DIFFERENCE METHOD FOR MULTI-DIMENSIONAL NONLINEAR SCHRÖDINGER EQUATIONS WITH WAVE OPERATOR*

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In solving physical problems the multi-dimensional nonlinear Schrödinger equations with wave operator are often obtained. Clearly, their solution requires the use of the numerical method. In this paper, we consider a difference method for the equations and prove the convergence and stability of the difference solution on the basis of prior estimates.

We consider the following periodic initial-value problem

$$\boldsymbol{u}_{tt}-\sum_{k,s=1}^{M}\frac{\partial}{\partial x_k}A_{k,s}(x)\frac{\partial \boldsymbol{u}}{\partial x_s}+\sum_{s=1}^{M}C_s(x)\frac{\partial^2\boldsymbol{u}}{\partial x_s\partial t}+R(x)\boldsymbol{u}+P(x)\boldsymbol{u}_t$$
$$+\sum_{s=1}^{M}B_s(x)\frac{\partial \boldsymbol{u}}{\partial x_s}+d(x)q(|\boldsymbol{u}|^2)\boldsymbol{u}=\boldsymbol{f}(x,t), \tag{1}$$

$$\boldsymbol{u}|_{t=0}=\boldsymbol{u}_0(x), \qquad \boldsymbol{u}_t|_{t=0}=\boldsymbol{u}_1(x), \tag{2}$$

$$\boldsymbol{u}(x_1, x_2, \cdots, x_m+E_m, \cdots, x_M, t)=\boldsymbol{u}(x_1, x_2, \cdots, x_m, \cdots, x_M, t), \quad 1\le m\le M, \tag{3}$$

where the unknown vector $\boldsymbol{u}(x, t)$ is the L-dimensional vector of the complex-valued functions, $\boldsymbol{f}(x, t), \boldsymbol{u}_0(x)$ and $\boldsymbol{u}_1(x)$ are given L-dimensional vectors of the complex-valued functions, $C_s(x)$, $d(x)$ and $q(s)$ are given real functions,

$$A_{k,s}(x)=\begin{pmatrix} a^1_{k,s}(x) & & & \\ & a^2_{k,s}(x) & & \\ & & \ddots & \\ & & & a^L_{k,s}(x)\end{pmatrix}$$

is the diagonal matrix of the real-valued functions, $P(x)=(p_{l,k})_{L\times L}$, $R(x)=(r_{l,k})_{L\times L}$ and $B_s(x)=(b^{(s)}_{l,k})_{L\times L}$ are the matrixes of the complex-valued functions. The constants E_m denote periods. Because of the periodic property, the region of numerical computation is $\Omega=[0, E_1]\times[0, E_2)\times\cdots\times[0, E_m]$. In $[0, E_m]$ the step size is $h_m=\frac{E_m}{J_m}$ and the points of the net are $0, h_m, \cdots, E_m$. Let

$$\Omega_h\equiv\{x_{j_1,j_2,\cdots,j_M}|0\le j_m\le J_m-1\},$$

$$(f^n_{j_1,j_2,\cdots,j_M})_{x_m}=\frac{1}{h_m}(f^n_{j_1,j_2,\cdots,j_m+1,\cdots,j_M}-f^n_{j_1,j_2,\cdots,j_m,\cdots,j_M}),$$

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$$(f^n_{j_1,j_2,\cdots,j_M})_{x_m}=\frac{1}{h_m}(f^n_{j_1,j_2,\cdots,j_m,\cdots,j_M}-f^n_{j_1,j_2,\cdots,j_m-1,\cdots,j_M}),$$

$$(f^n_{j_1,j_2,\cdots,j_M})_{\hat{x}_m}=\frac{1}{2h_m}(f^n_{j_1,j_2,\cdots,j_m+1,\cdots,j_M}-f^n_{j_1,j_2,\cdots,j_m-1,\cdots,j_M}).$$

The difference symbols for t are similar. Let the inner product

$$(\boldsymbol{u}^n_{j_1,j_2,\cdots,j_M},\ \boldsymbol{v}^n_{j_1,j_2,\cdots,j_M})=h_1h_2\cdots h_M\sum_{l=1}^{L}\sum_{\Omega_h}u^n_{l,j_1,j_2,\cdots,j_M}\cdot\bar{v}^n_{l,j_1,j_2,\cdots,j_M}.$$

$\|f\|_{H^m}$ denotes the norm of the space $H^m(\Omega)$, $\|f\|_{L_\infty}=\operatorname{ess\,sup}_{x\in\Omega}|f(x)|$.

For the periodic initial-value problem (1)—(3), we use the following implicit scheme

$$(\boldsymbol{u}^{n+1}_{j_1,j_2,\cdots,j_M})_{t\bar{t}}-\sum_{k,s=1}^{M}(A_{k,s,j_1,j_2,\cdots,j_M}(\boldsymbol{u}^{n+1}_{j_1,j_2,\cdots,j_M})_{x_s})_{\bar{x}_k}+\sum_{s=1}^{M}C_{s,j_1,j_2,\cdots,j_M}(\boldsymbol{u}^{n+1}_{j_1,j_2,\cdots,j_M})_{\hat{x}_s\bar{t}}$$
$$+R_{j_1,j_2,\cdots,j_M}\boldsymbol{u}^{n+1}_{j_1,j_2,\cdots,j_M}+P_{j_1,j_2,\cdots,j_M}(\boldsymbol{u}^{n+1}_{j_1,j_2,\cdots,j_M})_{\bar{t}}+\sum_{s=1}^{M}B_{s,j_1,j_2,\cdots,j_M}(\boldsymbol{u}^{n+1}_{j_1,j_2,\cdots,j_M})_{\hat{x}_s}$$
$$+d_{j_1,j_2,\cdots,j_M}q(|\boldsymbol{u}^{n+1}_{j_1,j_2,\cdots,j_M}|^2)\boldsymbol{u}^{n+1}_{j_1,j_2,\cdots,j_M}=\boldsymbol{f}^{n+1}_{j_1,j_2,\cdots,j_M},\quad (j_1,j_2,\cdots,j_M)\in\Omega_h,\tag{4}$$

