THE CONVERGENCE OF THE SPECTRAL SCHEME FOR SOLVING TWO-DIMENSIONAL VORTICITY EQUATIONS*

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Much work has been done for the spectral scheme of the P. D. E. (see[1]). The author proposed a technique to prove the strict error estimation of the spectral scheme for the K. D. V.-Burgers equation[2]. In this paper, the technique is generalized to two-dimensional vorticity equations. Under some conditions, the error estimation implies the convergence. The more smooth the solution of the vorticity equations, the more accurate the approximate solution.

I. The Scheme

Let $H(x_1, x_2, t)$ and $\Psi(x_1, x_2, t)$ be the vorticity and stream function respectively. $f_i(x_1, x_2, t)$ are given. All of them have the period 2π for variables x_1 and x_2.

Let
$$Q=\text{set}[(x_1, x_2)/-\pi\leqslant x_1,\ x_2\leqslant\pi],$$
$$F_p(Q)=\{\varphi/\varphi\in H^p,\ \varphi(x_1, x_2)=\varphi(x_1+2\pi, x_2)=\varphi(x_1, x_2+2\pi)\},$$
$$J(H, \Psi)=\frac{\partial\Psi}{\partial x_2}\frac{\partial H}{\partial x_1}-\frac{\partial\Psi}{\partial x_1}\frac{\partial H}{\partial x_2}.$$
We consider the following problem
$$\begin{cases}\dfrac{\partial H}{\partial t}+J(H, \Psi)-\nu\nabla^2H=f_1, & \text{in } Q\times(0, T],\\ -\nabla^2\Psi=H+f_2, & \text{in } Q\times[0, T],\\ H(x_1, x_2, 0)=H_0(x_1, x_2), & \text{in } Q,\end{cases}\tag{1}$$
where ν is a nonnegative constant. We suppose
$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}(H_0(x_1, x_2)+f_2(x_1, x_2, t))dx_1dx_2+\int_0^t\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}f_1(x_1, x_2, t')dx_1dx_2dt'=0.$$
Let
$$(\eta(t), \xi(t))=\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\eta(x_1, x_2, t)\xi(x_1, x_2, t)dx_1dx_2.$$
To fix $\Psi(x_1, x_2, t)$, we require
$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\Psi(x_1, x_2, t)dx_1dx_2=0,\qquad t\in[0, T].\tag{2}$$
We take the solution of (1) as follows:

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$$\begin{cases}\left(\frac{\partial H}{\partial t},\ \varphi\right)+(J(H,\ \Psi),\ \varphi)+\nu\sum_{j=1}^{2}\left(\frac{\partial H}{\partial x_j},\ \frac{\partial\varphi}{\partial x_j}\right)=(f_1,\ \varphi),\\ \sum_{j=1}^{2}\left(\frac{\partial\Psi}{\partial x_j},\ \frac{\partial\varphi}{\partial x_j}\right)=(H+f_2,\ \varphi),\end{cases}\tag{3}$$

where $\varphi\in F_1(Q)$. (3) is supposed to have a unique solution. Let

$$x=(x_1,\ x_2),\ l=(l_1,\ l_2),\ |l|=\sqrt{l_1^2+l_2^2},\ lx=l_1x_1+l_2x_2.$$

Put

$$H(x,\ t)=\sum_{|l|=0}^{\infty}H_l(t)e^{ilx},$$

$$\Psi(x,\ t)=\sum_{|l|=0}^{\infty}\Psi_l(t)e^{ilx},$$

$$f_j(x,\ t)=\sum_{|l|=0}^{\infty}f_{j,l}(t)e^{ilx},\qquad j=1,\ 2,$$

$$H^{(n)}(x,\ t)=\sum_{|l|<n}H_l(t)e^{ilx},$$

$$\Psi^{(n)}(x,\ t)=\sum_{|l|<n}\Psi_l(t)e^{ilx},$$

$$f_j^{(n)}(x,t)=\sum_{|l|<n}f_{j,l}(t)e^{ilx};\qquad j=1,\ 2,$$

and

$$R^{(n)}(H)=H(x,\ t)-H^{(n)}(x,\ t),$$

$$R^{(n)}(\Psi)=\Psi(x,\ t)-\Psi^{(n)}(x,\ t),$$

$$R^{(n)}(f_j)=f_j(x,\ t)-f_j^{(n)}(x,\ t),\qquad j=1,\ 2.$$

We assume that H, Ψ and f_j are so smooth that when $n\to\infty$, $R^{(n)}\left(\frac{\partial H}{\partial x_j}\right)$, $R^{(n)}\left(\frac{\partial\Psi}{\partial x_j}\right)$, $R^{(n)}(f_1)$ and $R^{(n)}(f_2)$ tend to zero in $Q\times[0,\ T]$.

Let τ be the mesh spacing of variable t and be sufficiently small,

$$\eta_t(x,\ K\tau)=\frac{1}{\tau}[\eta(x,\ K\tau+\tau)-\eta(x,\ K\tau)],\quad K\geqslant0.$$

Let $\eta^{(n)}(x,\ t)$ and $\psi^{(n)}(x,\ t)$ denote the approximation of $H^{(n)}(x,\ t)$ and $\Psi^{(n)}(x,\ t)$ respectively

$$\eta^{(n)}(x,\ K\tau)=\sum_{|l|<n}\eta_l^{(n)}(K\tau)e^{ilx},$$

$$\psi^{(n)}(x,\ K\tau)=\sum_{|l|<n}\psi_l^{(n)}(K\tau)e^{ilx}.$$

$\delta\geqslant0$ and $\sigma\geqslant0$ are parameters, $\varphi_l=e^{ilx}$.

