FINITE DIFFERENCE METHOD OF THE BOUNDARY PROBLEM FOR THE SYSTEMS OF SOBOLEV-GALPERN TYPE*

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§1

Many authors have paid great attention to the study of the linear and nonlinear pseudo-parabolic equations or the equations of Sobolev-Galpern type. The nonlinear pseudo-parabolic equations of ten occur in practical research, such as the equations for the long waves in nonlinear dispersion systems, the equations in the cooling process according to two-temperature of heat conduction, the equations for filtration of fluids in the broken rock and so forth[1-10]. These equations contain the differential operator u_t-u_{xxt} as the main part. Some fairly general family of nonlinear pseudo-parabolic systems[11,12], which contain the above mentioned equations as special cases, are considered by Galerkin's method.

Now let us consider in rectangular domain $Q_T=\{0\leqslant x\leqslant l, 0\leqslant t\leqslant T\}$ the nonlinear pseudo-parabolic system or the system of Sobolev-Galpern type

$$(-1)^M u_t+A(x, t)u_{x^{2M}t}=B(x, t, u, \cdots, u_{x^{2M-1}})u_{x^{2M}}+F(x, t, u, \cdots, u_{x^{2M-1}}) \quad (1)$$

with the boundary condition

$$u_{x^k}(0, t)=u_{x^k}(l, t)=0, \quad k=0, 1, \cdots, M-1 \quad (2)$$

and the initial condition

$$u(x, 0)=\varphi(x), \quad (3)$$

where $u(x, t)=(u_1(x, t), \cdots, u_m(x, t))$ is a m-dimensional vector valued unknown function.

Suppose that the following assumptions are fulfilled.

(I) $A(x, t)$ is a $m\times m$ symmetric positively definite continuous matrix and has bounded derivative $A_t(x, t)$; i.e., for any m-dimensional vectors $\xi\in\mathbb{R}^m$, $(\xi, A\xi)\geqslant a|\xi|^2$, where $a>0$.

(II) $B(x, t, p_0, p_1, \cdots, p_{2M-1})$ is a $m\times m$ semibounded continuous matrix of variables $(x, t)\in Q_T$ and $p_0, p_1, \cdots, p_{2M-1}\in\mathbb{R}^m$, i. e., there exists a constant b, such that

$$(\xi, B(x, t, p_0, p_1, \cdots, p_{2M-1})\xi)\leqslant b|\xi|^2 \quad (4)$$

for any m-dimensional vectors $\xi\in\mathbb{R}^m$ and $(x, t)\in Q_T$, $p_0, p_1, \cdots, p_{2M-1}\in\mathbb{R}^m$.

(III) $F(x, t, p_0, p_1, \cdots, p_{2M-1})$ is a m-dimensional vector valued continuous function satisfying the relation

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$$|F(x, t, p_0, p_1, \cdots, p_{2M-1})| \leqslant K_1 \left\{ \sum_{k=0}^{2M-1} |p_k| + 1 \right\}, \tag{5}$$

where K_1 is a constant.

(IV) $\varphi(x)$ is a m-dimensional vector valued initial function, belonging to $C^{(2M)}([0, l])$ and satisfying the homogeneous boundary condition (2).

In [13] it is proved by the fixed point technique, that under the conditions (I), (II), (III) and $\varphi(x) \in W_2^{(2M)}(0, l)$, the boundary problem (2), (3) for the nonlinear pseudo-parabolic system (1) has at least one m-dimensional vector valued global solution $u(x, t)$ in the functional space $W_\infty^{(1)}((0, T); W_2^{(2M)}(0, l))$. In addition that $B(x, t, p_0, p_1, \cdots, p_{2M-1})$ and $F(x, t, p_0, p_1, \cdots, p_{2M-1})$ are continuously differentiable or more precisely are locally Lipschitz continuous with respect to $p_0, p_1, \cdots, p_{2M-1} \in \mathbb{R}^m$, the solution $u(x, t)$ is unique.

The purpose of this note is to study the boundary problem (2), (3) for the system (1) by the finite difference method under the above mentioned assumptions (I), (II), (III) and (IV).

§2

The finite interval $[0, l]$ can be divided into the small segment grids by the points $x_j = jh (j=0, 1, \cdots, J)$, where $Jh = l$, J is an integer and h is the stepsize. The discrete function $\{u_j\} (j=0, 1, \cdots, J)$ is defined on the grid points $x_j (j=0, 1, \cdots, J)$. We denote the scalar product of two discrete functions $\{u_j\}$ and $\{v_j\}$ by $(u, v)_h = \sum_{j=0}^{J} u_j v_j h$. And $\|u\|_h = (u, u)_h$. Also we introduce the symbol $\|u\|_\infty = \max_{j=0,1,\cdots,J} |u_j|$.