$$\boldsymbol{u}^0_{j_1,j_2,\cdots,j_M}=\boldsymbol{u}_0(j_1h_1,j_2h_2,\cdots,j_Mh_M),\quad (\boldsymbol{u}^0_{j_1,j_2,\cdots,j_M})_{t}=\boldsymbol{u}_1(j_1h_1,j_2h_2,\cdots,j_Mh_M),\tag{5}$$

$$\boldsymbol{u}(x_1,x_2,\cdots,x_m+E_m,\cdots,x_M,t)=\boldsymbol{u}(x_1,x_2,\cdots,x_m,\cdots,x_M,t),\quad 1\leqslant m\leqslant M,\tag{6}$$

where $A_{k,s,j_1,j_2,\cdots,j_M}\equiv A_{k,s}\Big(j_1h_1,\ j_2h_2,\ \cdots,\ \Big(j_k+\frac{1}{2}\Big)h_k,\ \cdots,\ j_Mh_M\Big)$.

Lemma 1. *Assume that*

(i) $$a^l_{k,s}(x)=a^l_{s,k}(x),\quad \sum_{k,s=1}^{M}a^l_{k,s}(x)\xi_k\bar{\xi}_s\geqslant\gamma\sum_{k=1}^{M}|\xi_k|^2,$$

for $k,\ s=1,2,\cdots,M$, $1\leqslant l\leqslant L$, *where* γ *is a positive constant;*

(ii) $$d(x)\geqslant0,\ Q(s)\geqslant0,\ q'(s)\geqslant0,$$

for $s\in[0,\infty)$, $\int_\Omega d(x)Q(|\boldsymbol{u}_0|^2)\,dx<\infty$, *where* $Q_s=\int_0^s q(z)dz$;

(iii) $$|a^l_{k,s}(x)|\leqslant K_A,\ |b^{(s)}_{l,k}(x)|\leqslant K_B,\ |c_s(x)|\leqslant K_C,\ \left|\frac{\partial c_s(x)}{\partial x_\beta}\right|\leqslant K_C$$
$$|p_{l,k}(x)|\leqslant K_P,\ |r_{l,k}(x)|\leqslant K_R,\quad 1\leqslant s,\ \beta\leqslant M;$$

(iv) $$\boldsymbol{f}(x,t)\in C^0,\ \boldsymbol{u}_0(x)\in C^1,\ \boldsymbol{u}_1(x)\in C^0.$$

Then we have the estimates

$$\|\boldsymbol{u}^n_{j_1,j_2,\cdots,j_M}\|_{L_2}\leqslant C_1,\quad \|(\boldsymbol{u}^n_{j_1,j_2,\cdots,j_M})_{\bar{t}}\|_{L_2}\leqslant C_1,\quad \|(\boldsymbol{u}^n_{j_1,j_2,\cdots,j_M})_{x_k}\|_{L_2}\leqslant C_1,\qquad 1\leqslant k\leqslant M,$$

where $0\leqslant n\cdot\Delta t\leqslant T$, *the constant* C_1 *is independent of* Δt *and* h_m.

Proof. Multiplying (4) with $\overline{(\boldsymbol{u}^{n+1}_{j_1,j_2,\cdots,j_M})_{\bar{t}}}$ and taking the inner product, we obtain

$$((\boldsymbol{u}^{n+1}_{j_1,j_2,\cdots,j_M})_{t\bar{t}},\ (\boldsymbol{u}^{n+1}_{j_1,j_2,\cdots,j_M})_{\bar{t}})-\Big(\sum_{k,s=1}^{M}(A_{k,s,j_1,j_2,\cdots,j_M}(\boldsymbol{u}^{n+1}_{j_1,j_2,\cdots,j_M})_{x_s})_{\bar{x}_k},\ (\boldsymbol{u}^{n+1}_{j_1,j_2,\cdots,j_M})_{\bar{t}}\Big)$$
$$+\Big(\sum_{s=1}^{M}C_{s,j_1,j_2,\cdots,j_M}(\boldsymbol{u}^{n+1}_{j_1,j_2,\cdots,j_M})_{\hat{x}_s\bar{t}},\ (\boldsymbol{u}^{n+1}_{j_1,j_2,\cdots,j_M})_{\bar{t}}\Big)+(\boldsymbol{R}_{j_1,j_2,\cdots,j_M}\boldsymbol{u}^{n+1}_{j_1,j_2,\cdots,j_M},\ (\boldsymbol{u}^{n+1}_{j_1,j_2,\cdots,j_M})_{\bar{t}})$$
$$+(P_{j_1,j_2,\cdots,j_M}(\boldsymbol{u}^{n+1}_{j_1,j_2,\cdots,j_M})_{\bar{t}},\ (\boldsymbol{u}^{n+1}_{j_1,j_2,\cdots,j_M})_{\bar{t}})+\Big(\sum_{s=1}^{M}B_{s,j_1,j_2,\cdots,j_M}(\boldsymbol{u}^{n+1}_{j_1,j_2,\cdots,j_M})_{\hat{x}_s},\ (\boldsymbol{u}^{n+1}_{j_1,j_2,\cdots,j_M})_{\bar{t}}\Big)$$
$$+(d_{j_1,j_2,\cdots,j_M}\cdot q(|\boldsymbol{u}^{n+1}_{j_1,j_2,\cdots,j_M}|^2)\,\boldsymbol{u}^{n+1}_{j_1,j_2,\cdots,j_M},\ (\boldsymbol{u}^{n+1}_{j_1,j_2,\cdots,j_M})_{\bar{t}})=(\boldsymbol{f}^{n+1}_{j_1,j_2,\cdots,j_M},\ (\boldsymbol{u}^{n+1}_{j_1,j_2,\cdots,j_M})_{\bar{t}}).\tag{7}$$