The spectral scheme for solving (1) is the following

$$\begin{cases}(\eta_t^{(n)}(K\tau),\ \varphi_l)+(J(\eta^{(n)}(K\tau)+\delta\tau\eta_t^{(n)}(K\tau),\ \psi^{(n)}(K\tau)),\ \varphi_l)\\ \quad+\nu\sum_{j=1}^{2}\left(\frac{\partial}{\partial x_j}(\eta^{(n)}(K\tau)+\sigma\tau\eta_t^{(n)}(K\tau)),\ \frac{\partial\varphi_l}{\partial x_j}\right)\\ \quad=(f_1^{(n)}(K\tau),\ \varphi_l),\quad |l|\leqslant n,\quad 0\leqslant K\tau\leqslant T,\\ \sum_{j=1}^{2}\left(\frac{\partial}{\partial x_j}(\psi^{(n)}(K\tau)),\ \frac{\partial\varphi_l}{\partial x_j}\right)=(\eta^{(n)}(K\tau)+f_2^{(n)}(K\tau),\ \varphi_l),\ |l|\leqslant n,\quad 0\leqslant K\tau\leqslant T,\\ \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\psi^{(n)}(x,\ K\tau)dx_1\,dx_2=0,\quad 0\leqslant K\tau\leqslant T,\\ \eta^{(n)}(x,\ 0)=H_0^{(n)}(x),\quad x\in Q.\end{cases}\tag{4}$$

Clearly if $\delta=\sigma=0$, and $\eta_l^{(n)}(K\tau)$ and $\psi_l^{(n)}(K\tau)$ are known, we can calculate

$\eta_l^{(n)}(K\tau+\tau)$ and $\psi_l^{(n)}(K\tau+\tau)$ explicitly. If we adopt the finite element scheme for (1), we cannot get a really explicit scheme because of the existence of the mass matrix. This is one of the advantages of spectral schemes. If $\delta\neq 0$ or $\sigma\neq 0$, we must solve the linear algebraic equations to get $\eta_l^{(n)}(K\tau+\tau)$ and $\psi_l^{(n)}(K\tau+\tau)$ at each time $t=(K+1)\tau$.

II. Lemmas

We introduce the following notations

$$\|\eta(K\tau)\|^2=(\eta(K\tau),\ \eta(K\tau)),$$

$$|\eta(K\tau)|_1^2=\sum_{j=1}^{2}\left\|\frac{\partial\eta}{\partial x_j}\right\|^2,\qquad |\eta(K\tau)|_2^2=\sum_{j=1}^{2}\left|\frac{\partial\eta}{\partial x_j}\right|_1^2,$$

$$\|\eta\|_\infty=\max_{(x,t)\in Q\times[0,T]}|\eta(x,\ t)|,$$

$$|\eta|_{1,\infty}=\max_{1<j<2}\left\|\frac{\partial\eta}{\partial x_j}\right\|_\infty,\qquad \|\eta\|_{1,\infty}=\max(\|\eta\|_\infty,\ |\eta|_{1,\infty}).$$

Lemma 1. *If $\eta\in F_0(Q)$, then*

$$2(\eta(K\tau),\ \eta_t(K\tau))=(\|\eta(K\tau)\|^2)_t-\tau\|\eta_t(K\tau)\|^2.$$

Lemma 2. *If $\eta\in F_1(Q)$, then*

$$2\sum_{j=1}^{2}\left(\frac{\partial\eta(K\tau)}{\partial x_j},\ \frac{\partial\eta_t(K\tau)}{\partial x_j}\right)=(|\eta(K\tau)|_1^2)_t-\tau|\eta_t(K\tau)|_1^2.$$

Lemma 3. *If $\eta\in F_1(Q)$, $\xi\in F_1(Q)$, $\psi\in F_2(Q)$, then*

$$(J(\eta,\ \psi),\ \xi)+(J(\xi,\ \psi),\ \eta)=0.$$

Proof. We have

$$4\pi^2(J(\eta,\ \psi),\ \xi)=-\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\eta\left[\frac{\partial}{\partial x_1}\left(\xi\frac{\partial\psi}{\partial x_2}\right)-\frac{\partial}{\partial x_2}\left(\xi\frac{\partial\psi}{\partial x_1}\right)\right]dx_1dx_2$$

$$=-\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\eta\left(\frac{\partial\psi}{\partial x_2}\frac{\partial\xi}{\partial x_1}-\frac{\partial\psi}{\partial x_1}\frac{\partial\xi}{\partial x_2}\right)dx_1dx_2$$

$$=-4\pi^2(J(\xi,\ \psi),\ \eta).$$

Lemma 4. *If $\eta\in F_1(Q)$ and $\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\eta(x)dx_1dx_2=0$, then*

$$\|\eta^2\|^2\leqslant 4\pi\int_0^\infty\frac{dz}{(1+z)^2}\|\eta\|^2|\eta|_1^2.$$