Let us denote $\Delta_+ u_j = u_{j+1} - u_j$ and $\Delta_- u_j = u_j - u_{j-1} (j=0, 1, \cdots, J)$. Similarly we take

$$\left\| \frac{\Delta_+ u}{h} \right\|_h^2 = \sum_{j=0}^{J-1} \left| \frac{\Delta_+ u_j}{h} \right|^2 h = \sum_{j=1}^{J} \left| \frac{\Delta_- u_j}{h} \right|^2 h = \left\| \frac{\Delta_- u}{h} \right\|_h^2$$

and they can be denoted simply by $\|\delta_h u\|_h$. Also we have $\left\| \frac{\Delta_+ u}{h} \right\|_\infty = \left\| \frac{\Delta_- u}{h} \right\|_\infty = \|\delta_h u\|_\infty$. We adopt the similar notations for the difference quotients of higher order: $\|\delta_h^k u\|_h$ and $\|\delta_h^k u\|_\infty$ $(k=0, 1, \cdots)$.

Now we state some lemmas which are useful for later discussions and whose proof can be found in [14].

Lemma 1. *For any two discrete functions $\{u_j\}$ and $\{v_j\}$ $(j=0, 1, \cdots, J)$ on finite interval, there are identities*

$$\sum_{j=0}^{J-1} u_j \Delta_+ v_j = -\sum_{j=1}^{J} v_j \Delta_- u_j - u_0 v_0 + u_J v_J, \tag{6}$$

$$\sum_{j=1}^{J-1} u_j \Delta_+ \Delta_- v_j = -\sum_{j=0}^{J-1} (\Delta_+ u_j)(\Delta_+ v_j) - u_0 \Delta_+ v_0 + u_J \Delta_- v_J. \tag{7}$$

Lemma 2. *For any discrete function $\{u_j\}$ $(j=0, 1, \cdots, J)$ on the finite interval $[0, l]$, there are the interpolation relations*

$$\|\delta_h^k u\|_h \leqslant K_2 \|u\|_h^{1-\frac{k}{n}} \left(\|\delta_h^n u\|_h + \frac{\|u\|_h}{l^n} \right)^{\frac{k}{n}}, \quad k=0, 1, \cdots, n \tag{8}$$

and

$$\|\delta_h^k u\|_\infty \leqslant K_3 \|u\|_h^{1-\frac{k+\frac{1}{2}}{n}} \left(\|\delta_h^n u\|_h + \frac{\|u\|_h}{l^n}\right)^{\frac{k+\frac{1}{2}}{n}}, \quad k=0, 1, \cdots, n-1, \tag{9}$$

where K_2 *and* K_3 *are constants independent of* h, l *and* $\{u_j\}$.

Lemma 3. *For any discrete function* $\{u_j\}$ $(j=0, 1, \cdots, J)$ *on the finite interval* $[0, l]$ *and any given* $\varepsilon>0$, *there exists a constant* $K(\varepsilon, n)$ *depending only on* ε *and* n, *such that*

$$\|\delta_h^k u\|_h \leqslant \varepsilon \|\delta_h^n u\|_h + K(\varepsilon, n) \|u\|_h \tag{10}$$

and

$$\|\delta_h^k u\|_\infty \leqslant \varepsilon \|\delta_h^n u\|_h + K(\varepsilon, n) \|u\|_h, \tag{11}$$

where $k=0, 1, \cdots, n-1$ *and* $K(\varepsilon, n)$ *is independent of* $\{u_j\}$.

Lemma 4. *For any discrete function* $\{u_j\}$ $(j=0, 1, \cdots, J)$, *satisfying the homogeneous boundary conditions* $u_i=u_{J-i}=0$ $(i=0, 1, \cdots, M-1)$, *there is*

$$\|\delta_h^k u\|_h \leqslant K_4 l^{M-k} \|\delta_h^M u\|_h, \quad k=0, 1, \cdots, M-1, \tag{12}$$

where K_4 *is independent of* h, l *and* $\{u_j\}$.

As a simple consequence of the formula (6) of Lemma 1, we have the following lemma.

Lemma 5. *For any discrete function* $\{u_j\}$ $(j=0, 1, \cdots, J)$ *satisfying the homogeneous boundary conditions* $u_i=u_{J-i}=0$ $(i=0, 1, \cdots, M-1)$, *there is*

$$\sum_{j=M}^{J-M} u_j \Delta_+^M \Delta_-^M v_j = (-1)^M \sum_{j=0}^{J-M} (\Delta_+^M u_j)(\Delta_+^M v_j), \tag{6'}$$

where $M\geqslant 1$ *is an integer.*

§ 3

Suppose that the rectangular domain Q_T is divided into small grids by the parallel lines $x=x_j (j=0, 1, \cdots, J)$ and $t=t^n (n=0, 1, \cdots, N)$, where $x_j=jh$, $t^n=n\Delta t$ and $Jh=l$, $N\Delta t=T$. Denote the m-dimensional vector valued function on the grid point (x_j, t^n) by $v_j^n (j=0, 1, \cdots, J; n=0, 1, \cdots, N)$.

