem 4.2. *Under the assumptions of Theorem* 3.1*, for* $1\le j<l\le k\le n$ *and* $t>0$*, if* **N** *satisfies the MTP*$_2$ *property, we have*

$$(1) X^t_{k,j,l+1,n} \le_{st} X^t_{k,j,l,n}; \quad (2) X^t_{k,j,l,n} \le_{st} X^t_{k,j+1,l,n}. \tag{4.7}$$

Proof. Using (3.5), $\overline{F}^t_{k,j,l,n}-\overline{F}^t_{k,j,l+1,n}$ has the same sign as

$$\sum_{i=j}^{l-1}\binom{n}{i}P\{A^t_{i,n}\}\{\overline{G}^{(C^c_i)}_{k-i,n-i}(x,t)-\overline{G}^{(C^c_l)}_{k-l,n-l}(x,t)\}. \tag{4.8}$$

If **N** satisfies the MTP$_2$ property, from Lemma 4.1, we have **X** satisfies the MTP$_2$ property. [4] showed that $\overline{G}^{(C^c_i)}_{k-i,n-i}(x,t)-\overline{G}^{(C^c_l)}_{k-l,n-l}(x,t)>0$ when **X** is exchangeable and satisfies the MTP$_2$ property and hence $\overline{F}^t_{k,j,l,n}(x)>\overline{F}^t_{k,j,l+1,n}(x)$. So we complete the proof of part (1) and part (2) can be proved in a similar way. □

Example 4.1. We consider a 3-out-of-3 system with the lifetimes of components having an Erlang mixture. The shape parameters of the joint distribution are all permutations with repetition of the vector $(3,7,12)$. Then the number of the components of the joint distribution is 27 and all mixing weights are set to be a same value $\frac{1}{27}$. It is obvious that the corresponding primary counting random variables **N** satisfies the MTP$_2$ property.

We compare the random variables $X^t_{k,j,l,n}(x)$ and $X^t_{k,j,l+1,n}(x)$. According to the above theorem, $X^t_{3,1,3,3}\le_{st} X^t_{3,1,2,3}$. Figure 2 shows the survival curves for $X^t_{3,1,3,3}$ and $X^t_{3,1,2,3}$ at time $t=5$.

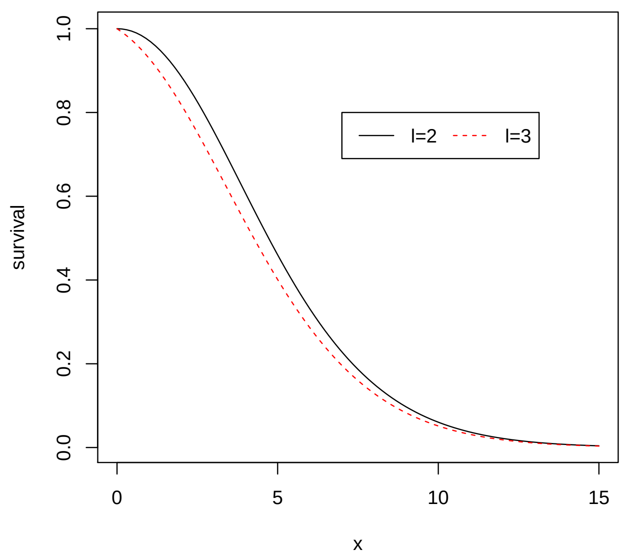

Figure 2: Survival curves for $l=2,3$.

