ON THE EXISTENCE OF FUNCTIONS WITH PRESCRIBED BEST L_1 APPROXIMATIONS*

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Abstract

This paper gives a partial answer to a problem of Rivlin[1] in L_1 approximation.

1. Introduction

In this paper we prove the following ($X\equiv[-1, 1]$)

Theorem. *Let V_1 and V_2 be Chebyshev subspaces of $C(X)$ with dimensions m and $n(m<n)$, respectively. Let $V_1\subset V_2$ and $v_j\in V_j(j=1, 2)$.*

(a) *If the function $v=v_2-v_1$ changes sign at least m times in X, then there exists an $f\in C(X)$ such that v_j is a best L_1 approximation to f from $V_j(j=1, 2)$;*

(b) *If there exists an $f\in C(X)$ such that v_j is a best L_1 approximation to f from V_j $(j=1, 2)$, then v has at least m zeros in $(-1, 1)$.*

This theorem provides a partial answer to a problem of Rivlin[1] in L_1 approximation. However, in the case $m=n-1$ if $v\neq 0$ has at least m zeros in $(-1, 1)$, then none of them can be a double zero and v, in fact, changes sign at least m times. Thus,. we can give the complete answer in this particular case, which is a generalization of the result[2] by the author, and we have

Corollary. Let V_1 and V_2 be Chebyshev subspaces of $C(X)$ with dimensions $n-1$ and $n(n>1)$, respectively. Let $V_1\subset V_2$ and $v_j\in V_j(j=1, 2)$. Then there exists an $f\in C(X)$ such that v_j is a best L_1 approximation to f from $V_j(j=1, 2)$ if and only if the function $v=v_2-v_1$ changes sign at least $n-1$ times in X or is identically zero.

Before proving the theorem we introduce some notation:

$$Z_+(g)=\{x\in X: g(x)>0\},$$

$$Z_-(g)=\{x\in X: g(x)<0\},$$

$$Z(g)=\{x\in X: g(x)=0\},$$

$$M(E)=\text{the Lebesgue measure of the set } E.$$

2. Proof of Part (a) of the Theorem

Let v change sign at points x^k, $k=1, 2, \cdots, l$ $(l\geqslant m)$,

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$$-1=x^0<x^1<\cdots<x^l<x^{l+1}=1.$$

By Lemma 2 in [3] there exist points

$$x^k=x_0^k<x_1^k<\cdots<x_n^k<x_{n+1}^k=x^{k+1},\quad k=0,1,\cdots,l,$$

such that

$$\sum_{i=0}^{n}(-1)^i\int_{x_i^k}^{x_{i+1}^k}u\,dx=0,\ \forall u\in V_2,\quad k=0,1,\cdots,l. \tag{1}$$

Write $n_j=\left[\frac{1}{2}(n+1-j)\right]$(the integral part of $\frac{1}{2}(n+1-j)$), $j=1, 2$ and denote

$$G_j^k=\bigcup_{i=0}^{n_j}[x_{2i+j-1}^k,\ x_{2i+j}^k],\quad k=0,1,\cdots,l,\ j=1,2,$$

$$G_j=\bigcup_{k=0}^{[l/2]}G_j^{2k},\quad j=1,2,$$

$$G_j^*=\bigcup_{k=0}^{[\frac{1}{2}(l-1)]}G_j^{2k+1},\quad j=1,2,$$

$$H_i^k=(x_i^k-h,\ x_i^k+h)\cap(G_2\cup G_2^*),\quad i=1,2,\cdots,n,\ k=0,1,\cdots,l,$$

$$H=\bigcup_{k=0}^{l}\bigcup_{i=1}^{n}H_i^k\cap G_2,$$

$$H^*=\bigcup_{k=0}^{l}\bigcup_{i=1}^{n}H_i^k\cap G_2^*,$$

where $0<h<\frac{1}{2}\min\limits_{\substack{1<i<n\\0<k<l}}(x_{i+1}^k-x_i^k)$ will be defined later. With this notation (1) becomes

$$\int_{G_1^k}u\,dx=\int_{G_2^k}u\,dx,\ \forall u\in V_2,\quad k=0,1,\cdots,l.$$

Whence

$$\int_{G_1}u\,dx=\int_{G_2}u\,dx,\ \int_{G_1^*}u\,dx=\int_{G_2^*}u\,dx,\ \forall u\in V_2. \tag{2}$$

Now put

$$f(x)=\begin{cases}v_2(x),\ x\in G_1\cup G_1^*,\\ v_1(x),\ x\in(G_2\cup G_2^*)\setminus(H\cup H^*),\\ \text{a continuous function on } H_i^k \text{ lying strictly between } v_1 \text{ and } v_2\\ \quad\text{almost everywhere on } \overline{H}_i^k,\quad i=1,2,\cdots,n,\ k=0,1,\cdots,l.\end{cases}$$

