ERROR BOUND FOR BERNSTEIN-BÉZIER TRIANGULAR APPROXIMATION*

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Abstract

Based upon a new error bound for the linear interpolant to a function defined on a triangle and having continuous partial derivatives of second order, the related error bound for n-th Bernstein tirangular approximation is obtained. The order of approximation is $1/n$.

1. Introduction

Bernstein-Bézier polynomials, or surfaces, have been studied extensively[1-5]. In this paper we first present an error bound on the right-hand side of (12) and show that the coefficient 1 is the best. Then, based on (12), the error bound for the Bernstein–Bézier triangular approximation is obtained and the coefficient 1 is again proved to be the best.

2. Definition and Notation

We begin with a brief discussion on the area coordinates of points with respect to a given triangle. Let T be a triangle with vertices $T_\alpha=(x_\alpha, y_\alpha)$, $\alpha=1, 2, 3$, and area $|\Delta|$. An internal point $P=(x, y)$ of T divides the triangle $T_1T_2T_3$ into three smaller ones, PT_2T_3, PT_3T_1, PT_1T_2, with respective areas $|\Delta_1|$, $|\Delta_2|$, $|\Delta_3|$, which may vary from zero to $|\Delta|$, depending on the position of P. In other words, the ratios $u:=\frac{|\Delta_1|}{|\Delta|}$, $v:=\frac{|\Delta_2|}{|\Delta|}$, $w:=\frac{|\Delta_3|}{|\Delta|}$ will take up any value between zero and unity. Here (u, v, w) with $u+v+w=1$ are called area coordinates of the point P.

It is easy to see that

$$\begin{bmatrix}1\\x\\y\end{bmatrix}=\begin{bmatrix}1&1&1\\x_1&x_2&x_3\\y_1&y_2&y_3\end{bmatrix}\begin{bmatrix}u\\v\\w\end{bmatrix}. \tag{1}$$

Let $F(x, y)$ be a function defined on T, where x and y are Cartesian coordinates; the related function f dependent on area coordinates u, v, w is given by

$$f(u, v, w)=F(x_1u+x_2v+x_3w,\ y_1u+y_2v+y_3w). \tag{2}$$

The n-th Bernstein–Bézier polynomial of the function f over the triangle T is given by

$$B_n(f; u, v, w)=\sum_{i+j+k=n} f\left(\frac{i}{n}, \frac{j}{n}, \frac{k}{n}\right)J^n_{i,j,k}(u, v, w), \tag{3}$$

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where

$$J^n_{i,j,k}(u,\ v,\ w)=\frac{n!}{i!j!k!}\,u^i v^j w^k \tag{4}$$

and i, j, k designate nonnegative integers such that $i+j+k=n$. Functions in (4) are called Bernstein basis polynomials of degree n, for they form a basis for all bivariate polynomials of degree n. In some cases $B_n(f; u, v, w)$ and $f\left(\frac{i}{n}, \frac{j}{n}, \frac{k}{n}\right)$ are replaced by $B_n(f)$ and $f_{i,j,k}$ for simplicity.

We outline the basic properties of $B_n(f)$ as follows:

(a) B_n is a positive and linear operator carrying every function defined on T to a bivariate polynomial of degree n.

(b) $B_n(f)$ interpolates to f at the three vertices of T, i. e. $B_n(f;\ T_\alpha)=f(T_\alpha)$, $\alpha=1, 2, 3$.

(c) Since the functions $J^n_{i,j,k}$ in (4) are nonnegative on T and sum to $(u+v+w)^n=1$, each point on the Bernstein–Bézier triangular surface is a convex combination of $f_{i,j,k}$. Hence we can say that the surface (3) lies within the convex hull of the points $P_{i,j,k}: =\left(\frac{i}{n}, \frac{j}{n}, \frac{k}{n}; f_{i,j,k}\right)$, $i+j+k=n$, which are on the surface associated with the function f.