Let us construct the finite difference scheme

$$(-1)^M \frac{v_j^{n+1}-v_j^n}{\Delta t} + \widetilde{A}_j^{n+\alpha} \frac{1}{\Delta t} \frac{\Delta_+^M \Delta_-^M (v_j^{n+1}-v_j^n)}{h^{2M}} = \widetilde{B}_j^{n+\alpha} \frac{\Delta_+^M \Delta_-^M v_j^{n+\alpha}}{h^{2M}} + F_j^{n+\alpha},$$
$$j=M, \cdots, J-M; \quad n=0, \cdots, N-1 \tag{1$_h$}$$

corresponding to the nonlinear pseudo-parabolic system (1), where

$$\begin{aligned} \widetilde{A}_j^{n+\alpha} &= A(x_{j*}, t^{n+\bar{\alpha}}); \\ \widetilde{B}_j^{n+\alpha} &= B(x_{j*}, t^{n+\bar{\alpha}}, \overset{*}{\delta}{}_h^0 v_j^{n+\alpha}, \cdots, \overset{*}{\delta}{}_h^{2M-1} v_j^{n+\alpha}); \\ F_j^{n+\alpha} &= F(x_{j*}, t^{n+\bar{\alpha}}, \widetilde{\delta}_h^0 v_j^{n+\alpha}, \cdots, \widetilde{\delta}_h^{2M-1} v_j^{n+\alpha}) \end{aligned} \tag{13}$$

and

$$\begin{aligned} \overset{*}{\delta}{}_h^k v_j^{n+\alpha} &= \sum_{i=j-M}^{j+M-k} \left(\alpha \bar{\beta}_{ki} \frac{\Delta_+^k v_i^{n+1}}{h^k} + \bar{\bar{\beta}}_{ki} \frac{\Delta_+^k v_i^n}{h^k}\right), \quad k=0, 1, \cdots, 2M-1; \\ \widetilde{\delta}_h^k v_j^{n+\alpha} &= \sum_{i=j-M}^{j+M-k} \widetilde{\beta}_{ki} \frac{\Delta_+^k v_i^{n+\alpha}}{h^k}, \quad k=0, 1, \cdots, 2M-1; \end{aligned} \tag{14}$$

here all $\bar{\beta}$'s, $\bar{\bar{\beta}}$'s and $\tilde{\beta}$'s are constants and $\sum_{i=j-M}^{j+M-k}(\alpha\bar{\beta}_{ki}+\bar{\bar{\beta}}_{ki})=1$ and $\sum_{i=j-M}^{i+M-k}\tilde{\beta}_{ki}=1$ ($k=0, 1, \cdots, 2M-1$), also $j-M\leqslant j^*\leqslant j+M$, $0\leqslant\alpha$, $\bar{\alpha}\leqslant 1$. The corresponding finite difference boundary conditions are

$$\Delta_+^k v_0^n=\Delta_-^k v_J^n=0 \quad (k=0, 1, \cdots, M-1). \tag{2}_h$$

The corresponding finite difference initial condition is

$$v_j^0=\bar{\varphi}_j \quad (j=0, 1, \cdots, J), \tag{3}_h$$

where $\bar{\varphi}_j=\varphi(x_j)$ ($j=M, \cdots, J-M$) and $\bar{\varphi}_k=\bar{\varphi}_{J-k}=0$ ($k=0, 1, \cdots, M-1$).

As to the existence of the m-dimensional vector valued discrete function v_j^n ($j=0, 1, \cdots, J$; $n=0, 1, \cdots, N$) for the finite difference system $(1)_h$, $(2)_h$ and $(3)_h$, we can step by step solve the nonlinear system $(1)_h$ and $(2)_h$ for the unknown vectors v_j^{n+1} ($j=0, 1, \cdots, J$), where v_j^n ($j=0, 1, \cdots, J$) are regarded as given vectors.

When $\alpha=0$, the system $(1)_h$ is an explicit difference scheme. So the solvability of the nonlinear system $(1)_h$ and $(2)_h$ for the unknown vectors v_j^n ($j=0, 1, \cdots, J$; $n=0, 1, \cdots, N$) is evident.