We deduce the terms of (7) as follows:

$$\mathrm{Re}((u^{n+1}_{j_1,j_2,\cdots,j_M})_{\bar{t}\bar{t}},\ (u^{n+1}_{j_1,j_2,\cdots,j_M})_{\bar{t}})=\frac{1}{2}(\|(u^{n+1}_{j_1,j_2,\cdots,j_M})_{\bar{t}}\|^2_{L_2})_{\bar{t}}+\frac{1}{2}\Delta t\|(u^{n+1}_{j_1,j_2,\cdots,j_M})_{\bar{t}\bar{t}}\|^2_{L_2},$$

$$\begin{aligned}&(A_{k,s,j_1,j_2,\cdots,j_M}(u^{n+1}_{j_1,j_2,\cdots,j_M})_{x_s})_{\bar{x}_k}\overline{(u^{n+1}_{j_1,j_2,\cdots,j_M})_{\bar{t}}}\\&=(A_{k,s,j_1,j_2,\cdots,j_M}(u^{n+1}_{j_1,j_2,\cdots,j_M})_{x_s}\overline{(u^{n+1}_{j_1,j_2,\cdots,j_k+1,\cdots,j_M})_{\bar{t}}})_{\bar{x}_k}\\&\quad-A_{k,s,j_1,j_2,\cdots,j_M}(u^{n+1}_{j_1,j_2,\cdots,j_M})_{x_s}\cdot\overline{(u^{n+1}_{j_1,j_2,\cdots,j_M})_{\bar{t}x_k}}.\end{aligned}$$

In view of the periodic condition (6), we have

$$\begin{aligned}&\mathrm{Re}\Big(\sum_{k,s=1}^{M}(A_{k,s,j_1,j_2,\cdots,j_M}(u^{n+1}_{j_1,j_2,\cdots,j_M})_{x_s})_{\bar{x}_k},\ (u^{n+1}_{j_1,j_2,\cdots,j_M})_{\bar{t}}\Big)\\&=-\mathrm{Re}\Big[\sum_{k,s=1}^{M}(A_{k,s,j_1,j_2,\cdots,j_M}(u^{n+1}_{j_1,j_2,\cdots,j_M})_{x_s},\ (u^{n+1}_{j_1,j_2,\cdots,j_M})_{\bar{t}x_k})\Big]\\&=-\frac{1}{2}\sum_{k,s=1}^{M}(A_{k,s,j_1,j_2,\cdots,j_M}(u^{n+1}_{j_1,j_2,\cdots,j_M})_{x_s},\ (u^{n+1}_{j_1,j_2,\cdots,j_M})_{x_k})_{\bar{t}}\\&\quad-\frac{1}{2}\Delta t\sum_{k,s=1}^{M}(A_{k,s,j_1,j_2,\cdots,j_M}(u^{n+1}_{j_1,j_2,\cdots,j_M})_{x_s\bar{t}},\ (u^{n+1}_{j_1,j_2,\cdots,j_M})_{x_k\bar{t}}),\end{aligned}$$

$$\begin{aligned}&\mathrm{Re}(d_{j_1,j_2,\cdots,j_M}q(|u^{n+1}_{j_1,j_2,\cdots,j_M}|^2)u^{n+1}_{j_1,j_2,\cdots,j_M},\ (u^{n+1}_{j_1,j_2,\cdots,j_M})_{\bar{t}})\\&=\frac{1}{2}h_1h_2\cdots h_M\sum_{\Omega_h}d_{j_1,j_2,\cdots,j_M}q(|u^{n+1}_{j_1,j_2,\cdots,j_M}|^2)[(|u^{n+1}_{j_1,j_2,\cdots,j_M}|^2)_{\bar{t}}\\&\quad+\Delta t|(u^{n+1}_{j_1,j_2,\cdots,j_M})_{\bar{t}}|^2].\end{aligned}$$

Making use of Taylor's expansion, we obtain

$$[Q(|u^{n+1}_{j_1,j_2,\cdots,j_M}|^2)]_{\bar{t}}\leqslant q(|u^{n+1}_{j_1,j_2,\cdots,j_M}|^2)(|u^{n+1}_{j_1,j_2,\cdots,j_M}|^2)_{\bar{t}},$$

$$\begin{aligned}&\mathrm{Re}\Big(\sum_{s=1}^{M}C_{s,j_1,j_2,\cdots,j_M}(u^{n+1}_{j_1,j_2,\cdots,j_M})_{\hat{x}_s\bar{t}},\ (u^{n+1}_{j_1,j_2,\cdots,j_M})_{\bar{t}}\Big)\\&=-\frac{1}{4}\Big(\sum_{s=1}^{M}((C_{s,j_1,j_2,\cdots,j_s+1,\cdots,j_M})_{\bar{x}_s}(u^{n+1}_{j_1,j_2,\cdots,j_s+1,\cdots,j_M})_{\bar{t}}\\&\quad+(C_{s,j_1,j_2,\cdots,j_M})_{\bar{x}_s}(u^{n+1}_{j_1,j_2,\cdots,j_s-1,\cdots,j_M})_{\bar{t}}),\ (u^{n+1}_{j_1,j_2,\cdots,j_M})_{\bar{t}}\Big).\end{aligned}$$

