A METHOD FOR CONSTRUCTING A CONTROLLABLE THIRD ORDER CURVE*

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1. Problem Description

There exists, on the interval $[a, b]$, a partition

$$\Delta:\ a=x_1<x_2<\cdots<x_n=b$$

and a set of real numbers y_j, y'_j, $j=1, 2, \cdots, n$, such that its corresponding piecewise third order Hermite interpolating function is as follows:

$$H(x)=y_jf_0(x, j)+y_{j+1}f_1(x, j)+y'_jh_{j+1}g_0(x, j)+y'_{j+1}h_{j+1}g_1(x, j), \tag{1}$$

$$x_j\leqslant x\leqslant x_{j+1},\ h_{j+1}=x_{j+1}-x_j, \quad j=1, 2, \cdots, n-1$$

in which

$$f_0(x, j)=2\left(\frac{x-x_j}{h_{j+1}}\right)^3-3\left(\frac{x-x_j}{h_{j+1}}\right)^2+1, \quad f_1(x, j)=-2\left(\frac{x-x_j}{h_{j+1}}\right)^3+3\left(\frac{x-x_j}{h_{j+1}}\right)^2,$$

$$g_0(x, j)=\left(\frac{x-x_j}{h_{j+1}}\right)^3-2\left(\frac{x-x_j}{h_{j+1}}\right)^2+\left(\frac{x-x_j}{h_{j+1}}\right), \quad g_1(x, j)=\left(\frac{x-x_j}{h_{j+1}}\right)^3-\left(\frac{x-x_j}{h_{j+1}}\right)^2.$$

For many engineering problems, as the derivatives y'_j at data points $P_j(x_j, y_j)$, $j=1, 2, \cdots, n$, are not given[2], formula (1) is not directly applicable.

To evaluate y'_j, a condition of C^2-continuity must be imposed at x_j, $j=2, 3, \cdots, n-1$. The resultant system of tridiagonal simultaneous equations ($n-2$ in number) plus two end conditions determine uniquely a set of $y'_j (j=1, 2, \cdots, n)$. This is the well-known third order spline function in terms of its first derivatives.

This kind of functions has extensive use in engineering, but some deficiencies are also felt. For instance, without preservation of convexity[3], unwanted points of inflexions would occur and the local properties would vanish. This will bring forth some inconvenience in design. The third order B-spline curve has some desirable properties[5,6,7], but it does not pass through data points (though we can make a curve passing through these points by finding the vertices first), and so the problem remains inconvenient. Consequently, it is a common desire to develop a sort of curves which preserves convexity and local properties, is easy to compute, and is suitable for interactive design.

This paper starts with formula (1) to define a third order curve and the control coefficients and then puts forward the necessary and sufficient conditions for the convexity of the curve. The curve is supposed to be C^1-continuous, manageable in getting straight line segments, cuspidal points and inflexion points, and can be expressed explicitly in terms of coordinates of data points (and end conditions).

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2. Representations of the Curve

Let there be a set of data points $P_j(x_j, y_j)$, $j=1, 2, \cdots, n$, with two given end conditions. It is required to construct a curve satisfying these conditions. As the first derivatives are not given at these data points, formula (1) is not directly applicable.

To provide additional constraints and to facilitate the adjustment and control of the shape of the curve, we take

$$y'_j=(1-\lambda)\frac{y_{j+1}-y_j}{h_{j+1}}+\lambda\frac{y_j-y_{j-1}}{h_j},\quad y'_{j+1}=(1-\mu)\frac{y_{j+2}-y_{j+1}}{h_{j+2}}+\mu\frac{y_{j+1}-y_j}{h_{j+1}},\tag{2}$$

$$j=1, 2, \cdots, n-1 \text{ and } 0\leqslant\lambda\leqslant1,\ 0\leqslant\mu\leqslant1,$$

where λ and μ are control coefficients. We will see later that by the adjustment of these two coefficients, the shape of a curve can be manipulated at will and unwanted points of inflexions can be eliminated.