Theorem 4.3. *Under the assumptions of Theorem* 3.1, *if* **N** *satisfies the MTP*$_2$ *property, we have*

$$X^t_{k-1,k,k,n} \leq_{st} X^t_{k-1,k,k,n-1} \tag{4.9}$$

Proof. It follows from (3.5) that $\overline{F}^t_{k-1,k,k,n-1}(x) - \overline{F}^t_{k-1,k,k,n}(x)$ has the same sign as

$$P\{A^t_{k-1,n-1}\}P\{A^t_{k-1,n}\}\{\overline{G}^{(C^c_{k-1})}_{1,n-1-k,t}(x) - \overline{G}^{(C^c_{k-1})}_{1,n-k,t}(x)\}, \tag{4.10}$$

where $\overline{G}^{(C^c_{k-1})}_{1,n-1-k,t}(x)$ is the survival function of minimum of conditional random vector $(X_k - t, \cdots, X_{n-1} - t | A^t_{k-1,n-1})$ and $\overline{G}^{(C^c_{k-1})}_{1,n-k,t}(x)$ is the survival function of minimum of conditional random vector $(X_k - t, \cdots, X_n - t | A^t_{k-1,n})$.

Let $\mathbf{x} = (x_1, \cdots, x_n)$, $A = \{\mathbf{x} : x_1 \leq t, \cdots, x_{k-1} \leq t, x_k > t, \cdots, x_{n-1} > t\}$ and $B = \{\mathbf{x} : x_1 \leq t, \cdots, x_{k-1} \leq t, x_k > t, \cdots, x_n > t, x_n > t\}$. We can easily obtain that

$$A \vee B = \{\mathbf{x} \vee \mathbf{y} : \mathbf{x} \in A, \mathbf{y} \in B\} = A, \quad A \wedge B = \{\mathbf{x} \wedge \mathbf{y} : \mathbf{x} \in A, \mathbf{y} \in B\} = B.$$

If **N** satisfies the MTP$_2$ property, from Lemma 4.1, we have **X** satisfies the MTP$_2$ property. Then we obtain (see [4])

$$(X_n - t, \cdots, X_k - t | \mathbf{X} \in B) \leq_{st} (X_n - t, \cdots, X_k - t | \mathbf{X} \in A) \tag{4.11}$$

From Lemma 4.2, we have $\overline{G}^{(C^c_{k-1})}_{1,n-1-k,t}(x) > \overline{G}^{(C^c_{k-1})}_{1,n-k,t}(x)$, and hence $X^t_{k-1,k,k,n} \leq_{st} X^t_{k-1,k,k,n-1}$. This completes the proof of the theorem. □

A Conditional residual lifetime

The following proofs are under the assumption that the joint distribution of the lifetimes is an arbitrary Erlang mixture.

Proof of Theorem 3.1. Note that for $1 \leq j < l \leq k \leq n$ and $t, x > 0$,

$$\overline{F}^t_{k,j,l,n}(x) = P(X_{k:n} - t > x | X_{l:n} > t \geq X_{j:n}) = \frac{P(X_{k:n} > t + x, X_{l:n} > t \geq X_{j:n})}{P(X_{l:n} > t \geq X_{j:n})},$$

and

$$
\begin{aligned}
&Pr(X_{k:n}>t+x, X_{l:n}>t\geq X_{j:n})\\
=&\sum_{i=j}^{l-1}\sum_{p=0}^{k-i-1} Pr(\text{exactly } i \text{ of } X's \text{ are } <t, t< \text{ exactly } p \text{ of } X's \text{ are } \leq t+x)\\
=&\sum_{i=j}^{l-1}\sum_{p=0}^{k-i-1}\sum_{C_i}\sum_{C_{i(p)}}\sum_{\mathbf{m}=1}^{\infty}\alpha_{\mathbf{m}}\prod_{s\in C_i}F(t|m_s)\prod_{s\in C_{i(p)}}(F(t+x|m_s)-F(t|m_s))\prod_{s\in C'_{i(p)}}\overline{F}(t+x|m_s)\\
=&\sum_{i=j}^{l-1}\sum_{p=0}^{k-i-1}\sum_{C_i}\sum_{C_{i(p)}}\sum_{\mathbf{m}=1}^{\infty}\alpha_{\mathbf{m}}\prod_{s=1}^{n}\overline{F}(t|m_s)\phi_i(t)\prod_{s\in C_{i(p)}}(1-\varphi(x,t|m_s))\prod_{s\in C'_{i(p)}}\varphi(x,t|m_s)\\
=&\sum_{\mathbf{m}=1}^{\infty}\alpha_{\mathbf{m}}w_{\mathbf{m}}\sum_{i=j}^{l-1}\sum_{C_i}\phi_i(t)\overline{F}^{(C_i^c)}_{k-i,n-i}(x,t|\mathbf{m}).
\end{aligned}
$$