Proof. Let
$$\eta(x)=\sum_{|l|=1}^{\infty}\eta_l e^{ilx},$$

we have from the Yong-Hausedouff inequality (see [4])

$$\|\eta^2\|^2=\frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|\eta|^4dx_1dx_2\leqslant\left(\sum_{|l|=1}^{\infty}|\eta_l|^{\frac{4}{3}}\right)^3$$

and

$$\sum_{|l|=1}^{\infty}|\eta_l|^{\frac{4}{3}}=\sum_{|l|=1}^{\infty}\left\{|\eta_l|^2\left[1+\frac{\|\eta\|^2|l|^2}{|\eta|_1^2}\right]\right\}^{\frac{2}{3}}\left[\frac{1}{1+\frac{\|\eta\|^2|l|^2}{|\eta|_1^2}}\right]^{\frac{2}{3}}$$

$$\leqslant\left\{\sum_{|l|=1}^{\infty}|\eta_l|^2\left[1+\frac{\|\eta\|^2|l|^2}{|\eta|_1^2}\right]\right\}^{\frac{2}{3}}\left[\sum_{|l|=1}^{\infty}\frac{1}{\left[1+\frac{\|\eta\|^2|l|^2}{|\eta|_1^2}\right]^2}\right]^{\frac{1}{3}}.$$

Moreover,

$$\sum_{|l|=1}^{\infty} |\eta_l|^2\left[1+\frac{\|\eta\|^2|l|^2}{|\eta|_1^2}\right]=2\|\eta\|^2,$$

$$\sum_{|l|=1}^{\infty} \frac{1}{\left[1+\frac{\|\eta\|^2|l|^2}{|\eta|_1^2}\right]^2}\leqslant 2\pi\int_0^{\infty} \frac{rdr}{\left(1+\frac{\|\eta\|^2r^2}{|\eta|_1^2}\right)^2}=\pi\frac{|\eta|_1^2}{\|\eta\|^2}\int_0^{\infty}\frac{dz}{(1+z)^2}.$$

Lemma 5. *If* $\psi\in F_1(Q)$ *and* $\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\psi(x_1, x_2)dx_1dx_2=0$, *then*

$$\|\psi\|^2\leqslant|\psi|_1^2.$$

Proof. We get

$$\psi(x)=\sum_{|l|=1}^{\infty}\psi_l e^{ilx}.$$

So

$$\|\psi\|^2=\sum_{|l|=1}^{\infty}\psi_l^2\leqslant\sum_{|l|=1}^{\infty}|l|^2\psi_l^2=|\psi|_1^2.$$

Lemma 6. *If the following conditions hold*

(i) $\xi(K\tau)$ *and* $\zeta(K\tau)$ *are nonnegative functions,*

(ii) ρ, a, M_1, M_2, *and* M_3 *are nonnegative constants,*

(iii) $A(\xi(K\tau))$ *is such a function that if* $\xi(K\tau)\leqslant M_3$, *then* $A(\xi(K\tau))\leqslant 0$,

(iv) $\xi(K\tau)\leqslant\rho+\tau\sum_{j=0}^{K-1}[M_1\xi(j\tau)+M_2n^a\xi^2(j\tau)+A(\xi(j\tau))\zeta(j\tau)]$,

(v) $\rho e^{(M_1+M_2)T}\leqslant\min\left(M_3, \frac{1}{n^a}\right)$, $\quad\xi(0)\leqslant\rho$, $K\tau\leqslant T$,

then

$$\xi(K\tau)\leqslant\rho e^{(M_1+M_2)K\tau}.$$

Especially, if $M_2=0$ *and* $A(\xi(j\tau))=0$, *then for all* ρ *and* k, *we have*

$$\xi(K\tau)\leqslant\rho e^{M_1K\tau}.$$

This lemma is a special case of Lemma 1 in [3].

III. The Basic Inequality of Error Estimation

From (3), we have

$$\begin{aligned}&(H_t(K\tau), \varphi_l)+(J(H(K\tau)+\delta\tau H_t(K\tau), \Psi(K\tau)), \varphi_l)\\&\quad+\nu\sum_{j=1}^{2}\left(\frac{\partial}{\partial x_j}(H(K\tau)+\sigma\tau H_t(K\tau)), \frac{\partial\varphi_l}{\partial x_j}\right)\\&=(f_1(K\tau)+E_1(K\tau)+E_2(K\tau), \varphi_l)\\&\quad+\nu\left(E_3(K\tau), \frac{\partial\varphi_l}{\partial x_1}\right)+\nu\left(E_4(K\tau), \frac{\partial\varphi_l}{\partial x_2}\right),\end{aligned}$$

where

$$\begin{aligned}&E_1=H_t-\frac{\partial H}{\partial t},\\&E_2=J(H+\delta\tau H_t, \Psi)-J(H, \Psi),\\&E_3=\frac{\partial}{\partial x_1}(H+\sigma\tau H_t)-\frac{\partial H}{\partial x_1},\\&E_4=\frac{\partial}{\partial x_2}(H+\sigma\tau H_t)-\frac{\partial H}{\partial x_2}.\end{aligned}\tag{5}$$

From (5), we obtain

$$
\begin{aligned}
&(H_t^{(n)}(K\tau),\ \varphi_l)+(J(H^{(n)}(K\tau)+\delta\tau H_t^{(n)}(K\tau),\ \Psi^{(n)}(K\tau)),\ \varphi_l)\\
&\quad+\nu\sum_{j=1}^{2}\Big(\frac{\partial}{\partial x_j}(H^{(n)}(K\tau)+\sigma\tau H_t^{(n)}(K\tau)),\ \frac{\partial\varphi_l}{\partial x_j}\Big)\\
&=(f_1^{(n)}(K\tau)+E_1(K\tau)+E_2(K\tau)+E_5(K\tau)+E_6(K\tau)+E_9(K\tau),\ \varphi_l)\\
&\quad+\nu\Big(E_3(K\tau)+E_7(K\tau),\ \frac{\partial\varphi_l}{\partial x_1}\Big)+\nu\Big(E_4(K\tau)+E_8(K\tau),\ \frac{\partial\varphi_l}{\partial x_2}\Big),\\
&\sum_{j=1}^{2}\Big(\frac{\partial\Psi^{(n)}}{\partial x_j},\frac{\partial\varphi_l}{\partial x_j}\Big)=(H^{(n)}(K\tau)+f_2^{(n)}(K\tau)+E_{10}(K\tau)+E_{11}(K\tau),\ \varphi_l)\\
&\qquad+\Big(E_{12}(K\tau),\frac{\partial\varphi_l}{\partial x_1}\Big)+\Big(E_{13}(K\tau),\ \frac{\partial\varphi_l}{\partial x_2}\Big),
\end{aligned}
\tag{6}
$$