In formula (2), owing to the presence of the four undetermined quantities y_0, y_{n+1}, h_1, h_{n+1}, we make extensions on both ends: $P_0(x_0, y_0)$, $x_0=x_1-h_1$ at the left end and $P_{n+1}(x_{n+1}, y_{n+1})$, $x_{n+1}=x_n+h_{n+1}$ at the right end, with $h_1=h_2$, $h_{n+1}=h_n$. As for y_0, y_{n+1}, they are determined as follows:

(I) If the end conditions are y'_1, y'_n, then y_0, y_{n+1} should satisfy

$$y'_1=(1-\lambda)\frac{y_2-y_1}{h_2}+\lambda\frac{y_1-y_0}{h_1},\quad y'_n=(1-\mu)\frac{y_{n+1}-y_n}{h_{n+1}}+\mu\frac{y_n-y_{n-1}}{h_n}.\tag{3}$$

(II) If the end conditions are y''_1, y''_n, then y_0, y_{n+1} should satisfy

$$y''_1=[4\lambda\quad -2(1-\mu)]\left[\frac{\frac{y_2-y_1}{h_2}-\frac{y_1-y_0}{h_1}}{h_2}\quad\frac{\frac{y_3-y_2}{h_3}-\frac{y_2-y_1}{h_2}}{h_2}\right]^T,$$

$$y''_n=[-2\lambda\quad 4(1-\mu)]\begin{bmatrix}\dfrac{\frac{y_n-y_{n-1}}{h_n}-\frac{y_{n-1}-y_{n-2}}{h_{n-1}}}{h_n}\\[2ex]\dfrac{\frac{y_{n+1}-y_n}{h_{n+1}}-\frac{y_n-y_{n-1}}{h_n}}{h_n}\end{bmatrix}.\tag{4}$$

(III) If the end conditions are not specified, y_0, y_{n+1} should satisfy

$$\frac{y_2-y_1}{h_2}-\frac{y_1-y_0}{h_1}=\left(\frac{y_3-y_2}{h_3}-\frac{y_2-y_1}{h_2}\right)^2\bigg/\left(\frac{y_4-y_3}{h_4}-\frac{y_3-y_2}{h_3}\right),$$

$$\frac{y_{n+1}-y_n}{h_{n+1}}-\frac{y_n-y_{n-1}}{h_n}=\left(\frac{y_n-y_{n-1}}{h_n}-\frac{y_{n-1}-y_{n-2}}{h_{n-1}}\right)^2\bigg/\left(\frac{y_{n-1}-y_{n-2}}{h_{n-1}}-\frac{y_{n-2}-y_{n-3}}{h_{n-2}}\right).\tag{5}$$

We can thus express piecewisely, in explicit form, the interpolating functions passing through the data points $P_j(x_j, y_j)$, $j=1, 2, \cdots, n$.

$$y=\overline{L}(x)=\overline{L}_j(x)=y_jf_0(x,\ j)+y_{j+1}f_1(x,\ j)$$
$$+\left((1-\lambda)\frac{y_{j+1}-y_j}{h_{j+1}}+\lambda\frac{y_j-y_{j-1}}{h_j}\right)g_0(x,\ j)h_{j+1}$$
$$+\left((1-\mu)\frac{y_{j+2}-y_{j+1}}{h_{j+1}}+\mu\frac{y_{j+1}-y_j}{h_{j+1}}\right)g_1(x,\ j)h_{j+1}$$
$$=[l_1(x,\ j,\ \lambda)\ \ l_2(x,\ j,\ \lambda,\ \mu)\ \ l_3(x,\ j,\ \lambda,\ \mu)\ \ l_4(x,\ j,\ \mu)]\,[y_{j-1}\ \ y_j\ \ y_{j+1}\ \ y_{j+2}]^T$$
$$=\left[\left(\frac{x-x_j}{h_{j+1}}\right)^3\ \ \left(\frac{x-x_j}{h_{j+1}}\right)^2\ \ \left(\frac{x-x_j}{h_{j+1}}\right)\ \ 1\right]M_j(\lambda,\ \mu)\,[y_{j-1}\ \ y_j\ \ y_{j+1}\ \ y_{j+2}]^T, \tag{6}$$
$$x_j\leqslant x\leqslant x_{j+1},\ h_{j+1}=x_{j+1}-x_j,\quad j=1,\ 2,\ \cdots,\ n-1,\ 0\leqslant\lambda\leqslant 1,\ 0\leqslant\mu\leqslant 1$$