It is clear that

$$\|u(^{n}_{j_1,j_2,\cdots,j_M})_{x_s}\|_{L_2}=\|(u^{n}_{j_1,j_2,\cdots,j_M})_{\bar{x}_s}\|_{L_2}\geqslant\|(u^{n}_{j_1,j_2,\cdots,j_M})_{\hat{x}_s}\|_{L_2}.$$

Taking the real parts of (7) and using the preceding deduction, we have

$$\begin{aligned}&[\|(u^{n+1}_{j_1,j_2,\cdots,j_M})_{\bar{t}}\|^2_{L_2}]_{\bar{t}}+\Big[\sum_{k,s=1}^{M}(A_{k,s,j_1,j_2,\cdots,j_M}(u^{n+1}_{j_1,j_2,\cdots,j_M})_{x_s},\ (u^{n+1}_{j_1,j_2,\cdots,j_M})_{x_k})\Big]_{\bar{t}}\\&\quad+h_1h_2\cdots h_M\sum_{\Omega_h}d_{j_1,j_2,\cdots,j_M}[Q(|u^{n+1}_{j_1,j_2,\cdots,j_M}|^2)]_{\bar{t}}\\&\leqslant K_1\Big[\|(u^{n+1}_{j_1,j_2,\cdots,j_M})_{\bar{t}}\|^2_{L_2}+\sum_{s=1}^{M}\|(u^{n+1}_{j_1,j_2,\cdots,j_M})_{x_s}\|^2_{L_2}\\&\quad+\|u^{n+1}_{j_1,j_2,\cdots,j_M}\|^2+\|f^{n+1}_{j_1,j_2,\cdots,j_M}\|^2_{L_2}\Big],\end{aligned}$$

where the constant K_1 is positive. Summing up the last formula for n from 0 to $N-1$ and using the conditions of uniform positive definiteness, we obtain

$$\|(u^N_{j_1,j_2,\cdots,j_M})_{\bar{t}}\|^2_{L_2}+\gamma\sum_{s=1}^{M}\|(u^N_{j_1,j_2,\cdots,j_M})_{x_s}\|^2_{L_2}$$
$$\leqslant\|(u^0_{j_1,j_2,\cdots,j_M})_{\bar{t}}\|^2_{L_2}+\left|\sum_{k,s=1}^{M}(A_{k,s,j_1,j_2,\cdots,j_M}(u^0_{j_1,j_2,\cdots,j_M})_{x_s},\ (u^0_{j_1,j_2,\cdots,j_M})_{x_k})\right|$$
$$+h_1h_2\cdots h_M\sum_{\Omega_h}d_{j_1,j_2,\cdots,j_M}Q(|u^0_{j_1,j_2,\cdots,j_M}|^2)+K_1\Delta t\sum_{n=0}^{N-1}[\|(u^{n+1}_{j_1,j_2,\cdots,j_M})_{\bar{t}}\|^2_{L_2}$$
$$+\sum_{s=1}^{M}\|(u^{n+1}_{j_1,j_2,\cdots,j_M})_{x_s}\|^2_{L_2}+\|u^{n+1}_{j_1,j_2,\cdots,j_M}\|^2_{L_2}+\|f^{n+1}_{j_1,j_2,\cdots,j_M}\|^2_{L_2}], \tag{8}$$

where

$$\left|\sum_{k,s=1}^{M}(A_{k,s,j_1,j_2,\cdots,j_M}(u^0_{j_1,j_2,\cdots,j_M})_{x_s},\ (u^0_{j_1,j_2,\cdots,j_M})_{x_k})\right|\leqslant K_AM^2\sum_{s=1}^{M}\|(u^0_{j_1,j_2,\cdots,j_M})_{x_s}\|^2_{L_2}.$$

Without loss of generality, we can assume that Δt and h_m are so small that there exist

$$\|(u^0_{j_1,j_2,\cdots,j_M})_{\bar{t}}\|^2_{L_2}\leqslant 2\int_\Omega[u_1(x)]^2dx,\quad \|u^0_{j_1,j_2,\cdots,j_M}\|^2_{L_2}\leqslant 2\int_\Omega[u_0(x)]^2dx,$$
$$\|(u^0_{j_1,j_2,\cdots,j_M})_{x_s}\|^2_{L_2}\leqslant 2\int_\Omega\left(\frac{\partial u_0(x)}{\partial x_s}\right)^2dx,\quad 1\leqslant s\leqslant M,$$
$$h_1h_2\cdots h_M\sum_{\Omega_h}d_{j_1,j_2,\cdots,j_M}Q(|u^0_{j_1,j_2,\cdots,j_M}|^2)\leqslant 2\int_\Omega d(x)Q(|u_0|^2)dx,$$
$$\Delta t\sum_{k=0}^{N-1}\|f^{n+1}_{j_1,j_2,\cdots,j_M}\|^2_{L_2}\leqslant 2\int_0^T\int_\Omega|f(x,\ t)|^2dx\,dt.$$

Thus, the conclusions of the lemma are obtained from Gronwall's inequality of discrete operator and Sobolev's lemma.