where

$$
\begin{aligned}
&E_5=-R_t^{(n)}(H),\\
&E_6=-J(R^{(n)}(H)+\delta\tau R_t^{(n)}(H),\ \Psi)-J(H+\delta\tau H_t,\ R^{(n)}(\Psi))\\
&\qquad+J(R^{(n)}(H)+\delta\tau R_t^{(n)}(H),\ R^{(n)}(\Psi)),\\
&E_7=-\frac{\partial}{\partial x_1}(R^{(n)}(H)+\sigma\tau R_t^{(n)}(H)),\\
&E_8=-\frac{\partial}{\partial x_2}(R^{(n)}(H)+\sigma\tau R_t^{(n)}(H)),\\
&E_9=R^{(n)}(f_1), \qquad E_{10}=R^{(n)}(H),\\
&E_{11}=R^{(n)}(f_2), \qquad E_{12}=-R^{(n)}\Big(\frac{\partial\Psi}{\partial x_1}\Big),\\
&E_{13}=-R^{(n)}\Big(\frac{\partial\Psi}{\partial x_2}\Big).
\end{aligned}
$$

Let

$$
\begin{aligned}
&\tilde{\eta}^{(n)}(x,\ K\tau)=\eta^{(n)}(x,\ K\tau)-H^{(n)}(x,\ K\tau)=\sum_{|l|<n}\tilde{\eta}_l^{(n)}(K\tau)e^{ilx},\\
&\tilde{\psi}^{(n)}(x,\ K\tau)=\psi^{(n)}(x,\ K\tau)-\Psi^{(n)}(x,\ K\tau)=\sum_{|l|<n}\tilde{\psi}_l^{(n)}(K\tau)e^{ilx}.
\end{aligned}
$$

Let ε denote a suitably small positive constant; C is a positive constant which may depends on $\|H\|_\infty$, $\|H\|_{1,\infty}$ and $\|\psi\|_{1,\infty}$. From (4) and (6) we have

$$
\begin{aligned}
&(\tilde{\eta}_t^{(n)}(K\tau),\ \varphi_l)+(J(\tilde{\eta}^{(n)}(K\tau)+\delta\tau\tilde{\eta}_t^{(n)}(K\tau),\ \Psi^{(n)}(K\tau)\\
&\quad+\tilde{\psi}^{(n)}(K\tau)),\ \varphi_l)+(J(H^{(n)}(K\tau)+\delta\tau H_t^{(n)}(K\tau),\ \tilde{\psi}^{(n)}(K\tau)),\ \varphi_l)\\
&\quad+\nu\sum_{j=1}^{2}\Big(\frac{\partial}{\partial x_j}(\tilde{\eta}^{(n)}(K\tau)+\sigma\tau\tilde{\eta}_t^{(n)}(K\tau)),\ \frac{\partial\varphi_l}{\partial x_j}\Big)\\
&=-(E_1(K\tau)+E_2(K\tau)+E_5(K\tau)+E_6(K\tau)+E_9(K\tau),\ \varphi_l)\\
&\quad-\nu\Big(E_3(K\tau)+E_7(K\tau),\ \frac{\partial\varphi_l}{\partial x_1}\Big)\\
&\quad-\nu\Big(E_4(K\tau)+E_8(K\tau),\ \frac{\partial\varphi_l}{\partial x_2}\Big),
\end{aligned}
\tag{7}
$$

and

$$
\begin{aligned}
&\sum_{j=1}^{2}\Big(\frac{\partial\tilde{\psi}^{(n)}(K\tau)}{\partial x_j},\ \frac{\partial\varphi_l}{\partial x_j}\Big)=(\tilde{\eta}^{(n)}(K\tau)-E_{10}(K\tau)-E_{11}(K\tau),\ \varphi_l)\\
&\quad-\Big(E_{12}(K\tau),\frac{\partial\varphi_l}{\partial x_1}\Big)-\Big(E_{13}(K\tau),\ \frac{\partial\varphi_l}{\partial x_2}\Big).
\end{aligned}
\tag{8}
$$

Multiplying (7) by 2 $\tilde{\eta}_l^{(n)}(K\tau)$ and summing them for all $|l|\le n$, we obtain from Lemmas 1, 2, 3