(d) If f is a continuous function on T, then

$$\lim_{n\to\infty} B_n(f;\ u,\ v,\ w)=f(u,\ v,\ w) \tag{5}$$

uniformly on T (see, e. g., p. 51 of [1]).

(e) Simple calculation shows

$$J^n_{i,j,k}=\frac{1}{n+1}[(i+1)J^{n+1}_{i+1,j,k}+(j+1)J^{n+1}_{i,j+1,k}+(k+1)J^{n+1}_{i,j,k+1}], \tag{6}$$

which enables us to express $B_n(f)$ in terms of the Bernstein basis polynomials of degree $n+1$:

$$B_n(f)=\frac{1}{n+1}\sum_{i+j+k=n+1}(if_{i-1,j,k}+jf_{i,j-1,k}+kf_{i,j,k-1})J^{n+1}_{i,j,k}. \tag{7}$$

Concerning the surface points $P_{i,j,k}$, $i+j+k=n$, in (c), we further note that there are altogether $\frac{(n+1)(n+2)}{2}$ such points in the space. If a line segment is connected each two of the three points

$$P_{i+1,j,k},\quad P_{i,j+1,k},\quad P_{i,j,k+1},$$

where $i+j+k=n-1$, a piecewise linear function on T is obtained, which is denoted by $\hat{f}_n(u,\ v,\ w)$. $\hat{f}_n$ is called the n-th Bézier net of f, in accordence with literature in CAGD (see Farin[3]).

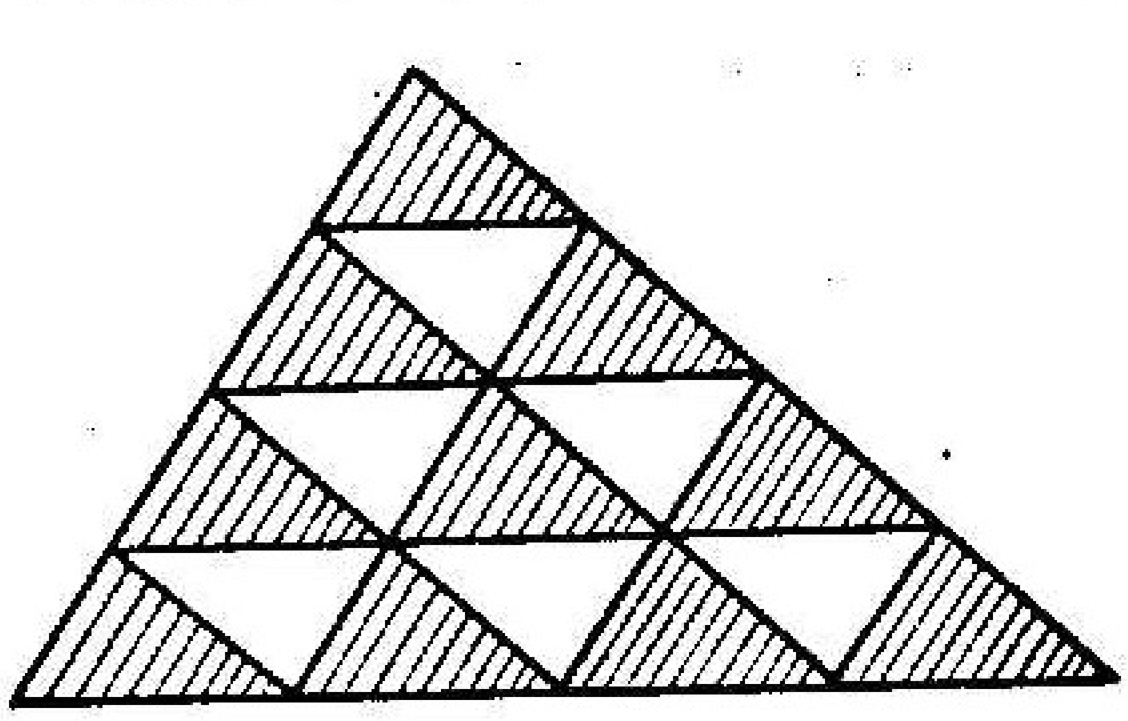

Fig. 1

The projection of $\hat{f}_n$ onto the triangle T produces a subdivision of T, denoted by $S_n(T)$. Each of the points $\left(\frac{i}{n}, \frac{j}{n}, \frac{k}{n}\right)$ satisfying $i+j+k=n$ is called a node of $S_n(T)$. $S_4(T)$ is illustrated in Fig. 1.