When $0<\alpha\leqslant 1$, the system $(1)_h$ can be written in the form

$$\frac{v_j^{n+1}-v_j^n}{\Delta t}=(-1)^{M+1}\left(\frac{\tilde{A}^{n+\alpha}}{\alpha\Delta t}-\tilde{B}_j^{n+\alpha}\right)\frac{\Delta_+^M\Delta_-^M v_j^{n+\alpha}}{h^{2M}}+G_j^{n+\alpha}, \tag{15}$$

where

$$G_j^{n+\alpha}=(-1)^M\left(\frac{\tilde{A}^{n+\alpha}}{\alpha\Delta t}\,\frac{\Delta_+^M\Delta_-^M v_j^n}{h^{2M}}+F_j^{n+\alpha}\right). \tag{16}$$

Hence (15) can be considered as the finite implicit difference scheme of certain nonlinear ordinary parabolic system. The existence of v_j^{n+1} ($j=0, 1, \cdots, J$; $n=0, 1, \cdots, N-1$) for the nonlinear system $(1)_h$, $(2)_h$ or (15), $(2)_h$ can be established by the fixed point technique as in [14].

Lemma 6. *Under the conditions* (I), (II) *and* (III), *for the sufficient small Δt, the finite difference system $(1)_h$, $(2)_h$ and $(3)_h$ has at least one solution v_j^n ($j=0, 1, \cdots, J$; $n=0, 1, \cdots, N$), where $0\leqslant\alpha\leqslant 1$.*

§ 4

Now we turn to estimate the solutions v_j^n ($j=0, 1, \cdots, J$; $n=0, 1, \cdots, N$) of the finite difference system $(1)_h$, $(2)_h$ and $(3)_h$.

Taking the scalar product of the vectors $\frac{\Delta_+^M\Delta_-^M v_j^{n+\alpha}}{h^{2M}}h\Delta t$ and the vector equations $(1)_h$ and summing up the resulting relations for $j=M, \cdots, J-M$, we get

$$\begin{aligned}(-1)^M\sum_{j=M}^{J-M}\left(\frac{\Delta_+^M\Delta_-^M v_j^{n+\alpha}}{h^{2M}},\ v_j^{n+1}-v_j^n\right)h+\sum_{j=M}^{J-M}\left(\frac{\Delta_+^M\Delta_-^M v_j^{n+\alpha}}{h^{2M}},\ \tilde{A}_j^{n+\alpha}\frac{\Delta_+^M\Delta_-^M(v_j^{n+1}-v_j^n)}{h^{2M}}\right)h\\ =\Delta t\sum_{j=M}^{J-M}\left(\frac{\Delta_+^M\Delta_-^M v_j^{n+\alpha}}{h^{2M}},\ \tilde{B}_j^{n+\alpha}\frac{\Delta_+^M\Delta_-^M v_j^{n+\alpha}}{h^{2M}}\right)h+\Delta t\sum_{j=M}^{J-M}\left(\frac{\Delta_+^M\Delta_-^M v_j^{n+\alpha}}{h^{2M}},\ F_j^{n+\alpha}\right)h,\\ j=M, \cdots, J-M;\quad n=0, 1, \cdots, N-1.\end{aligned} \tag{17}$$

From (6′) of Lemma 5, we have

$$(-1)^M \sum_{j=M}^{J-M} \left(\frac{\Delta_+^M \Delta_-^M v_j^{n+\alpha}}{h^{2M}},\ v_j^{n+1}-v_j^n \right) h = \sum_{j=M}^{J-M} \left(\frac{\Delta_+^M v_j^{n+\alpha}}{h^M},\ \frac{\Delta_+^M (v_j^{n+1}-v_j^n)}{h^M} \right) h$$
$$= \frac{1}{2} \|\delta_h^M v^{n+1}\|_h^2 - \frac{1}{2} \|\delta^M v^n\|_h^2 + \left(\alpha - \frac{1}{2}\right) \|\delta_h^M (v^{n+1}-v^n)\|_h^2. \tag{18}$$

Since A is a symmetric positively definite matrix, we have

$$\sum_{j=M}^{J-M} (z_j^{n+\alpha},\ \tilde{A}_j^{n+\alpha}(z_j^{n+1}-z_j^n))h = \frac{1}{2}(z^{n+1},\ \tilde{A}_j^{n+\alpha} z^{n+1})_h - \frac{1}{2}(z^n,\ \tilde{A}^{n+\alpha} z^n)_h$$
$$+ \left(\alpha - \frac{1}{2}\right)(z^{n+1}-z^n,\ \tilde{A}^{n+\alpha}(z^{n+1}-z^n))_h, \tag{19}$$

where
$$z_j^{n+\alpha} = \frac{\Delta_+^M \Delta_-^M v_j^{n+\alpha}}{h^{2M}} \quad (j = M, \cdots, J-M).$$

Assume that $\frac{1}{2} \leqslant \alpha \leqslant 1$.