in which

$$\begin{cases} l_1(x,\ j,\ \lambda)=\lambda\dfrac{h_{j+1}}{h_j}\left(-\left(\dfrac{x-x_j}{h_{j+1}}\right)^3+2\left(\dfrac{x-x_j}{h_{j+1}}\right)^2-\left(\dfrac{x-x_j}{h_{j+1}}\right)\right), \\ l_2(x,\ j,\ \lambda,\ \mu)=\left(1+\lambda\dfrac{h_{j+1}}{h_j}+\lambda-\mu\right)\left(\dfrac{x-x_j}{h_{j+1}}\right)^3 \\ \qquad +\left(-1-2\lambda-2\lambda\dfrac{h_{j+1}}{h_j}+\mu\right)\left(\dfrac{x-x_j}{h_{j+1}}\right)^2+\left(\lambda+\lambda\dfrac{h_{j+1}}{h_j}-1\right)\dfrac{x-x_j}{h_{j+1}}+1, \\ l_3(x,\ j,\ \lambda,\ \mu)=\left(\mu\dfrac{h_{j+1}}{h_{j+2}}-\dfrac{h_{j+1}}{h_{j+2}}-1+\mu-\lambda\right)\left(\dfrac{x-x_j}{h_{j+1}}\right)^3 \\ \qquad +\left(1-\mu\dfrac{h_{j+1}}{h_{j+2}}+\dfrac{h_{j+1}}{h_{j+2}}-\mu+2\lambda\right)\left(\dfrac{x-x_j}{h_{j+1}}\right)^2+(1-\lambda)\dfrac{x-x_j}{h_{j+1}}, \\ l_4(x,\ j,\ \mu)=(1-\mu)\dfrac{h_{j+1}}{h_{j+2}}\left(\left(\dfrac{x-x_j}{h_{j+1}}\right)^3-\left(\dfrac{x-x_j}{h_{j+1}}\right)^2\right), \end{cases} \tag{7}$$

$$M_j(\lambda,\ \mu)$$
$$=\begin{bmatrix} -\lambda\dfrac{h_{j+1}}{h_j} & 1+\lambda\dfrac{h_{j+1}}{h_j}+\lambda-\mu & \mu\dfrac{h_{j+1}}{h_{j+2}}-\dfrac{h_{j+1}}{h_{j+2}}-1+\mu-\lambda & (1-\mu)\dfrac{h_{j+1}}{h_{j+2}} \\ 2\lambda\dfrac{h_{j+1}}{h_j} & -2\lambda\dfrac{h_{j+1}}{h_j}-1-2\lambda+\mu & 1+\dfrac{h_{j+1}}{h_{j+2}}-\mu\dfrac{h_{j+1}}{h_{j+2}}-\mu+2\lambda & (\mu-1)\dfrac{h_{j+1}}{h_{j+2}} \\ -\lambda\dfrac{h_{j+1}}{h_j} & \lambda+\lambda\dfrac{h_{j+1}}{h_j}-1 & 1-\lambda & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