Lemma 2. *Suppose the conditions of Lemma* 1 *are satisfied. Assume that*

(i) $\left|\frac{\partial a_{k,s}(x)}{\partial x_\beta}\right|\leqslant K_A,\quad \left|\frac{\partial^2 a_{k,s}(x)}{\partial x_\beta\cdot\partial x_j}\right|\leqslant K_A,\quad \left|\frac{\partial b^{(s)}_{l,k}(x)}{\partial x_\beta}\right|\leqslant K_B,\ |d(x)|\leqslant K_D,$

$\left|\frac{\partial d(x)}{\partial x_\beta}\right|\leqslant K_D,\quad \left|\frac{\partial p_{l,k}(x)}{\partial x_\beta}\right|\leqslant K_P,\quad \left|\frac{\partial r_{l,k}(x)}{\partial x_\beta}\right|\leqslant K_R,\quad 1\leqslant\beta,\ j\leqslant M;$

(ii) $M\leqslant 3$;

(iii) $|q'(s)|\leqslant K_q,\ |q(s)|\leqslant K_q\cdot s,\ s\in[0,\ \infty)$;

(iv) $u_0(x)\in H^2,\ u_1(x)\in H^1,\ \int_0^T\int_\Omega\left|\frac{\partial f(x,\ t)}{\partial x_\beta}\right|^2dx\,dt<\infty.$

Then there are estimates

$$\|u^{n+1}_{j_1,j_2,\cdots,j_M}\|_{H^1}\leqslant C_2,\quad \|u^{n+1}_{j_1,j_2,\cdots,j_M}\|_{L_\infty}\leqslant C_2,\quad \|(u^{n+1}_{j_1,j_2,\cdots,j_M})_{tx_\beta}\|_{L_2}\leqslant C_2,$$
$$1\leqslant\beta\leqslant M,$$

where the constant C_2 is independent of Δt and h_m.

Proof. Let

$$\Phi^{n+1,\beta}_{j_1,j_2,\cdots,j_M}\equiv(u^{n+1}_{j_1,j_2,\cdots,j_M})_{x_\beta}.$$

Taking forward difference of the scheme (4) for x_β, we obtain

$$(\Phi^{n+1,\beta}_{j_1,j_2,\cdots,j_M})_{t\bar{t}}-\sum_{k,s=1}^{M}(A_{k,s,j_1,j_2,\cdots,j_M}(u^{n+1}_{j_1,j_2,\cdots,j_M})_{x_s})_{\bar{x}_kx_\beta}+\sum_{s=1}^{M}C_{s,j_1,j_2,\cdots,j_\beta+1,\cdots,j_M}$$
$$\cdot(\Phi^{n+1,\beta}_{j_1,j_2,\cdots,j_M})_{\hat{x}_s\bar{t}}+\sum_{s=1}^{M}(C_{s,j_1,j_2,\cdots,j_M})_{x_\beta}(u^{n+1}_{j_1,j_2,\cdots,j_M})_{\hat{x}_s\bar{t}}+R_{j_1,j_2,\cdots,j_M}\Phi^{n+1,\beta}_{j_1,j_2,\cdots,j_M}$$
$$+(R_{j_1,j_2,\cdots,j_M})_{x_\beta}\cdot u^{n+1}_{j_1,j_2,\cdots,j_\beta+1,\cdots,j_M}+P_{j_1,j_2,\cdots,j_\beta+1,\cdots,j_M}(\Phi^{n+1,\beta}_{j_1,j_2,\cdots,j_M})_{\bar{t}}$$

$$+(P_{j_1,j_2,\cdots,j_M})_{x_\beta}(u^{n+1}_{j_1,j_2,\cdots,j_M})_{\bar t}+\sum_{s=1}^{M}B_{s,j_1,j_2,\cdots,j_s+1,\cdots,j_M}(\Phi^{n+1,\beta}_{j_1,j_2,\cdots,j_M})_{\hat x_s}$$

$$+\sum_{s=1}^{M}(B_{s,j_1,j_2,\cdots,j_M})_{x_\beta}(u^{n+1}_{j_1,j_2,\cdots,j_M})_{\hat x_s}+(d_{j_1,j_2,\cdots,j_M}q(|u^{n+1}_{j_1,j_2,\cdots,j_M}|^2)u^{n+1}_{j_1,j_2,\cdots,j_M})_{x_\beta}$$

$$=(f^{n+1}_{j_1,j_2,\cdots,j_M})_{x_\beta}.$$

Multiplying the above formula with $\overline{(\Phi^{n+1,\beta}_{j_1,j_2,\cdots,j_M})_{\bar t}}$ and taking the inner product, we obtain

$$[\|(\Phi^{n+1,\beta}_{j_1,j_2,\cdots,j_M})_{\bar t}\|^2_{L_2}]_{\bar t}-2\,\mathrm{Re}\Big(\sum_{k,s=1}^{M}(A_{k,s,j_1,j_2,\cdots,j_M}(u^{n+1}_{j_1,j_2,\cdots,j_M})_{x_s})_{\bar x_k x_\beta},\ (\Phi^{n+1,\beta}_{j_1,j_2,\cdots,j_M})_{\bar t}\Big)$$

$$+2\mathrm{Re}((d_{j_1,j_2,\cdots,j_M}q(|u^{n+1}_{j_1,j_2,\cdots,j_M}|^2)u^{n+1}_{j_1,j_2,\cdots,j_M})_{x_\beta},\ (\Phi^{n+1,\beta}_{j_1,j_2,\cdots,j_M})_{\bar t})$$