It follows from (18) and (19) that

$$\frac{1}{2} \|\delta^M v^{n+1}\|^2 - \frac{1}{2} \|\delta_h^M v^n\|_h^2 \leqslant (-1)^M (\delta_h^{2M} v^{n+\alpha},\ v^{n+1}-v^n)_h \tag{20}$$

and

$$\frac{1}{2}(\delta_h^{2M} v^{n+1},\ \tilde{A}^{n+\alpha} \delta_h^{2M} v^{n+1})_h - \frac{1}{2}(\delta_h^{2M} v^n,\ \tilde{A}^{n+\alpha} \delta_h^{2M} v^n)_h$$
$$\leqslant (\delta_h^{2M} v^{n+\alpha},\ \tilde{A}^{n+\alpha}(\delta_h^{2M} v^{n+1} - \delta_h^{2M} v^n))_h. \tag{21}$$

Since B is a semibounded matrix

$$\sum_{j=M}^{J-M} (z_j^{n+\alpha},\ \tilde{B}_j^{n+\alpha} z_j^{n+\alpha}) h \leqslant b \|\delta_h^{2M} v^{n+\alpha}\|_h^2. \tag{22}$$

For the last term of (17), we have

$$\sum_{j=M}^{J-M} (z_j^{n+\alpha},\ F_j^{n+\alpha}) h \leqslant \frac{1}{2} \|\delta_h^{2M} v^{n+\alpha}\|_h^2 + \frac{1}{2} \|F^{n+\alpha}\|_h^2. \tag{23}$$

From the assumption (III),

$$\|F^{n+\alpha}\|_h^2 \leqslant C_1 \left\{ \sum_{k=0}^{2M-1} \sum_{j=M}^{J-M} |\tilde{\delta}_h^k v_j^{n+\alpha}|^2 h + 1 \right\}.$$

Here by the expression (14) and the formulae (10) and (12),

$$\sum_{j=M}^{J-M} |\tilde{\delta}_h^k v_j^{n+\alpha}|^2 h = \sum_{j=M}^{J-M} \left| \sum_{i=j-M}^{j+M-k} \tilde{B}_{ki} \frac{\Delta_+^k v_j^{n+\alpha}}{h^k} \right|^2 h$$
$$\leqslant C_2 \sum_{i=0}^{2M-k} \sum_{j=i}^{J-2M+i} \left| \frac{\Delta_+^k v_j^{n+\alpha}}{h^k} \right|^2 h \leqslant C_3 \|\delta_h^k v^{n+\alpha}\|_h^2$$
$$\leqslant C_4 \|\delta_h^{2M} v^{n+\alpha}\|_h^2 + C_5 \|\delta_h^M v^{n+\alpha}\|_h^2, \quad k = 0, 1, \cdots, 2M-1.$$

Hence,

$$\|F^{n+\alpha}\|_h^2 \leqslant C_6 \|\delta_h^{2M} v^{n+\alpha}\|_h^2 + C_7 \|\delta_h^M v^{n+\alpha}\|_h^2 + C_8. \tag{24}$$

Therefore (17) can be replaced by

$$\|\delta_h^M v^{n+1}\|_h^2 - \|\delta_h^M v^n\|_h^2 + (\delta_h^{2M} v^{n+1},\ \tilde{A}^{n+\alpha} \delta_h^{2M} v^{n+1})_h - (\delta_h^{2M} v^n,\ \tilde{A}^{n+\alpha} \delta_h^{2M} v^n)_h$$
$$\leqslant (2b+1+C_6)\Delta t \|\delta_h^{2M} v^{n+\alpha}\|_h^2 + C_7 \Delta t \|\delta_h^M v^{n+\alpha}\|_h^2 + C_8 \Delta t$$
$$\leqslant C_9 \Delta t \|\delta_h^{2M} v^{n+1}\|^2 + C_{10} \Delta t \|\delta_h^{2M} v^n\|_h^2 + C_{11} \Delta t \|\delta_h^M v^{n+1}\|_h^2 + C_{12} \Delta t \|\delta_h^M v^n\|_h^2 + C_{13} \Delta t.$$

Summing up the above inequality for $n = 0, 1, \cdots, m$, we get

$$\|\delta_h^M v^{m+1}\|_h^2-\|\delta_h^M v^0\|_h^2+(\delta_h^{2M}v^{m+1},\ \widetilde{A}^{m+\alpha}\delta_h^{2M}v^{m+1})_h-(\delta_h^{2M}v^0,\ \widetilde{A}^\alpha\delta_h^{2M}v^0)_h$$
$$-\sum_{n=1}^{m}\Big(\delta_h^{2M}v^n,\ \frac{\widetilde{A}^{n+\alpha}-\widetilde{A}^{n-1+\alpha}}{\Delta t}\delta_h^{2M}v^n\Big)_h\Delta t$$
$$\leqslant C_{14}\sum_{n=0}^{m+1}\|\delta_h^{2M}v^n\|_h^2\Delta t+C_{15}\sum_{n=0}^{m+1}\|\delta_h^M v^n\|_h^2+C_{16}\Delta t,$$