It is easy to verify that:

$$l_1(x,\ j,\ \lambda)+l_2(x,\ j,\ \lambda,\ \mu)+l_3(x,\ j,\ \lambda,\ \mu)+l_4(x,\ j,\ \mu)\equiv 1, \tag{8}$$
$$l_2(x_j,\ j,\ \lambda,\ \mu)=1,\ \ l_1(x_j,\ j,\ \lambda)=l_3(x_j,\ j,\ \lambda,\ \mu)=l_4(x_j,\ j,\ \mu)=0 \text{ for } x=x_j,$$
$$l_3(x_{j+1},\ j,\ \lambda,\ \mu)=1,\ l_1(x_{j+1},\ j,\ \lambda)=l_2(x_{j+1},\ j,\ \lambda,\ \mu)=l_4(x_{j+1},\ j,\ \mu)=0 \text{ for } x=x_{j+1},$$
$$y'=\overline{L}'(x)=\overline{L}'_j(x)=[G_1(x,\ j,\ \lambda)\quad G_2(x,\ j,\ \lambda,\ \mu)\quad G_3(x,\ j,\ \mu)]$$
$$\cdot\left[\frac{y_j-y_{j-1}}{h_j}\quad\frac{y_{j+1}-y_j}{h_{j+1}}\quad\frac{y_{j+2}-y_{j+1}}{h_{j+2}}\right]^T$$
$$=\left[\left(\frac{x-x_j}{h_{j+1}}\right)^2\ \ \frac{x-x_j}{h_{j+1}}\ \ 1\right]\overline{M}_j(\lambda,\ \mu)\left[\frac{y_j-y_{j-1}}{h_j}\quad\frac{y_{j+1}-y_j}{h_{j+1}}\quad\frac{y_{j+2}-y_{j+1}}{h_{j+2}}\right]^T \tag{9}$$

in which

$$\begin{cases} G_1(x,\ j,\ \lambda)=3\lambda\left(\frac{x-x_j}{h_{j+1}}\right)^2-4\lambda\left(\frac{x-x_j}{h_{j+1}}\right)+\lambda, \\ G_2(x,\ j,\ \lambda,\ \mu)=3(-1+\mu-\lambda)\left(\frac{x-x_j}{h_{j+1}}\right)^2+2(1-\mu+2\lambda)\frac{x-x_j}{h_{j+1}}+(1-\lambda), \\ G_3(x,\ j,\ \mu)=3(1-\mu)\left(\frac{x-x_j}{h_{j+1}}\right)^2+2(\mu-1)\frac{x-x_j}{h_{j+1}}, \end{cases} \tag{10}$$

$$\overline{M}_j(\lambda,\ \mu)=\begin{bmatrix} 3\lambda & 3(-1+\mu-\lambda) & 3(1-\mu) \\ -4\lambda & 2(1-\mu+2\lambda) & 2(\mu-1) \\ \lambda & 1-\lambda & 0 \end{bmatrix}.$$

We can also verify that

$$G_1(x,\ j,\ \lambda)+G_2(x,\ j,\ \lambda,\ \mu)+G_3(x,\ j,\ \mu)\equiv 1, \tag{11}$$

$$y''=\overline{L}''(x)=\overline{L}''_j(x)$$

$$=[H_1(x,\ j,\ \lambda)\quad H_2(x,\ j,\ \mu)]\begin{bmatrix} \dfrac{\frac{y_{j+1}-y_j}{h_{j+1}}-\frac{y_j-y_{j-1}}{h_j}}{h_{j+1}} & \dfrac{\frac{y_{j+2}-y_{j+1}}{h_{j+2}}-\frac{y_{j+1}-y_j}{h_{j+1}}}{h_{j+1}} \end{bmatrix}^T$$

where

$$H_1(x,\ j,\ \lambda)=-6\lambda\frac{x-x_j}{h_{j+1}}+4\lambda,\ H_2(x,\ j,\ \mu)=6(1-\mu)\frac{x-x_j}{h_{j+1}}-2(1-\mu). \tag{12}$$

Note that (6) and (9) in form, and (8) and (11) in nature, bear striking resemblance to their counterparts of Bézier-Bernstein curves and B-spline curves. Furthermore, as $\lambda\neq 0$ and $\mu\neq 1$, $M_j(\lambda,\ \mu)$ and $\overline{M}_j(\lambda,\ \mu)$ are all non-singular.