It is easy to see that $f\in C(X)$. Now take $x^*<x^1$ such that $v(x^*)\neq0$ and let $s=\operatorname{sgn} v(x^*)$. Thus

$$\operatorname{sgn}(f(x)-v_1(x))=\begin{cases}s, & x\in G_1\cup H,\\ -s, & x\in G_1^*\cup H^*,\\ 0, & x\in(G_2\cup G_2^*)\setminus(H\cup H^*)\end{cases}$$

and

$$\operatorname{sgn}(f(x)-v_2(x))=\begin{cases}-s, & x\in G_2,\\ s, & x\in G_2^*,\\ 0, & x\in G_1\cup G_1^*\end{cases}$$

are valid almost everywhere.

Since by (2) for any $u\in V_2$

$$\left|\int_X u\,\mathrm{sgn}(f-v_2)dx\right|=\left|\int_{G_2}udx-\int_{G_2^*}udx\right|=\left|\int_{G_1}udx-\int_{G_1^*}udx\right|$$
$$\leqslant\int_{G_1\cup G_1^*}|u|\,dx=\int_{Z(f-v_1)}|u|\,dx,$$

by Theorem 4-2 in [4] v_2 is a best approximation to f from V_2.

On the other hand, for any $u\in V_1$ satisfying $\|u\|_\infty\equiv\max\limits_{x\in X}|u(x)|=1$ we have

$$\left|\int_X u\,\mathrm{sgn}(f-v_1)dx\right|=\left|\int_{G_1}udx-\int_{G_1^*}udx+\int_H udx-\int_{H^*}udx\right|$$
$$=\left|\int_{G_2}udx-\int_{G_2^*}udx+\int_H udx-\int_{H^*}udx\right|$$
$$\leqslant\left|\int_{G_2}udx-\int_{G_2^*}udx\right|+\int_{H\cup H^*}|u|\,dx.$$

We claim that

$$c=\inf_{u\in V_1,\|u\|_\infty=1}\left(\int_{G_2\cup G_2^*}|u|\,dx-\left|\int_{G_2}udx-\int_{G_2^*}udx\right|\right)>0. \tag{3}$$

Whence taking $h\leqslant c/2n(l+1)$, it follows that

$$\int_{H\cup H^*}|u|\,dx\leqslant M(H\cup H^*)\leqslant\frac{1}{2}c$$

and

$$\left|\int_X u\,\mathrm{sgn}(f-v_1)dx\right|\leqslant\int_{G_2\cup G_2^*}|u|\,dx+\int_{H\cup H^*}|u|\,dx-c$$
$$=\int_{(G_2\cup G_2^*)\setminus(H\cup H^*)}|u|\,dx+2\int_{H\cup H^*}|u|\,dx-c$$
$$\leqslant\int_{(G_2\cup G_2^*)\setminus(H\cup H^*)}|u|\,dx=\int_{Z(f-v_1)}|u|\,dx.$$

This means that v_1 is a best approximation to f from V_1, because the restriction of $\|u\|_\infty=1$ may be removed.

The remainder of the proof is devoted to showing (3). In fact, if $c=0$, we can find a $u\in V_1$ such that $\|u\|_\infty=1$ and

$$\int_{G_2\cup G_2^*}|u|\,dx=\left|\int_{G_2}udx-\int_{G_2^*}udx\right|.$$

It implies that

$$\mathrm{sgn}\,u(x)=\begin{cases}s^*, & x\in G_2,\\ -s^*, & x\in G_2^*\end{cases}$$

is true almost everywhere, where $s^*=1$ or -1. Hence u has at least $l\geqslant m$ zeros in X, a contradiction.

3. Proof of Part (b) of the Theorem

We need the following

Lemma. *If $f\in C(X)$ satisfies that v_j is a best approximation to f from V_j ($j=$*

1, 2), *then there exists a* $g\in C(X)$ *satisfying that* v_j *is a best approximation to* g *from* $V_j(j=1, 2)$ *and*

$$\min\{v_1(x), v_2(x)\}\leqslant g(x)\leqslant\max\{v_1(x), v_2(x)\}. \tag{4}$$

Proof. Put

$$g(x)=\begin{cases}\min\{v_1(x), v_2(x)\}, & f(x)<\min\{v_1(x), v_2(x)\},\\ \max\{v_1(x), v_2(x)\}, & f(x)>\max\{v_1(x), v_2(x)\},\\ f(x), & \text{for the other } x.\end{cases}$$

Obviously, $g\in C(X)$ and satisfies (4).