where $\dfrac{\widetilde{A}_j^{n+\alpha}-\widetilde{A}_j^{n-1+\alpha}}{\Delta t}$ is bounded. Since $\varphi(x)\in C^{(2M)}([0,\ l])$, then the second and the fourth terms of the left part of above inequality are bounded. So this becomes

$$\|\delta_h^M v^{m+1}\|_h^2+a\|\delta_h^{2M}v^{m+1}\|_h^2\leqslant C_{17}\sum_{n=0}^{m+1}\|\delta_h^{2M}v^n\|_h^2\Delta t+C_{15}\sum_{n=0}^{m+1}\|\delta_h^M v^n\|_h^2+C_{18}. \tag{25}$$

Let us denote

$$z_m=\sum_{n=0}^{m}\{\|\delta_h^M v^n\|_h^2+a\|\delta_h^{2M}v^n\|_h^2\}\Delta t.$$

Then (25) can be simplified to

$$\frac{z_{m+1}-z_m}{\Delta t}\leqslant C_{19}z_{m+1}+C_{18}. \tag{26}$$

This follows that z_m is uniformly bounded for sufficiently small Δt. Then the right hand side of (25) or (26) is also uniformly bounded.

Lemma 7. *Under the conditions* (I), (II), (III) *and* (IV), *the solution* v_j^n ($j=0, 1, \cdots, J$; $n=0, 1, \cdots, N$) *of the finite difference system* $(1)_h$, $(2)_h$ *and* $(3)_h$ *has the estimation relation*

$$\max_{n=0,1,\cdots,N}\|\delta_h^k v^n\|_h\leqslant K_5,\quad k=0,\ 1,\ \cdots,\ 2M, \tag{27}$$

where Δt *is sufficiently small and* K_5 *is independent of* h *and* Δt.

Now making the scalar product of the vector $\dfrac{\Delta_+^M\Delta_-^M(v_j^{n+1}-v_j^n)}{h^{2M}\Delta t}h$ and the finite difference system $(1)_h$ and then summing up the resulting relations for $j=M, \cdots, J-M$, we obtain by simple calculation

$$\Big\|\frac{\delta_h^M(v^{n+1}-v^n)}{\Delta t}\Big\|_h^2+\Big(\frac{\delta_h^{2M}(v^{n+1}-v^n)}{\Delta t},\ \widetilde{A}^{n+\alpha}\frac{\delta_h^{2M}(v^{n+1}-v^n)}{\Delta t}\Big)_h$$
$$=\Big(\frac{\delta_h^{2M}(v^{n+1}-v^n)}{\Delta t},\ \widetilde{B}^{n+\alpha}\delta_h^{2M}v^{n+\alpha}\Big)_h+\Big(\frac{\delta_h^{2M}(v^{n+1}-v^n)}{\Delta t},\ F_j^{n+\alpha}\Big). \tag{28}$$

From (27), we know that $\max\limits_{n=0,1,\cdots,N}\|\delta_h^k v^n\|_\infty$ is uniformly bounded for $k=0, 1, \cdots, 2M-1$. Hence $\widetilde{B}_j^{n+\alpha}$ and $F_j^{n+\alpha}$ are bounded. So (28) easily becomes

$$\Big\|\frac{\delta_h^M(v^{n+1}-v^n)}{\Delta t}\Big\|_h^2+\frac{a}{2}\Big\|\frac{\delta_h^{2M}(v^{n+1}-v^n)}{\Delta t}\Big\|_h^2\leqslant C_{20}\|\delta_h^{2M}v^{n+\alpha}\|_h^2+C_{21}\|\delta_h^M v^{n+\alpha}\|_h^2+C_{22}.$$

Lemma 8. *Under the conditions of Lemma 7, there is the estimation*

$$\max_{n=0,1,\cdots,N-1}\Big\|\frac{\delta_h^k(v^{n+1}-v^n)}{\Delta t}\Big\|_h\leqslant K_6,\quad k=0,\ 1,\ \cdots,\ 2M, \tag{29}$$

where K_6 *is independent of* h *and* Δt.

Lemma 9. *Under the conditions of Lemma 7, there is the estimations for the solution* v_j^n ($j=0, 1, \cdots, J$; $n=0, 1, \cdots, N$) *of the finite difference system* $(1)_h$, $(2)_h$ *and* $(3)_h$:

$$\max_{j=0,1,\cdots,J-k}|\Delta_+^k v_j^n|\leqslant K_7h^k,\quad k=0,\ 1,\ \cdots,\ 2M-1, \tag{30}$$

$$\max_{j=0,1,\cdots,J-2M} |\Delta_+^{2M} v_j^n| \leqslant K_8 h^{2M-\frac{1}{2}}, \tag{30'}$$

where $n=0, 1, \cdots, N$;

$$\max_{n=0,1,\cdots,N-1} |\Delta_+^k v_j^{n+1} - \Delta_+^k v_j^n| \leqslant K_9 h^k \Delta t, \quad k=0, 1, \cdots, 2M-1, \tag{31}$$

where $j=0, 1, \cdots, J-k$.