The subtriangles in $S_n(T)$ can be divided into two categories:

(i) With vertices

$$\left(\frac{i+1}{n},\ \frac{j}{n},\ \frac{k}{n}\right),\ \left(\frac{i}{n},\ \frac{j+1}{n},\ \frac{k}{n}\right),\ \left(\frac{i}{n},\ \frac{j}{n},\ \frac{k+1}{n}\right), \tag{8}$$

where $i+j+k=n-1$.

(ii) With vertices

$$\left(\frac{i-1}{n},\ \frac{j}{n},\ \frac{k}{n}\right),\ \left(\frac{i}{n},\ \frac{j-1}{n},\ \frac{k}{n}\right),\ \left(\frac{i}{n},\ \frac{j}{n},\ \frac{k-1}{n}\right), \tag{9}$$

where $i+j+k=n+1$.

For example, the shaded subtriangles in Fig. 1 belong to the first category.

It is easy to verify that the Bézier net $\hat{f}_n$ restricted on the subtriangle with vertices in (9) has the following equation

$$Z=(i-nu)f_{i-1,j,k}+(j-nv)f_{i,j-1,k}+(k-nw)f_{i,j,k-1},$$

where $i+j+k=n+1$. The following equality

$$\left(\frac{i}{n+1},\ \frac{j}{n+1},\ \frac{k}{n+1}\right)=\frac{i}{n+1}\left(\frac{i-1}{n},\ \frac{j}{n},\ \frac{k}{n}\right)+\frac{j}{n+1}\left(\frac{i}{n},\ \frac{j-1}{n},\ \frac{k}{n}\right)$$
$$+\frac{k}{n+1}\left(\frac{i}{n},\ \frac{j}{n},\ \frac{k-1}{n}\right)$$

indicates that the point $\left(\frac{i}{n+1},\ \frac{j}{n+1},\ \frac{k}{n+1}\right)$ lies inside the subtriangle with vertices in (9). Thus (7) can be rewritten as

$$B_n(f)=\sum_{i+j+k=n+1}\hat{f}_n\left(\frac{i}{n+1},\ \frac{j}{n+1},\ \frac{k}{n+1}\right)J^{n+1}_{i,j,k}$$

and hence we have

$$|B_n(f)-B_{n+1}(f)|\leqslant\sum_{i+j+k=n+1}\left|\hat{f}_n\left(\frac{i}{n+1},\ \frac{j}{n+1},\ \frac{k}{n+1}\right)\right.$$
$$\left.-f\left(\frac{i}{n+1},\ \frac{j}{n+1},\ \frac{k}{n+1}\right)\right|J^{n+1}_{i,j,k}. \tag{10}$$

3. Lemma

From (10) we see that to estimate $|(B_n-B_{n+1})(f)|$ it suffices to estimate the difference between f and its n-th Bézier net $\hat{f}_n$.