3. The Geometrical Properties of Curve (6)

For ease of manipulating the curve in CAGD, we now show how to produce varieties of curves from (6) by selecting proper control coefficients λ and μ. (In the following cases, curves pass through all the data points $P_j\ (x_j,\ y_j)$, $j=1,\ 2,\ \cdots,\ n$.)

(I) If the curve derived from (6) is C^1-continuous, set $\lambda=\mu$.

(II) If the curve derived from (6) is C^1-continuous and singly convex, set $\lambda=\mu$ satisfying

$$\frac{\mu}{2(1-\mu)}<\frac{A_{j+1}}{A_j}<\frac{2\mu}{1-\mu},\quad j=1,\ 2,\ \cdots,\ n-1,\quad 0<\mu<1,$$

where

$$A_j=\frac{y_{j+1}-y_j}{h_{j+1}}-\frac{y_j-y_{j-1}}{h_j},\quad j=2,\ 3,\ \cdots,\ n-1.$$

Substitute μ thus obtained into (3) or (4) to evaluate $P_0(x_0,\ y_0)$, $P_{n+1}(x_{n+1}, y_{n+1})$ and finally A_1, A_n. If A_1, A_n satisfy the following inequalities respectively

$$\frac{\mu}{2(1-\mu)}<\frac{A_2}{A_1}<\frac{2\mu}{1-\mu}\quad\text{and}\quad\frac{\mu}{2(1-\mu)}<\frac{A_n}{A_{n-1}}<\frac{2\mu}{1-\mu}$$

then the curve derived from (6) is a smooth curve with single convexity and C^1-continuity, which passes through all the data points and satisfies the end conditions.

If the end conditions are not given, $P_0(x_0,\ y_0)$ and $P_{n+1}(x_{n+1},\ y_{n+1})$, and consequently A_1 and A_n can be evaluated from (5).

(III) If we need a C^1-continuous curve with a straight line segment between P_k and P_{k+1} (Fig. 1), let