$$
\begin{aligned}
&(\|\tilde{\eta}^{(n)}(K\tau)\|^2)_t-\tau\|\tilde{\eta}_t^{(n)}(K\tau)\|^2+(2\nu-\nu\varepsilon)|\tilde{\eta}^{(n)}(K\tau)|_1^2+\nu\sigma\tau(|\tilde{\eta}^{(n)}(K\tau)|_1^2)_t\\
&\quad-\nu\sigma\tau^2|\tilde{\eta}_t^{(n)}(K\tau)|_1^2-2\delta\tau(\tilde{\eta}_t^{(n)}(K\tau),\ J(\tilde{\eta}^{(n)}(K\tau),\ \Psi^{(n)}(K\tau)+\tilde{\psi}^{(n)}(K\tau)))\\
&\quad+2(\tilde{\eta}^{(n)}(K\tau),\ J(H^{(n)}(K\tau)+\delta\tau H_t^{(n)}(K\tau),\ \tilde{\psi}^{(n)}(K\tau)))\\
&\le C\|\tilde{\eta}^{(n)}(K\tau)\|^2+C(\|E_1(K\tau)\|^2+\|E_2(K\tau)\|^2\\
&\quad+\|E_5(K\tau)\|^2+\|E_6(K\tau)\|^2+\|E_9(K\tau)\|^2)\\
&\quad+\frac{\nu C}{\varepsilon}(\|E_3(K\tau)\|^2+\|E_4(K\tau)\|^2+\|E_7(K\tau)\|^2+\|E_8(K\tau)\|^2).
\end{aligned}
$$

Let m be a positive constant. We multiply (7) by $m\tau(\tilde{\eta}_l^{(n)}(K\tau))_t$ and sum them. From Lemmas 2, 3 we get

$$
\begin{aligned}
&m\tau\|\tilde{\eta}_t^{(n)}(K\tau)\|^2+m\nu\sigma\tau^2|\tilde{\eta}_t^{(n)}(K\tau)|_1^2+\frac{m\nu\tau}{2}(|\tilde{\eta}^{(n)}(K\tau)|_1^2)_t-\frac{m\nu\tau^2}{2}|\tilde{\eta}_t^{(n)}(K\tau)|_1^2\\
&\quad+m\tau(\tilde{\eta}_t^{(n)}(K\tau),\ J(\tilde{\eta}(K\tau),\ \Psi^{(n)}(K\tau)+\tilde{\psi}^{(n)}(K\tau)))\\
&\quad+m\tau(\tilde{\eta}_t^{(n)}(K\tau),\ J(H^{(n)}(K\tau)+\delta\tau H_t^{(n)}(K\tau),\ \tilde{\psi}^{(n)}(K\tau)))\\
&\le\varepsilon\tau\|\tilde{\eta}_t^{(n)}(K\tau)\|^2+\varepsilon\nu n^2\tau^2\|\tilde{\eta}_t^{(n)}(K\tau)\|^2\\
&\quad+\frac{C\tau m^2}{\varepsilon}(\|E_1(K\tau)\|^2+\|E_2(K\tau)\|^2+\|E_5(K\tau)\|^2+\|E_6(K\tau)\|^2\\
&\quad+\|E_9(K\tau)\|^2)+\frac{Cm^2\nu}{\varepsilon}(\|E_3(K\tau)\|^2+\|E_4(K\tau)\|^2\\
&\quad+\|E_7(K\tau)\|^2+\|E_8(K\tau)\|^2).
\end{aligned}
\tag{10}
$$

By putting (9) and (10) together, we obtain

$$
\begin{aligned}
&(\|\tilde{\eta}^{(n)}(K\tau)\|^2)_t+\tau(m-1-\varepsilon-\varepsilon\nu\tau n^2)\|\tilde{\eta}_t^{(n)}(K\tau)\|^2+(2\nu-\nu\varepsilon)|\tilde{\eta}^{(n)}(K\tau)|_1^2\\
&\quad+\nu\tau\left(\sigma+\frac{m}{2}\right)(|\tilde{\eta}^{(n)}(K\tau)|_1^2)_t+\nu\tau^2\left(m\sigma-\sigma-\frac{m}{2}\right)|\tilde{\eta}_t^{(n)}(K\tau)|_1^2\\
&\quad+F_1(K\tau)+F_2(K\tau)+F_3(K\tau)\le C\|\tilde{\eta}^{(n)}(K\tau)\|^2+\varepsilon_1^2(K\tau),
\end{aligned}
\tag{11}
$$

where

$$
\begin{aligned}
\varepsilon_1^2(K\tau)&=C(1+m^2)\left(1+\frac{1}{\varepsilon}\right)(\|E_1(K\tau)\|^2+\|E_2(K\tau)\|^2\\
&\quad+\nu\|E_3(K\tau)\|^2+\nu\|E_4(K\tau)\|^2+\|E_5(K\tau)\|^2\\
&\quad+\|E_6(K\tau)\|^2+\nu\|E_7(K\tau)\|^2+\nu\|E_8(K\tau)\|^2+\|E_9(K\tau)\|^2),\\
F_1(K\tau)&=2(\tilde{\eta}^{(n)}(K\tau),\ J(H^{(n)}(K\tau)+\delta\tau H_t^{(n)}(K\tau),\ \tilde{\psi}^{(n)}(K\tau))),\\
F_2(K\tau)&=m\tau(\tilde{\eta}_t^{(n)}(K\tau),\ J(H^{(n)}(K\tau)+\delta\tau H_t^{(n)}(K\tau),\ \tilde{\psi}^{(n)}(K\tau)))\\
&\quad+\tau(m-2\delta)(\tilde{\eta}_t^{(n)}(K\tau),\ J(\tilde{\eta}^{(n)}(K\tau),\ \Psi^{(n)}(K\tau))),\\
F_3(K\tau)&=\tau(m-2\delta)(\tilde{\eta}_t^{(n)}(K\tau),\ J(\tilde{\eta}^{(n)}(K\tau),\ \tilde{\psi}^{(n)}(K\tau))).
\end{aligned}
$$