We need the following

Lemma. *Let $T\equiv T_1T_2T_3$ be a triangle with sides $h_1\leqslant h_2\leqslant h_3$ and vertices $T_\alpha=(x_\alpha,\ y_\alpha)$, $\alpha=1,\ 2,\ 3$ respectively. Let $F(x,\ y)$ be a function having continuous partial derivatives of second order on T. Set*

$$M:=\sup_{(x,y)\in T}\left\|\left(\frac{\partial^2F}{\partial x^2},\ \frac{\partial^2F}{\partial x\,\partial y},\ \frac{\partial^2F}{\partial y^2}\right)\right\|_\infty:=\sup_{(x,y)\in T}\max\left(\left|\frac{\partial^2F}{\partial x^2}\right|,\ \left|\frac{\partial^2F}{\partial x\,\partial y}\right|,\ \left|\frac{\partial^2F}{\partial y^2}\right|\right). \tag{11}$$

Then

$$\|f(u\ ,\ v,\ w)-Z(f)(u,\ v,\ w)\|_\infty\leqslant Mh_2^2 \tag{12}$$

and the coefficient 1 is the best, where f is determined by (2) and $Z(f)$ is the linear function interpolating to f at the three vertices of T.

Proof. Let

$$\varphi(u, v)=f(u, v, 1-u-v),$$

where $0\leqslant u, v, u+v\leqslant 1$. The plane interpolating to f at the three vertices now has the equation

$$Z(\varphi)(u, v)=u\varphi(1, 0)+v\varphi(0, 1)+(1-u-v)\varphi(0, 0).$$

Since

$$\varphi(1, 0)=\varphi(0, 0)+\int_0^1\varphi_u(\tau, 0)d\tau,$$

$$\varphi(0, 1)=\varphi(0, 0)+\int_0^1\varphi_v(0, \sigma)d\sigma,$$

$$\varphi(u, v)=\varphi(0, 0)+u\int_0^1\varphi_u(u\tau, 0)d\tau+v\int_0^1\varphi_v(0, v\sigma)d\sigma$$
$$+uv\int_0^1\int_0^1\varphi_{uv}(u\tau, v\sigma)d\tau\, d\sigma,$$

where $\varphi_u=\frac{\partial\varphi}{\partial u}$, $\varphi_v=\frac{\partial\varphi}{\partial v}$ and $\varphi_{uv}=\frac{\partial^2\varphi}{\partial u\partial v}$, we have

$$\varphi(u, v)-Z(\varphi)(u, v)=u\int_0^1\left[\int_\tau^{u\tau}\varphi_{uu}(\xi, 0)d\xi\right]d\tau$$
$$+v\int_0^1\left[\int_\sigma^{v\sigma}\varphi_{vv}(0, \eta)d\eta\right]d\sigma+uv\int_0^1\int_0^1\varphi_{uv}(u\tau, v\sigma)d\tau\, d\sigma. \tag{13}$$

Let

$$M_0:=\sup_{(u,v)\in T}\|(\varphi_{uu}, \varphi_{uv}, \varphi_{vv})\|_\infty. \tag{14}$$

Hence

$$\|\varphi-Z(\varphi)\|_\infty\leqslant\frac{1}{2}M_0u(1-u)+\frac{1}{2}M_0v(1-v)+M_0uv\leqslant\frac{1}{2}M_0. \tag{15}$$

From (1), it is easy to verify the relation

$$\begin{pmatrix}\varphi_{uu}\\ \varphi_{uv}\\ \varphi_{vv}\end{pmatrix}=M_2\begin{pmatrix}F_{xx}\\ F_{xy}\\ F_{yy}\end{pmatrix} \tag{16}$$

with

$$M_2:=\begin{pmatrix}(x_1-x_3)^2 & 2(x_1-x_3)(y_1-y_3) & (y_1-y_3)^2\\ (x_1-x_3)(x_2-x_3) & (x_1-x_3)(y_2-y_3)+(x_2-x_3)(y_1-y_3) & (y_1-y_3)(y_2-y_3)\\ (x_2-x_3)^2 & 2(x_2-x_3)(y_2-y_3) & (y_2-y_3)^2\end{pmatrix}. \tag{17}$$