$$\leqslant K_2\Big[\|(\Phi^{n+1,\beta}_{j_1,j_2,\cdots,j_M})_{\bar t}\|^2_{L_2}+\sum_{s=1}^{M}\|(\Phi^{n+1,s}_{j_1,j_2,\cdots,j_M})_{\bar t}\|^2_{L_2}$$

$$+\sum_{s=1}^{M}\|(\Phi^{n+1,\beta}_{j_1,j_2,\cdots,j_M}\|^2_{L_2}+\|\Phi^{n+1,\beta}_{j_1,j_2,\cdots,j_M}\|^2_{L_2}+\|(u^{n+1}_{j_1,j_2,\cdots,j_M})_{\bar t}\|^2_{L_2}$$

$$+\sum_{s=1}^{M}\|(u^{n+1}_{j_1,j_2,\cdots,j_M})_{x_s}\|^2_{L_2}+\|u^{n+1}_{j_1,j_2,\cdots,j_M}\|^2_{L_2}+\|(f^{n+1}_{j_1,j_2,\cdots,j_M})_{x_\beta}\|^2_{L_2}\Big],\qquad(9)$$

where K_2 is a positive constant. We deduce the terms of (9) as follows:

$$(A_{k,s,j_1,j_2,\cdots,j_M}(u^{n+1}_{j_1,j_2,\cdots,j_M})_{x_s})_{\bar x_k x_\beta}$$

$$=(A_{k,s,j_1,j_2,\cdots,j_M}(\Phi^{n+1,\beta}_{j_1,j_2,\cdots,j_M})_{x_s})_{\bar x_k}+(A_{k,s,j_1,j_2,\cdots,j_M})_{x_\beta\bar x_k}(u^{n+1}_{j_1,j_2,\cdots,j_\beta+1,\cdots,j_M})_{x_s}$$

$$+(A_{k,s,j_1,j_2,\cdots,j_k-1,\cdots,j_M})_{x_\beta}\cdot(u^{n+1}_{j_1,j_2,\cdots,j_\beta+1,\cdots,j_M})_{x_s\bar x_k},$$

$$(d_{j_1,j_2,\cdots,j_M}\cdot q(|u^{n+1}_{j_1,j_2,\cdots,j_M}|^2)u^{n+1}_{j_1,j_2,\cdots,j_M})_{x_\beta}$$

$$=(d_{j_1,j_2,\cdots,j_M})_{x_\beta}q(|u^{n+1}_{j_1,j_2,\cdots,j_M}|^2)u^{n+1}_{j_1,j_2,\cdots,j_M}+d_{j_1,j_2,\cdots,j_\beta+1,\cdots,j_M}\cdot q(|u^{n+1}_{j_1,j_2,\cdots,j_M}|^2)$$

$$\cdot(\Phi^{n+1,\beta}_{j_1,j_2,\cdots,j_M})+d_{j_1,j_2,\cdots,j_\beta+1,\cdots,j_M}u^{n+1}_{j_1,j_2,\cdots,j_\beta+1,\cdots,j_M}(q(|u^{n+1}_{j_1,j_2,\cdots,j_M}|^2))_{x_\beta},$$

$$|(q(|u^{n+1}_{j_1,j_2,\cdots,j_M}|^2))_{x_\beta}|\leqslant K_q\cdot(|u^{n+1}_{j_1,j_2,\cdots,j_\beta+1,\cdots,j_M}|+|u^{n+1}_{j_1,j_2,\cdots,j_M}|)\,|\Phi^{n+1,\beta}_{j_1,j_2,\cdots,j_M}|.$$

Using Schwartz's and Hölder's inequalities, we have

$$|2\cdot\mathrm{Re}((d_{j_1,j_2,\cdots,j_M}q(|u^{n+1}_{j_1,j_2,\cdots,j_M}|^2)u^{n+1}_{j_1,j_2,\cdots,j_M})_{x_\beta},\ (\Phi^{n+1,\beta}_{j_1,j_2,\cdots,j_M})_{\bar t})|$$

$$\leqslant K_D\cdot K_q(\|u^{n+1}_{j_1,j_2,\cdots,j_M}\|^6_{L_6}+\|(\Phi^{n+1,\beta}_{j_1,j_2,\cdots,j_M})_{\bar t}\|^2_{L_2})$$

$$+K_D\cdot K_q\Big(\frac{3}{2}\|\,|u^{n+1}_{j_1,j_2,\cdots,j_M}|^2\Phi^{n+1,\beta}_{j_1,j_2,\cdots,j_M}\|^2_{L_2}$$

$$+\frac{3}{2}\|\,|u^{n+1}_{j_1,j_2,\cdots,j_\beta+1,\cdots,j_M}\|^2\cdot\Phi^{n+1,\beta}_{j_1,j_2,\cdots,j_M}\|^2_{L_2}+3\|(\Phi^{n+1,\beta}_{j_1,j_2,\cdots,j_M})_{\bar t}\|^2_{L_2}\Big)$$

$$\leqslant K_D\cdot K_q(4\|(\Phi^{n+1,\beta}_{j_1,j_2,\cdots,j_M})_{\bar t}\|^2_{L_2}+\|u^{n+1}_{j_1,j_2,\cdots,j_M}\|^6_{L_6}+3\|u^{n+1}_{j_1,j_2,\cdots,j_M}\|^4_{L_6}\|\cdot\Phi^{n+1,\beta}_{j_1,j_2,\cdots,j_M}\|^2_{L_6}).$$