On the other hand, since by (4) $g(x)>(<)v_j(x)$ implies that $v_{3-j}(x)\geqslant(\leqslant)g(x)>(<)v_j(x)$ and $f(x)>(<)v_j(x)(j=1, 2)$, we have

$$\operatorname{sgn}(g(x)-v_j(x))=\operatorname{sgn}(f(x)-v_j(x)),\ \forall x\in X\backslash Z(g-v_j),\quad j=1, 2.$$

Thus for any $u\in V_j(j=1, 2)$

$$\begin{aligned}\left|\int_X u\operatorname{sgn}(g-v_j)dx\right| &= \left|\int_{X\backslash Z(g-v_j)} u\operatorname{sgn}(g-v_j)dx\right| = \left|\int_{X\backslash Z(g-v_j)} u\operatorname{sgn}(f-v_j)dx\right|\\ &\leqslant\left|\int_{X\backslash Z(g-v_j)} u\operatorname{sgn}(f-v_j)dx+\int_{Z(g-v_j)\backslash Z(f-v_j)} u\operatorname{sgn}(f-v_j)dx\right|\\ &\quad+\int_{Z(g-v_j)\backslash Z(f-v_j)}|u|dx\\ &=\left|\int_{X\backslash Z(f-v_j)} u\operatorname{sgn}(f-v_j)dx\right|+\int_{Z(g-v_j)\backslash Z(f-v_j)}|u|dx\\ &\leqslant\int_{Z(f-v_j)}|u|dx+\int_{Z(g-v_j)\backslash Z(f-v_j)}|u|dx=\int_{Z(g-v_j)}|u|dx,\end{aligned}$$

which means that v_j is a best approximation to g from $V_j(j=1, 2)$.

This completes the proof of the lemma.

Now assume that there exists an $f\in C(X)$ such that v_j is a best approximation to f from $V_j(j=1, 2)$, but the condition of the theorem is not satisfied, i. e. v has at most $m-1$ zeros in $(-1, 1)$. By the lemma we can without loss of generality suppose f satisfies (4).

It is not hard to see that

$$v(x)\begin{cases}>0, & x\in Z_+(f-v_1)\cup Z_-(f-v_2),\\ <0, & x\in Z_-(f-v_1)\cup Z_+(f-v_2).\end{cases}$$

Since v has at most $m-1$ zeros in $(-1, 1)$, we can find a $u\in V_1$ such that

$$\operatorname{sgn} u(x)=\operatorname{sgn} v(x),\ \forall x\in X\backslash Z(v).$$

Whence by Theorem 4-2 in [4]

$$\begin{aligned}\sum_{j=1}^{2}\int_{Z(f-v_j)}|u|dx &\geqslant\sum_{j=1}^{2}\left|\int_X u\operatorname{sgn}(f-v_j)dx\right|\\ &=\sum_{j=1}^{2}\int_{X\backslash Z(f-v_j)}|u|dx\geqslant\sum_{j=1}^{2}\int_{Z(f-v_{3-j})}|u|dx=\sum_{j=1}^{2}\int_{Z(f-v_j)}|u|dx.\end{aligned}$$

This gives that

$$(-1)^{j-1}\int_X u\operatorname{sgn}(f-v_j)dx=\int_{Z(f-v_j)}|u|dx,\quad j=1, 2$$

and

$$\int_{X\setminus Z(f-v_j)} |u|\,dx=\int_{Z(f-v_{3-j})} |u|\,dx, \quad j=1,\ 2,$$

which implies that

$$Z(f-v_1)\cup Z(f-v_2)=X.$$

Thus

$$\overline{Z_-(f-v_1)\cup Z_-(f-v_2)}\ \cap\ \overline{Z_+(f-v_1)\cup Z_+(f-v_2)}\ \subset\ Z(v)$$

and there also exists a $u^*\in V_1$ such that

$$u^*(x)\begin{cases}>0, & x\in Z_+(f-v_1)\cup Z_+(f-v_2),\\ <0, & x\in Z_-(f-v_1)\cup Z_-(f-v_2).\end{cases}$$

The similar arguments as above give that

$$\int_X u^*\operatorname{sgn}(f-v_j)\,dx=\int_{Z(f-v_j)} |u^*|\,dx, \quad j=1,\ 2.$$

Hence, noting that $\operatorname{sgn} u^*(x)=\operatorname{sgn} u(x)$ on $Z(f-v_2)\setminus Z(v)=Z_+(f-v_1)\cup Z_-(f-v_1)$, we have

$$\int_X (u^*-u)\operatorname{sgn}(f-v_2)\,dx=\int_{Z(f-v_2)} (|u^*|+|u|)\,dx>\int_{Z(f-v_2)} |u^*-u|\,dx,$$

a contradiction.

References

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