Multiplying (8) by $\tilde{\psi}_l^{(n)}$ and summing them for all $|l|\le n$, we have

$$|\tilde{\psi}^{(n)}(K\tau)|_1^2\le\|\tilde{\eta}^{(n)}(K\tau)\|^2+\frac{1}{4}\|\tilde{\psi}^{(n)}(K\tau)\|^2+\frac{1}{4}|\tilde{\psi}^{(n)}(K\tau)|_1^2+\varepsilon_2^2(K\tau),$$

where

$$\varepsilon_2^2(K\tau)=C(\|E_{10}(K\tau)\|^2+\|E_{11}(K\tau)\|^2+\|E_{12}(K\tau)\|^2+\|E_{13}(K\tau)\|^2).$$

From Lemma 5, we get

$$|\tilde{\psi}^{(n)}(K\tau)|_1^2\le 2(\|\tilde{\eta}^{(n)}(K\tau)\|^2+\varepsilon_2^2(K\tau)). \tag{12}$$

IV. The Case $\nu>0$

We are going to estimate $|F_i|$. From (12), we first obtain

$$|F_1(K\tau)|\leqslant C'\|H^{(n)}\|^2_{1,\infty}\|\tilde\eta^{(n)}(K\tau)\|^2+|\tilde\psi^{(n)}(K\tau)|_1^2$$
$$\leqslant C(1+\|R^{(n)}(H)\|^2_{1,\infty})\|\tilde\eta^{(n)}(K\tau)\|^2+C\varepsilon_2^2(K\tau). \tag{13}$$

We have

$$|F_2(K\tau)|\leqslant\varepsilon\tau\|\tilde\eta_t^{(n)}(K\tau)\|^2+\frac{C\tau m^2}{\varepsilon}(\|R^{(n)}(H)\|^2_{1,\infty}+1)(\|\tilde\eta^{(n)}(K\tau)\|^2+\varepsilon_2^2(K\tau))$$
$$+\frac{C\tau(m-2\delta)^2}{\varepsilon}(1+\|R^{(n)}(\Psi)\|^2_{1,\infty})|\tilde\eta^{(n)}(K\tau)|_1^2. \tag{14}$$

From Lemmas 4, 5 and $|\tilde\eta^{(n)}(K\tau)|_2^2\leqslant Cn^2|\tilde\eta^{(n)}(K\tau)|_1^2$, we have

$$|F_3(K\tau)|\leqslant\varepsilon\tau\|\tilde\eta_t^{(n)}(K\tau)\|^2+\frac{C\tau(m-2\delta)^2}{\varepsilon}|\tilde\eta^{(n)}(K\tau)|_1|\tilde\eta^{(n)}(K\tau)|_2$$
$$\cdot|\tilde\psi^{(n)}(K\tau)|_1|\tilde\psi^{(n)}(K\tau)|_2\leqslant\varepsilon\tau\|\tilde\eta_t^{(n)}(K\tau)\|^2$$
$$+\frac{2C\tau n(m-2\delta)^2}{\varepsilon}(\|\tilde\eta^{(n)}(K\tau)\|^2+\varepsilon_2^2(K\tau))|\tilde\eta^{(n)}(K\tau)|_1^2. \tag{15}$$

Substituting (13)—(15) into (11), we get the basic inequality

$$(\|\tilde\eta^{(n)}(K\tau)\|^2)_t+\tau(m-1-3\varepsilon-\varepsilon\nu\tau n^2)\|\tilde\eta_t^{(n)}(K\tau)\|^2+\nu|\tilde\eta^{(n)}(K\tau)|_1^2$$
$$+\nu\tau\Big(\sigma+\frac{m}{2}\Big)(|\tilde\eta^{(n)}(K\tau)|_1^2)_t+\nu\tau^2\Big(m\sigma-\sigma-\frac{m}{2}\Big)|\tilde\eta_t^{(n)}(K\tau)|_1^2$$
$$\leqslant M^*\|\tilde\eta^{(n)}(K\tau)\|^2+B(\tilde\eta^{(n)}(K\tau))|\tilde\eta^{(n)}(K\tau)|_1^2+\varepsilon^2(K\tau), \tag{16}$$

where

$$M^*=C(1+\|R^{(n)}(H)\|^2_{1,\infty})\Big(1+\frac{1}{\varepsilon}\Big)(1+\tau m^2),$$

$$B(\tilde\eta^{(n)}(K\tau))=-\nu+\nu\varepsilon+\frac{C\tau(m-2\delta)^2}{\varepsilon}(1+\|R^{(n)}(\psi)\|^2_{1,\infty})$$
$$+\frac{2C\tau n(m-2\delta)^2}{\varepsilon}(\|\tilde\eta^{(n)}(K\tau)\|^2+\varepsilon_2^2(K\tau)),$$

$$\varepsilon^2(K\tau)=C\varepsilon_1^2(K\tau)+C\varepsilon_3^2(K\tau)+\frac{C\tau m^2}{\varepsilon}(\|R^{(n)}(H)\|^2_{1,\infty}+1)\varepsilon_2^2(K\tau).$$

Now we are going to choose m for three different cases.