which is the numerator of (3.1). Similarly, we can obtain that

$$Pr(X_{j:n}\leq t<X_{l:n})=\sum_{m=1}^{\infty}\alpha_{\mathbf{m}}w_{\mathbf{m}}\sum_{i=j}^{l-1}\sum_{C_i}\phi_i(t).$$

the result follows immediately. □

Remark A.1. Similar to (3.5), we have another version about the conditional residual lifetime (also see [18]):

$$\overline{F}^{t}_{k,j,l,n}(x)=\frac{\sum_{i=j}^{l-1}\sum_{C_i}P^{(C_i)}(t)\overline{G}^{(C_i^c)}_{k-i,n-i}(x,t)}{\sum_{i=j}^{l-1}\sum_{C_i}P^{(C_i)}(t)} \tag{A.1}$$

where $C_i=\{c_1,\cdots,c_i\}$ is the set of all permutations of $\{1,2,\cdots,n\}$ satisfies $c_1\leq c_2\leq\cdots\leq c_i$ and $c_{i+1}\leq\cdots\leq c_n$,

$$P^{(C_i)}(t)=Pr(X_{c_1}<t,\cdots,X_{c_i}<t,X_{c_{i+1}}\geq t,\cdots,X_{c_n}>t),$$

we denote event $[X_{c_1}<t,\cdots,X_{c_i}<t,X_{c_{i+1}}\geq t,\cdots,X_{c_n}>t]$ by $A^{(t,C_i)}$ and $\overline{G}^{(C_i^c)}_{k-i,n-i}(x,t)$ is the survival function of $(k-i)$th order statistic of conditional random vector $(X_{c_{i+1}}-t,\cdots,X_{c_n}-t|A^{(t,C_i)})$.

Proof of Theorem. Theorem4.2 Using (A.1), $\overline{F}^{t}_{k,j,l,n}-\overline{F}^{t}_{k,j,l+1,n}$ has the same sign as

$$\sum_{i=j}^{l-1}\sum_{C_i}\sum_{C_l}P^{(C_i)}(t)P^{(C_l)}(t)\overline{G}^{(C_i^c)}_{k-i,n-i}(x,t)-\sum_{i=j}^{l-1}\sum_{C_i}\sum_{C_l}P^{(C_i)}(t)P^{(C_l)}(t)\overline{G}^{(C_l^c)}_{k-l,n-l}(x,t).$$

For each C_i, there exists C'_l such that $C_i \subset C'_l$ and $C_i + C_l = C'_i + C'_l$. Therefore, the above equation could be written as

$$\sum_{i=j}^{l-1}\sum_{C_i}\sum_{C'_l} P^{(C_i)}(t)P^{(C'_l)}(t)(\overline{G}_{k-i,n-i}^{(C_i^c)}(x,t)-\overline{G}_{k-l,n-l}^{(C_l'^c)}(x,t)).$$

Let $A=\{\mathbf{x}:x_{c_1}\le t,\cdots,x_{c_i}\le t,x_{c_{i+1}}>t,\cdots,x_{c_n}>t\}$ and $B=\{\mathbf{x}:x_{c_1}\le t,\cdots,x_{c_l}\le t,x_{c_{l+1}}>t,\cdots,x_{c_n}>t\}$. Because $C_i\subset C'_l$, then we have $A\vee B=A$ and $A\wedge B=B$. The next procedures are similar to the proof in Theorem 4.2 and we have $X^t_{k,j,l+1,n}\le_{st} X^t_{k,j,l,n}$. □

Proof of Theorem 4.3. The proof is similar to the procedure of Theorem 4.2 and we omit it here. □

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