In view of Sobolev's inequality (see [3])

$$\|u^{n+1}_{j_1,j_2,\cdots,j_M}\|^2_{L_6}\leqslant\mathrm{const}\Big(\sum_{k=1}^{M}\|(u^{n+1}_{j_1,j_2,\cdots,j_M})_{x_k}\|^2_{L_2}+\|u^{n+1}_{j_1,j_2,\cdots,j_M}\|^2_{L_2}\Big)$$

and Lemma 1, we obtain

$$|2\cdot\mathrm{Re}((d_{j_1,j_2,\cdots,j_M}q(|u^{n+1}_{j_1,j_2,\cdots,j_M}|^2)u^{n+1}_{j_1,j_2,\cdots,j_M})_{x_\beta},\ (\Phi^{n+1,\beta}_{j_1,j_2,\cdots,j_M})_{\bar t})|$$

$$\leqslant K_3\Big(\|(\Phi^{n+1,\beta}_{j_1,j_2,\cdots,j_M})_{\bar t}\|^2_{L_2}+(M+1)^3C_1^6+(M+1)^2C_1^4\sum_{k=1}^{M}\|(\Phi^{n+1,\beta}_{j_1,j_2,\cdots,j_M})_{x_k}\|^2_{L_2}\Big),$$

where K_3 is a positive constant.

Summing (9) for n from 0 to $N-1$ and using the preceding deduction, we have

$$\|(\Phi^{N,\beta}_{j_1,j_2,\cdots,j_M})_{\bar{t}}\|^2_{L_2}+\gamma\sum_{s=1}^{M}\|(\Phi^{N,\beta}_{j_1,j_2,\cdots,j_M})_{x_s}\|^2_{L_2}$$

$$\leqslant\|(\Phi^{0,\beta}_{j_1,j_2,\cdots,j_M})_{\bar{t}}\|^2_{L_2}+\left|\sum_{k,s=1}^{M}(A_{k,s,j_1,j_2,\cdots,j_M}(\Phi^{0,\beta}_{j_1,j_2,\cdots,j_M})_{x_s},\ (\Phi^{0,\beta}_{j_1,j_2,\cdots,j_M})_{x_k})\right|$$

$$+\Delta t\sum_{n=0}^{N-1}\Big\{K_4M^2(C_1^2+\|(\Phi^{n+1,\beta}_{j_1,j_2,\cdots,j_M})_{\bar{t}}\|^2_{L_2})+K_4\Big(\sum_{k,s=1}^{M}\|(\Phi^{n+1,s}_{j_1,j_2,\cdots,j_M})_{\bar{x}_k}\|^2_{L_2}$$

$$+M^2\|(\Phi^{n+1,\beta}_{j_1,j_2,\cdots,j_M})_{\bar{t}}\|^2_{L_2}\Big)+K_3\Big(\|(\Phi^{n+1,\beta}_{j_1,j_2,\cdots,j_M})_{\bar{t}}\|^2_{L_2}+(M+1)^3C_1^6$$

$$+(M+1)^2C_1^4\sum_{k=1}^{M}\|(\Phi^{n+1,\beta}_{j_1,j_2,\cdots,j_M})_{x_k}\|^2_{L_2}\Big)+K_2\Big(\|(\Phi^{n+1,\beta}_{j_1,j_2,\cdots,j_M})_{\bar{t}}\|^2_{L_2}$$

$$+\sum_{s=1}^{M}\|(\Phi^{n+1,s}_{j_1,j_2,\cdots,j_M})_{\bar{t}}\|^2_{L_2}+\sum_{s=1}^{M}\|(\Phi^{n+1,\beta}_{j_1,j_2,\cdots,j_M})_{x_s}\|^2_{L_2}+(3+M)C_1^2$$

$$+\|(f^{n+1}_{j_1,j_2,\cdots,j_M})_{x_\beta}\|^2_{L_2}\Big)\Big\}.$$

Thus, summing the last formula for β from 1 to M and choosing Δt and h_m properly, we obtain

$$\|u^N_{j_1,j_2,\cdots,j_M}\|^2_{H^1}+\sum_{\beta=1}^{M}\|(u^N_{j_1,j_2,\cdots,j_M})_{\bar{t}x_\beta}\|^2_{L_2}$$

$$\leqslant K_4+K_5\Delta t\sum_{n=0}^{N-1}\Big(\|u^{n+1}_{j_1,j_2,\cdots,j_M}\|^2_{H^1}+\sum_{\beta=1}^{M}\|(u^{n+1}_{j_1,j_2,\cdots,j_M})_{\bar{t}x_\beta}\|^2\Big),$$

where K_4 and K_5 are positive constants. Finally, in view of Gronwall's inequality and Sobolev's imbeding theorem, we obtain the conclusions of Lemma 2.