Proof. Applying the formula (9) to the discrete function $\left\{\frac{\Delta_+^k v_j^n}{h^k}\right\}$ $(j=0, 1, \cdots, J-k; n=0, 1, \cdots, N)$ for $k=0, 1, \cdots, 2M-1$, we have

$$\left|\frac{\Delta_+^k v_j^n}{h^k}\right| \leqslant K_3 \|v^n\|_h^{1-\frac{k+\frac{1}{2}}{2M}} \left\{\|\delta_h^{2M} v^n\|_h + \frac{\|v^n\|}{l^{2M}}\right\}^{\frac{k+\frac{1}{2}}{2M}},$$

for $j=0, 1, \cdots, J-k$, $n=0, 1, \cdots, N$ and $k=0, 1, \cdots, 2M-1$. From Lemma 7, the right hand side of the above inequality is uniformly bounded for $k=0, 1, \cdots, 2M-1$. This proves the formula (30).

The formula (31) can be proved similarly.

The estimation relation (30′) is a immediate consequence of the following inequality

$$|\Delta_+^{2M} v_j^n| = h^{2M-\frac{1}{2}} \left(\left|\frac{\Delta_+^{2M} v_j^n}{h^{2M}}\right|^2 h\right)^{\frac{1}{2}} \leqslant h^{2M-\frac{1}{2}} \|\delta_h^{2M} v^n\|_h.$$

Hence the proof of this lemma is complete.

§5

For the m-dimensional vector valued discrete function v_j^n $(j=0, 1, \cdots, J; n=0, 1, \cdots, N)$, we define a set of m-dimensional vector valued piecewise constant functions as follows. Let $v_{h\Delta t}^{(k)}(x, t) = \frac{\Delta_+^k v_j^{n+1}}{h^k}$ in $Q_j^n = \{jh < x \leqslant (j+1)h;\ n\Delta t < t \leqslant (n+1)\Delta t\}$ and $v_{h\Delta t}^{(k)}(x, t) = \frac{\Delta_+^k v_{J-k}^{n+1}}{h^k} \{(J-k)h < x \leqslant l;\ n\Delta t < t \leqslant (n+1)\Delta t\}$, where $j=0, 1, \cdots, J-k$, $n=0, 1, \cdots, N-1$ and $k=0, 1, \cdots, 2M$. Let $\tilde{v}_{h\Delta t}^{(k)}(x, t) = \frac{\Delta_+^k (v_j^{n+1} - v_j^n)}{h^k \Delta t}$ in Q_j^n and $\tilde{v}_{h\Delta t}^{(k)}(x, t) = \frac{\Delta_+^k (v_{J-k}^{n+1} - v_{J-k}^n)}{h^k \Delta t} \{(J-k)h < x \leqslant l;\ n\Delta t < t \leqslant (n+1)\Delta t\}$. Denote $\bar{v}_{h\Delta t}^{(k)}(x, t) = \alpha v_{h\Delta t}^{(k)}(x, t) + (1-\alpha) v_{h\Delta t}^{(k)}(x, t-\Delta t)$.

From the estimations (27) and (29), we have the estimations for the m-dimensional vector valued piecewise constant functions $v_{h\Delta t}^{(k)}(x, t)$ and $\tilde{v}_{h\Delta t}^{(k)}(x, t)$ $(k=0, 1, \cdots, 2M)$ as follows

$$\begin{aligned} &\sup_{0\leqslant t\leqslant T} \|v_{h\Delta t}^{(k)}(\cdot, t)\|_{L_2(0,l)} \leqslant K_5, \\ &\sup_{0\leqslant t\leqslant T} \|\tilde{v}_{h\Delta t}^{(k)}(\cdot, t)\|_{L_2(0,l)} \leqslant K_6, \end{aligned} \tag{32}$$

where $k=0, 1, \cdots, 2M$.

Hence there is a sequence h_i and Δt_i, such that as $h_i^2 + \Delta t_i^2 \to 0$, the sequences $\{v_{h_i\Delta t_i}^{(k)}(x, t)\}$ and $\{\tilde{v}_{h_i\Delta t_i}^{(k)}(x, t)\}$ weakly converge to $u^{(k)}(x, t)$ and $\tilde{u}^{(k)}(x, t)$ respectively for $k=0, 1, \cdots, 2M$. At same time $\{\bar{v}_{h\Delta t}^{(k)}(x, t)\}$ converges to $u^{(k)}(x, t)$ respectively for $k=0, 1, \cdots, 2M$.