Case 1. $\sigma>\frac{1}{2}$, we take

$$m=m_1=\max\Big(1+3\varepsilon+C_0+\varepsilon\nu\tau n^2,\ \frac{2\sigma}{2\sigma-1}\Big),$$

where C_0 is a nonnegative constant. Hence $m\sigma-\sigma-\frac{m}{2}\geqslant0$. So from (16) we get

$$(\|\tilde\eta^{(n)}(K\tau)\|^2)_t+C_0\tau\|\tilde\eta_t^{(n)}(K\tau)\|^2+\nu|\tilde\eta^{(n)}(K\tau)|_1^2+\nu\tau\Big(\sigma+\frac{m}{2}\Big)(|\tilde\eta^{(n)}(K\tau)|_1^2)_t$$
$$\leqslant M^*\|\tilde\eta^{(n)}(K\tau)\|^2+B(\tilde\eta^{(n)}(K\tau))|\tilde\eta^{(n)}(K\tau)|_1^2+\varepsilon^2(K\tau). \tag{17}$$

Case 2. $\sigma=\frac{1}{2}$, we take

$$m=m_2=1+3\varepsilon+C_0+\varepsilon\nu\tau n^2+\nu\tau n^2.$$

Therefore

$$\tau(m-1-3\varepsilon-\varepsilon\nu\tau n^2)\|\tilde{\eta}_t^{(n)}(K\tau)\|^2+\nu\tau^2\left(m\sigma-\sigma-\frac{m}{2}\right)|\tilde{\eta}_t^{(n)}(K\tau)|_1^2$$
$$\geqslant C_0\tau\|\tilde{\eta}_t^{(n)}(K\tau)\|^2, \tag{18}$$

whence (17) holds still.

Case 3. $\sigma<\frac{1}{2}$, $\tau n^2<\frac{1}{\nu(1-2\sigma)}$. We take

$$m=m_3=\frac{1+3\varepsilon+C_0+\varepsilon\nu\tau n^2+2\sigma\nu\tau n^2}{1+2\sigma\nu\tau n^2-\nu\tau n^2}.$$

Hence (17) and (18) hold too.

Let $\tilde{Q}^{(n)}(K\tau)=\|\tilde{\eta}^{(n)}(K\tau)\|^2+C_0\tau^2\sum_{j=0}^{K-1}\|\tilde{\eta}_t^{(n)}(j\tau)\|^2+\nu\tau\sum_{j=0}^{K-1}|\tilde{\eta}^{(n)}(j\tau)|_1^2,$

$$\tilde{\rho}^{(n)}(K\tau)=\tau\sum_{j=0}^{K-1}\varepsilon^2(j\tau).$$

We sum (17) for $t=0, \tau, 2\tau, \cdots, (K-1)\tau$; then

$$\tilde{Q}^{(n)}(K\tau)\leqslant\tilde{\rho}^{(n)}(K\tau)+\tau\sum_{j=0}^{K-1}(M^*\tilde{Q}^{(n)}(j\tau)+B(\tilde{Q}^{(n)}(j\tau))|\tilde{\eta}^{(n)}(j\tau)|_1^2).$$

Finally we use Lemma 6 with

$$\xi(K\tau)=\tilde{Q}^{(n)}(K\tau),\ \zeta(K\tau)=|\tilde{\eta}^{(n)}(K\tau)|_1^2,\ A(\xi(K\tau))=B(Q^{(n)}(K\tau)),$$
$$\rho=\tilde{\rho}^{(n)}(K\tau),\ M_1=M^*,\ M_2=0,\ M_3=\frac{\nu\varepsilon}{4C\tau n(m-2\delta)^2}.$$

Theorem 1. *If the following conditions are satisfied*

(i) $\tau=O\left(\frac{1}{n^2}\right)$,

(ii) $\sigma>\frac{1}{2}$ *or* $\tau n^2<\frac{1}{\nu(1-2\sigma)}$,

(iii) $\|R^{(n)}(H)\|_{1,\infty}^2\leqslant C$, $\tau\|R^{(n)}(\Psi)\|_{1,\infty}^2\leqslant\frac{\nu\varepsilon}{8C(m-2\delta)^2}$,

and $$\varepsilon_2^2(K\tau)\leqslant\frac{\nu\varepsilon}{8C\tau n(m-2\delta)^2},$$

(iv) $\tilde{\rho}^{(n)}(K\tau)e^{M^*K\tau}\leqslant\frac{\nu\varepsilon}{4C\tau n(m-2\delta)^2}$,

then

$$\tilde{Q}^{(n)}(K\tau)\leqslant\tilde{\rho}^{(n)}(K\tau)e^{M^*K\tau}.$$

Now we assume

$$\begin{cases}\delta\geqslant\frac{m_1}{2}, & \text{if } \sigma>\frac{1}{2},\\ \delta\geqslant\frac{m_2}{2}, & \text{if } \sigma=\frac{1}{2},\\ \delta\geqslant\frac{m_3}{2}, & \text{if } \sigma<\frac{1}{2}.\end{cases} \tag{19}$$

Then we can take $m=2\delta$ in (17) and use Lemma 6 with $A(\xi(K\tau))=0$.

Theorem 2. *If the following conditions hold*

(i) $\sigma>\frac{1}{2}$ *or* $\tau n^2<\frac{1}{\nu(1-2\sigma)}$,

(ii) (19) *holds,*

(iii) $\|R^{(n)}(H)\|_{1,\infty}\leqslant C$,

then for all $\tilde{\rho}^{(n)}(K\tau)$ *and* k, *we have*

$$\tilde{Q}^{(n)}(K\tau)\leqslant\tilde{\rho}^{(n)}(K\tau)e^{M^*K\tau}.$$

Remark 1. If $\tilde{\rho}^{(n)}(K\tau)\to0$ as $n\to\infty$, then under the conditions of Theorem 1 or Theorem 2, we have

$$\tilde{Q}^{(n)}(K\tau)\to0,$$

i. e. the scheme (4) is convergent.

Clearly the more smooth the solution H and Ψ, the more accurate the approximate solutin $\eta^{(n)}$ and $\Psi^{(n)}$. If we use the finite difference method, the accuracy of the approximate solution is limited for the same scheme. This is another advantage of spectral schemes.