Theorem 1. *Suppose the conditions of Lemma 2 are satisfied. Assume that the solution $u(x, t)$ of the problem (1)—(3) possesses the bounded partial derivatives of fourth order for x and of second order for t, $|u(x, t)|\leqslant C_0$. Then the solution of the difference scheme (4)—(6) converges to the solution of the problem (1)—(3) with order $O(\Delta t+h^2)$ by square norm.*

Proof. Let

$$\varepsilon^n_{j_1,j_2,\cdots,j_M}=u(j_1h_1,\ j_2h_2,\ \cdots,\ j_Mh_M,\ n\Delta t)-u^n_{j_1,j_2,\cdots,j_M}.$$

We have the error equations

$$(\varepsilon^{n+1}_{j_1,j_2,\cdots,j_M})_{\bar{t}\bar{t}}-\sum_{k,s=1}^{M}(A_{k,s,j_1,j_2,\cdots,j_M}(\varepsilon^{n+1}_{j_1,j_2,\cdots,j_M})_{x_s})_{\bar{x}_k}+\sum_{s=1}^{M}C_{s,j_1,j_2,\cdots,j_M}(\varepsilon^{n+1}_{j_1,j_2,\cdots,j_M})_{\hat{x}_s\bar{t}}$$

$$+R_{j_1,j_2,\cdots,j_M}\varepsilon^{n+1}_{j_1,j_2,\cdots,j_M}+P_{j_1,j_2,\cdots,j_M}(\varepsilon^{n+1}_{j_1,j_2,\cdots,j_M})_{\bar{t}}+\sum_{s=1}^{M}B_{s,j_1,j_2,\cdots,j_M}(\varepsilon^{n+1}_{j_1,j_2,\cdots,j_M})_{\hat{x}_s}$$

$$+d_{j_1,j_2,\cdots,j_M}[q(|u(j_1h_1,\ j_2h_2,\ \cdots,\ j_Mh_M,\ (n+1)\Delta t)|^2)$$

$$\cdot u(j_1h_1,\ j_2h_2,\ \cdots,\ j_Mh_M,\ (n+1)\Delta t)$$

$$-q(|u^{n+1}_{j_1,j_2,\cdots,j_M}|^2)u^{n+1}_{j_1,j_2,\cdots,j_M}]=R^{n+1}_{j_1,j_2,\cdots,j_M}, \tag{10}$$

$$\varepsilon^0_{j_1,j_2,\cdots,j_M}=0,\ (\varepsilon^0_{j_1,j_2,\cdots,j_M})_{\bar{t}}=R^0_{j_1,j_2,\cdots,j_M}, \tag{11}$$

$$\varepsilon(x_1,\ x_2,\ \cdots,\ x_m+E_m,\ \cdots,\ x_M,\ t)=\varepsilon(x_1,\ x_2,\ \cdots,\ x_m,\ \cdots,\ x_M,\ t), \tag{12}$$

where $\quad |R^n_{j_1,j_2,\cdots,j_M}|\leqslant K_6(\Delta t+h^2),\ h\equiv\max_{1\leqslant m\leqslant M}|h_m|,\quad n=0,\ 1,\ \cdots$

and K_6 is a positive constant. We have

$$|q(|u(j_1h_1, j_2h_2, \cdots, j_Mh_M, (n+1)\Delta t)|^2)u(j_1h_1, j_2h_2, \cdots, j_Mh_M, (n+1)\Delta t) - q(|u^{n+1}_{j_1,j_2,\cdots,j_M}|^2)u^{n+1}_{j_1,j_2,\cdots,j_M}|$$
$$\leqslant K_q[|u^{n+1}_{j_1,j_2,\cdots,j_M}|^2\cdot|\varepsilon^{n+1}_{j_1,j_2,\cdots,j_M}| + |u(j_1h_1, j_2h_2, \cdots, j_Mh_M, (n+1)\Delta t)| \cdot(|u(j_1h_1, j_2h_2, \cdots, j_Mh_M, (n+1)\Delta t)| + |u^{n+1}_{j_1,j_2,\cdots,j_M}|)\cdot|\varepsilon^{n+1}_{j_1,j_2,\cdots,j_M}|].$$

Multiplying (10) with $\overline{(\varepsilon^{n+1}_{j_1,j_2,\cdots,j_M})_{\bar{t}}}$, taking the inner product and using the estimate of L_∞ norm, we obtain

$$\|(\varepsilon^N_{j_1,j_2,\cdots,j_M})_{\bar{t}}\|^2_{L_2} + \gamma\sum_{s=1}^{M}\|(\varepsilon^N_{j_1,j_2,\cdots,j_M})_{x_s}\|^2_{L_2}$$
$$\leqslant K_7\cdot(\Delta t + h^2) + K_8\cdot\Delta t\sum_{n=0}^{N-1}\Big[\|(\varepsilon^{n+1}_{j_1,j_2,\cdots,j_M})_{\bar{t}}\|^2_{L_2} + \sum_{s=1}^{M}\|\varepsilon^n_{j_1,j_2,\cdots,j_M})_{x_s}\|^2_{L_2} + \|(\varepsilon^n_{j_1,j_2,\cdots,j_M}\|^2_{L_2}\Big],$$

where K_7 and K_8 are positive constants. From Gronwall's inequality we obtain

$$\|\varepsilon^N_{j_1,j_2,\cdots,j_M}\|^2_{L_2} \leqslant \text{const}\cdot(\Delta t + h^2)^2,$$

i.e. the difference solution is convergent by L_2 norm.

Theorem 2. *Suppose the conditions of Lemma 2 are satisfied. Then the solution of the difference scheme* (4)—(6) *is stable for the initial value by L_2 norm.*

Proof. Suppose that there are the solutions of the difference equations $u^n_{j_1,j_2,\cdots,j_M}$ and $v^n_{j_1,j_2,\cdots,j_M}$, which satisfy the difference equations (4) and the periodic condition (6). But, their initial conditions are different. Let

$$\varepsilon^n_{j_1,j_2,\cdots,j_M} = u^n_{j_1,j_2,\cdots,j_M} - v^n_{j_1,j_2,\cdots,j_M}.$$

Similar to the proof of Theorem 1, we can establish equations and initial conditions satisfied by $\varepsilon^n_{j_1,j_2,\cdots,j_M}$ and prove the stability.

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