It follows that

$$\|(\varphi_{uu}, \varphi_{uv}, \varphi_{vv})\|_\infty\leqslant\|M_2\|_\infty\cdot\|(F_{xx}, F_{xy}, F_{yy})\|_\infty\leqslant 2Mh_2^2,$$

and then

$$M_0\leqslant 2Mh_2^2. \tag{18}$$

Now, from (15) and (18), we conclude that

$$\|\varphi-Z(\varphi)\|_\infty\leqslant Mh_2^2.$$

In order to complete the proof of the lemma, let

$$T_1=(-1, 1),\ T_2=(1, -1),\ T_3=(a, a),\quad a>0, \tag{19}$$

and

$$F(x, y)=(x-y)^2/2. \tag{20}$$

It is easy to see that the linear interpolant to $F(x, y)$ is

$$Z(F)(x, y)=2-(x+y)/a$$

on the triangle $T_1T_2T_3$. For any $\varepsilon>0$, let $a^2<\varepsilon/2$. Then

$$Mh_2^2=2+2a^2<2+\varepsilon \tag{21}$$

and

$$\|F(x, y)-Z(F)(x, y)\|_\infty \geqslant \left|(x-y)^2/2-2+\frac{1}{a}(x+y)\right|_{x=y=0}=2\geqslant Mh_2^2-\varepsilon. \tag{22}$$

(22) shows that the coefficient 1 is the best.

4. Main Theorem

Theorem.

$$\|B_n(f)-f\|_\infty=Mh_2^2/n+O(1/n^2)$$

and the coefficient 1 *is the best.*

Proof. Applying the lemma to f and $\hat{f}_n$, the latter is restricted to the subtriangle with vertices in (9), we obtain

$$\left|\hat{f}_n\left(\frac{i}{n+1}, \frac{j}{n+1}, \frac{k}{n+1}\right)-f\left(\frac{i}{n+1}, \frac{j}{n+1}, \frac{k}{n+1}\right)\right|\leqslant M(h_2/n)^2.$$

From (10) we get

$$|B_n(f)-B_{n+1}(f)|\leqslant M(h_2/n)^2\sum_{i+j+k=n+1}J_{i,j,k}^{n+1}(u, v, w)=M(h_2/n)^2.$$

Thus we have

$$\|B_n(f)-B_{n+1}(f)\|_\infty\leqslant M(h_2/n)^2 \tag{23}$$

for $n=1, 2, 3, \cdots$. By the triangle inequality

$$\|B_n(f)-B_{n+m}(f)\|_\infty\leqslant\sum_{i=1}^{m}\|B_{n+i-1}(f)-B_{n+i}(f)\|_\infty$$

and then by (23) we have

$$\|B_n(f)-B_{n+m}(f)\|_\infty\leqslant Mh_2^2\sum_{i=1}^{m}1/(n+i-1)^2. \tag{24}$$

Let $m\to+\infty$ on both sides of (24), we obtain by (5)

$$\|B_n(f)-f\|_\infty\leqslant Mh_2^2\cdot\sum_{k=n}^{\infty}1/k^2,$$

and

$$\|B_n(f)-f\|_\infty=Mh_2^2/n+O(1/n^2) \tag{25}$$

by the well-known result that

$$\sum_{k=n}^{\infty}1/k^2=1/n+O(1/n^2).$$

For the function $F(x, y)$ and $T_\alpha(\alpha=1, 2, 3)$ defined in (20) and (19) respectively, using relation (1), it follows that

$$\varphi(u, v)=2(u-v)^2. \tag{26}$$

Noting identities

$$\sum_{i+j+k=n}\frac{n!}{i!j!k!}i^2u^iv^jw^k=nu(1-u)+n^2u^2 \tag{27}$$

and

$$\sum_{i+j+k=n} \frac{n!}{i!j!k!} iju^i v^j w^k = n(n-1)uv, \tag{28}$$

from (3), (26), (27) and (28), we have

$$B_n(\varphi; u, v) = \varphi(u, v) + \frac{2}{n}(u(1-u) + v(1-v) + 2uv).$$

It follows that

$$\| B_n(\varphi; u, v) - \varphi(u, v) \|_\infty = 2/n.$$

The same kind of argument as in the proof of the lemma shows that coefficient 1 in (25) is the best.

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