V. The Case $\nu=0$

In this case we have from (11)

$$(\|\tilde{\eta}^{(n)}(K\tau)\|^2)_t+\tau(m-1-\varepsilon)\|\tilde{\eta}_t^{(n)}(K\tau)\|^2+F_1(K\tau)+F_2(K\tau)+F_3(K\tau)$$
$$\leqslant C\|\tilde{\eta}^{(n)}(K\tau)\|^2+\varepsilon_1^2(K\tau). \tag{20}$$

As (13) holds still, we can get the following estimations

$$|F_2(K\tau)|\leqslant\varepsilon\tau\|\tilde{\eta}_t^{(n)}(K\tau)\|^2+\frac{C\tau}{\varepsilon}(m^2+n^2(m-2\delta)^2$$
$$+m^2\|R^{(n)}(H)\|_{1,\infty}^2+n^2(m-2\delta)^2\|R^{(n)}(\Psi)\|_{1,\infty}^2)\|\tilde{\eta}^{(n)}(K\tau)\|^2$$
$$+\frac{C\tau m^2}{\varepsilon}(\|R^{(n)}(H)\|_{1,\infty}^2+1)\varepsilon_2^2(K\tau), \tag{21}$$

$$|F_3(K\tau)|\leqslant\varepsilon\tau\|\tilde{\eta}_t^{(n)}(K\tau)\|^2+\frac{C\tau n^3(m-2\delta)^2}{\varepsilon}(\|\tilde{\eta}^{(n)}(K\tau)\|^2+\varepsilon_2^2(K\tau))\|\tilde{\eta}^{(n)}(K\tau)\|^2. \tag{22}$$

Substituting (13), (21), (22) into (20), we obtain

$$(\|\tilde{\eta}^{(n)}(K\tau)\|^2)_t+\tau(m-1-3\varepsilon)\|\tilde{\eta}_t^{(n)}(K\tau)\|^2$$
$$\leqslant M^{**}\|\tilde{\eta}^{(n)}(K\tau)\|^2+\frac{C\tau n^3(m-2\delta)^2}{\varepsilon}\|\tilde{\eta}^{(n)}(K\tau)\|^4+\varepsilon^2(K\tau), \tag{23}$$

where $\varepsilon^2(K\tau)$ is given in Section III, and

$$M^{**}=C(1+\|R^{(n)}(H)\|_{1,\infty}^2)\left(1+\frac{\tau m^2}{\varepsilon}\right)$$
$$+\frac{C\tau n^2}{\varepsilon}(m-2\delta)^2[1+\|R^{(n)}(\Psi)\|_{1,\infty}^2+n\varepsilon_2^2(K\tau)].$$

We take

$$m=m_0=1+3\varepsilon+C_0$$

and sum (21) for $t=0,\cdots,(K-1)\tau$; then

$$\tilde{Q}^{(n)}(K\tau)\leqslant\tilde{\rho}^{(n)}(K\tau)+\tau\sum_{j=0}^{K-1}\left(M^{**}\tilde{Q}^{(n)}(j\tau)+\frac{C\tau n^3(m-2\delta)^2}{\varepsilon}[\tilde{Q}^{(n)}(j\tau)]^2\right).$$

Finally by using Lemma 6 with

$$\xi(K\tau)=\tilde{Q}^{(n)}(K\tau),\quad A(\xi(K\tau))=0,$$

$$\rho=\tilde{\rho}^{(n)}(K\tau),\quad M_1=M^{**},\quad M_2=\frac{C\tau n^2(m-2\delta)^2}{8},\quad M_3^{-1}=0,\quad a=1,$$

we get

Theorem 3. *If the following conditions hold*

(i) $\tau=O\left(\frac{1}{n^2}\right)$,

(ii) $\|R^{(n)}(H)\|_{1,\infty}^2\leqslant C$, $\|R^{(n)}(\Psi)\|_{1,\infty}^2\leqslant C$, *and* $ns_2^2(K\tau)\leqslant C$,

(iii) $\tilde{\rho}^{(n)}(K\tau)e^{C^*K\tau}\leqslant\frac{C^{**}}{n}$,

where C^*, C^{**} *are positive constants, then*

$$\tilde{Q}^{(n)}(K\tau)\leqslant\tilde{\rho}^{(n)}(K\tau)e^{C^*K\tau}.$$

Remark 2. When $n\to\infty$, we obtain

$$\tilde{Q}^{(n)}(K\tau)\to 0.$$

If $\delta>\frac{1}{2}$, then we can take $m=2\delta$ in (23) and use Lemma 6 with $M_2=0$. So we get

Theorem 4. *If the following conditions hold*

(i) $\delta>\frac{1}{2}$,

(ii) $\|R^{(n)}(H)\|_{1,\infty}^2\leqslant C$, *and* $ns_2^2(K\tau)\leqslant C$,

then for all $\tilde{\rho}^{(n)}(K\tau)$ *and* K, *we have*

$$\tilde{Q}^{(n)}(K\tau)\leqslant\tilde{\rho}^{(n)}(K\tau)e^{C^*K\tau}.$$

Remark 3. If $\tilde{\rho}^{(n)}(K\tau)\to 0$, when $n\to\infty$, we have $\tilde{Q}^{(n)}(K\tau)\to 0$.

Finally we point out that if we use the spherical mean summation (see [5], [6])

$$\eta^{(n)}(x,t)=\sum_{|l|<n}\left(1-\frac{|l|^2}{n^2}\right)^b\eta_l^{(n)}e^{ilx},\quad b\geqslant 0,$$

then we can get a better result.

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