It can be easily verified[14], that $u^{(k)}(x, t) = u_{x^k}(x, t)$ and $\widetilde{u}^{(k)}(x, t) = u_t^{(k)}(x, t) = u_{x^k t}(x, t)$, for $k=0, 1, \cdots, 2M$, where $u(x, t) = u^{(0)}(x, t)$.

From the estimations (30) and (30′), we know that the convergences of the sequences $\{v_{h_i\Delta t_i}^{(k)}(x, t)\}$ to $\{u_{x^k}(x, t)\}$ are all uniform in Q_T for $k=0, 1, \cdots, 2M-1$.

Similarly, let us define $A_{h\Delta t}(x, t) = \widetilde{A}_j^{n+\alpha}$ in Q_j^n ($j=M, \cdots, J-M$; $n=0, 1, \cdots, N-1$) to be a $m\times m$ piecewise constant matrix. Let $B_{h\Delta t}(x, t) = \widetilde{B}_j^{n+\alpha}$ in Q_j^n be also a $m\times m$ piecewise constant matrix and let $F_{h\Delta t}(x, t) = F_j^{n+\alpha}$ in Q_j^n be a m-dimensional vector valued piecewise constant function($j=M, \cdots, J-M$; $n=0, 1, \cdots, N-1$). It is clear that as $h_i^2+\Delta t_i^2 \to 0$, $\{A_{h_i\Delta t_i}(x, t)\}$, $\{B_{h_i\Delta t_i}(x, t)\}$ and $\{F_{h_i\Delta t_i}(x, t)\}$ are uniformly convergent to $A(x, t)$, $B(x, t, u(x, t), \cdots, u_{x^{2M-1}}(x, t))$ and $F(x, t, u(x, t), \cdots, u_{x^{2M-1}}(x, t))$ respectively in Q_T.

By the construction of the all above piecewise constant functions, we see that

$$(-1)^M \widetilde{v}_{h\Delta t}(x, t) + A_{h\Delta t}(x, t)\widetilde{v}_{h\Delta t}^{(2M)}(x, t) = B_{h\Delta t}(x, t) v_{h\Delta t}^{(2M)}(x, t) + F_{h\Delta t}(x, t).$$

Hence for any smooth test function $\Phi(x, t)$, we have

$$\iint_{Q_T} \Phi[(-1)^M u_t + A(x, t) u_{x^{2M}t} - B(x, t, u, \cdots, u_{x^{2M-1}}) u_{x^{2M}} - F(x, t, u, \cdots, u_{x^{2M-1}})]\,dx\,dt = 0.$$

So $u(x, t)$ satisfies the nonlinear pseudo-parabolic system (1) in generalized sense. And $u(x, t)$ satisfies the boundary conditions (2) and the initial condition (3) in classical sense. Also we can see that $u(x, t) \in W_\infty^{(1)}((0, T); W_2^{(2M)}(0, l))$.

Suppose that besides the conditions (I), (II), (III) and (IV), $B(x, t, p_0, p_1, \cdots, p_{2M-1})$ and $F(x, t, p_0, p_1, \cdots, p_{2M-1})$ are continuously differentiable or more precisely are Lipshitz continuous with respect to $p_0, p_1, \cdots, p_{2M-1}$ on any bounded domain of m-dimensional Euclidean space $\mathbb{R}^m$. Then the solution of the boundary problem (2), (3) for the system (1) is unique. Therefore under there conditions the solution v_j^n ($j=0, 1, \cdots, J$; $n=0, 1, \cdots, N$) of the finite difference system $(1)_h$, $(2)_h$ and $(3)_h$ converges to $u(x, t)$ in $W_\infty^{(1)}((0, T), W_2^{(2M)}(0, l))$, as $h^2+\Delta t^2 \to 0$.

Theorem. *Suppose that the conditions* (I), (II), (III) *and* (IV) *are satisfied and suppose that* $B(x, t, p_0, p_1, \cdots, p_{2M-1})$ *and* $F(x, t, p_0, p_1, \cdots, p_{2M-1})$ *are locally Lipschitz continuous with respect to* $p_0, p_1, \cdots, p_{2M-1} \in \mathbb{R}^m$. *Then the solution* v_j^n ($j=0, 1, \cdots, J$; $n=0, 1, \cdots, N$) *of the finite difference system* $(1)_h$, $(2)_h$ *and* $(3)_h$ *converges to* $u(x, t) \in W_\infty^{(1)}((0, T); W_2^{(2M)}(0, l))$ *as* $h^2+\Delta t^2 \to 0$, *which satisfies nonlinear pseudo-parabolic system* (1) *in generalized sense and satisfies the boundary conditions* (2) *and the initial condition* (3) *in classical sense.*

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