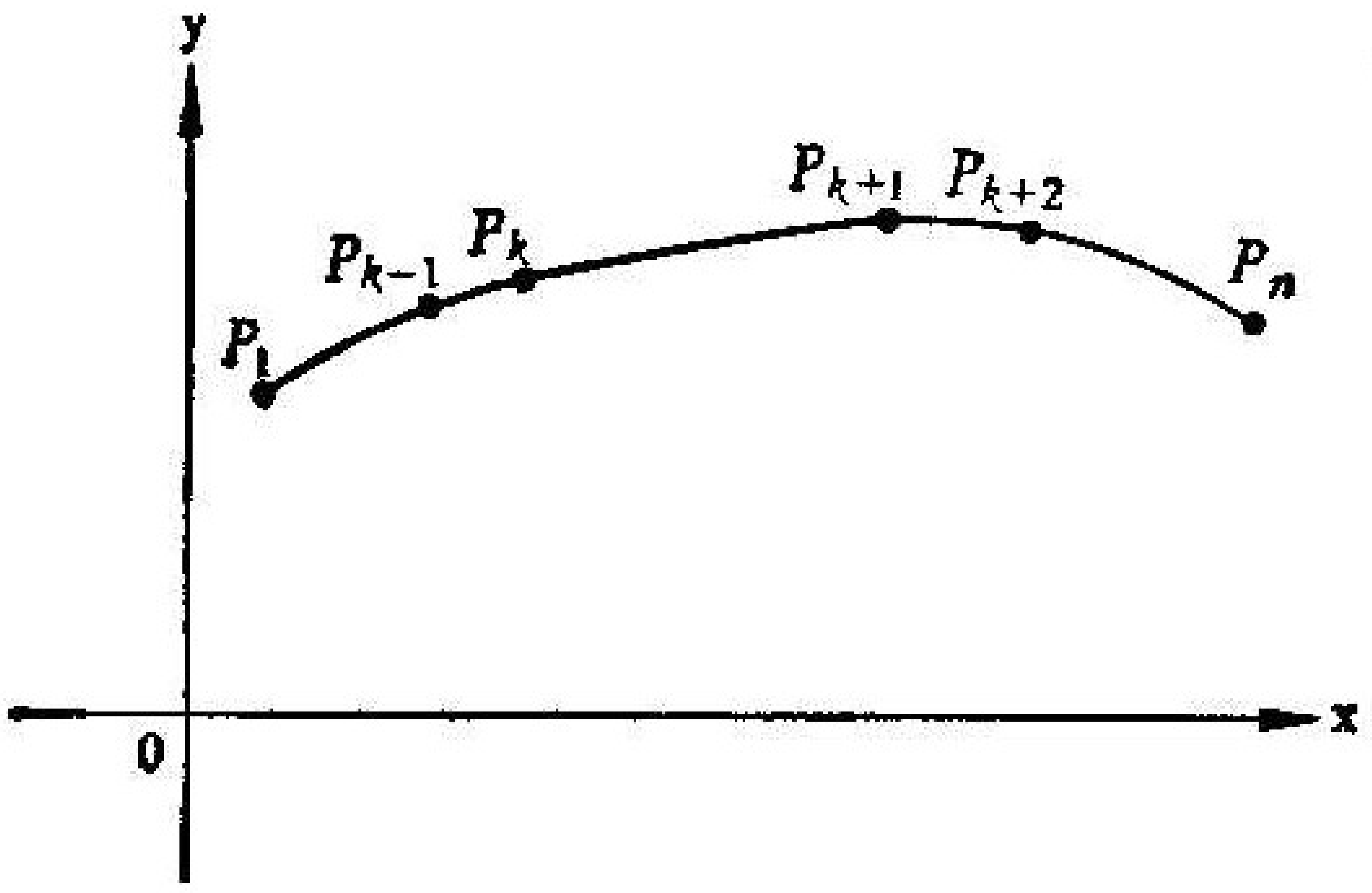

Fig. 1

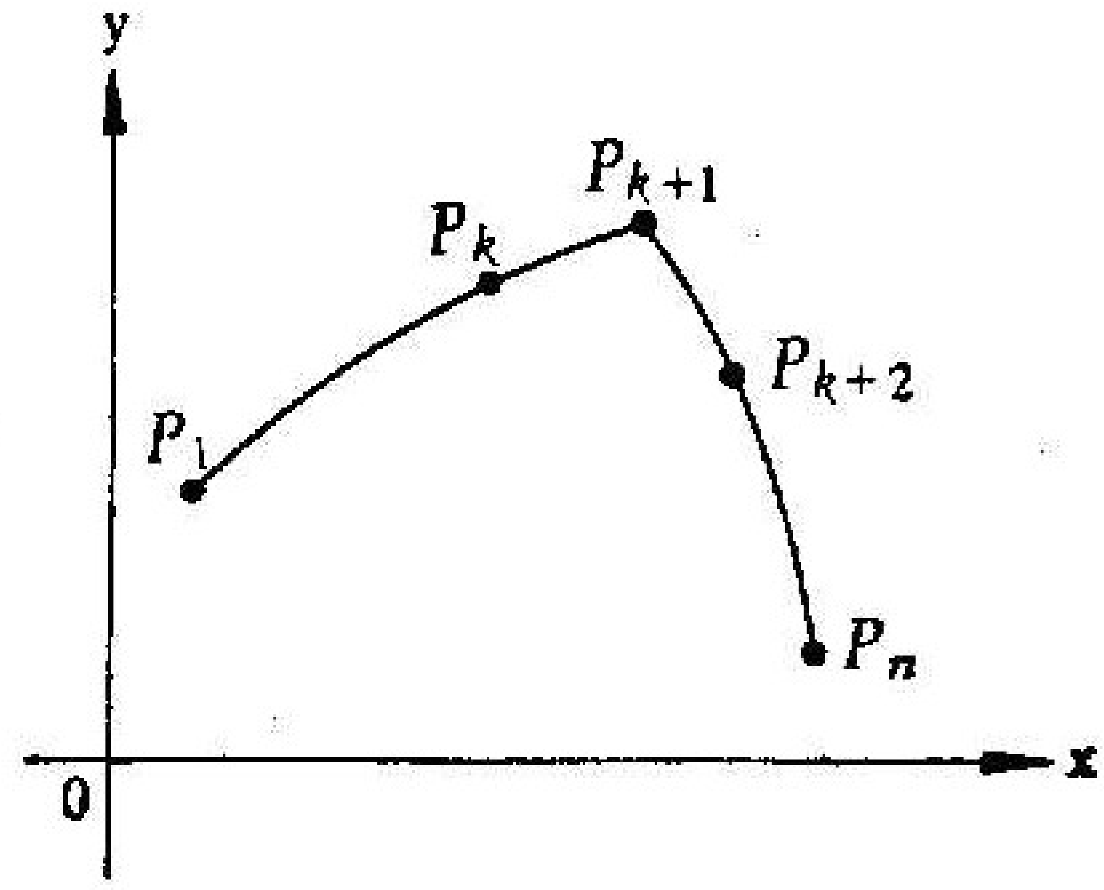

Fig. 2

$$\begin{aligned} &\lambda=0,\ \mu=1 && \text{in } M_k(\lambda,\ \mu),\\ &\mu=0,\ \lambda=\bar{\mu} && \text{in } M_{k-1}(\lambda,\ \mu),\\ &\lambda=1,\ \mu=\bar{\mu} && \text{in } M_{k+1}(\lambda,\ \mu),\\ &\lambda=\mu=\bar{\mu} \ \text{ with } && j=\begin{cases} k-1,\\ k,\\ k+1. \end{cases} \end{aligned}$$

(IV) If we want the curve to have a cusp at P_{k+1} and is C^1-continuous on $[x_1, x_{k+1})$, $(x_{k+1},\ x_n]$ (Fig. 2), we should take $\lambda=\mu=\mu_k$ on $[x_1,\ x_{k+1})$ and $\lambda=\mu=\mu_{k+1}$ on $(x_{k+1},\ x_n]$ with $\mu_k \neq \mu_{k+1}$.

(V) To require straight line segments between P_k, P_{k+1} and between P_{k+1}, P_{k+2}, a cusp at P_{k+1}, and C^1-continuity on $(x_1,\ x_{k+1})$ and $(x_{k+1},\ x_n)$ (Fig. 3), we take

$$\begin{aligned} &\lambda=0,\ \mu=1 && \text{in } M_k(\lambda,\ \mu),\ M_{k+1}(\lambda,\ \mu),\\ &\mu=0,\ \mu=\bar{\mu} && \text{in } M_{k-1}(\lambda,\ \mu),\\ &\lambda=1,\ \mu=\bar{\bar{\mu}} && \text{in } M_{k+2}(\lambda,\ \mu),\\ &M_j(\lambda,\ \mu)=M_j(\bar{\mu},\ \bar{\mu}) && \text{with } \ j<k-1,\\ &M_j(\lambda,\ \mu)=M_j(\bar{\bar{\mu}},\ \bar{\bar{\mu}}) && \text{with } \ j>k+2 \end{aligned}$$

and

$$0<\bar{\mu}<1,\ 0<\bar{\bar{\mu}}<1.$$

(VI) To require a C^1-continuous curve with an inflexion on $[x_k,\ x_{k+1}]$ (Fig. 4), we take

$$\begin{aligned} &\lambda=\bar{\mu},\ \mu=1 && \text{in } M_k(\lambda,\ \mu),\\ &\lambda=1,\ \mu=\bar{\mu} && \text{in } M_{k+1}(\lambda,\ \mu), \end{aligned}$$

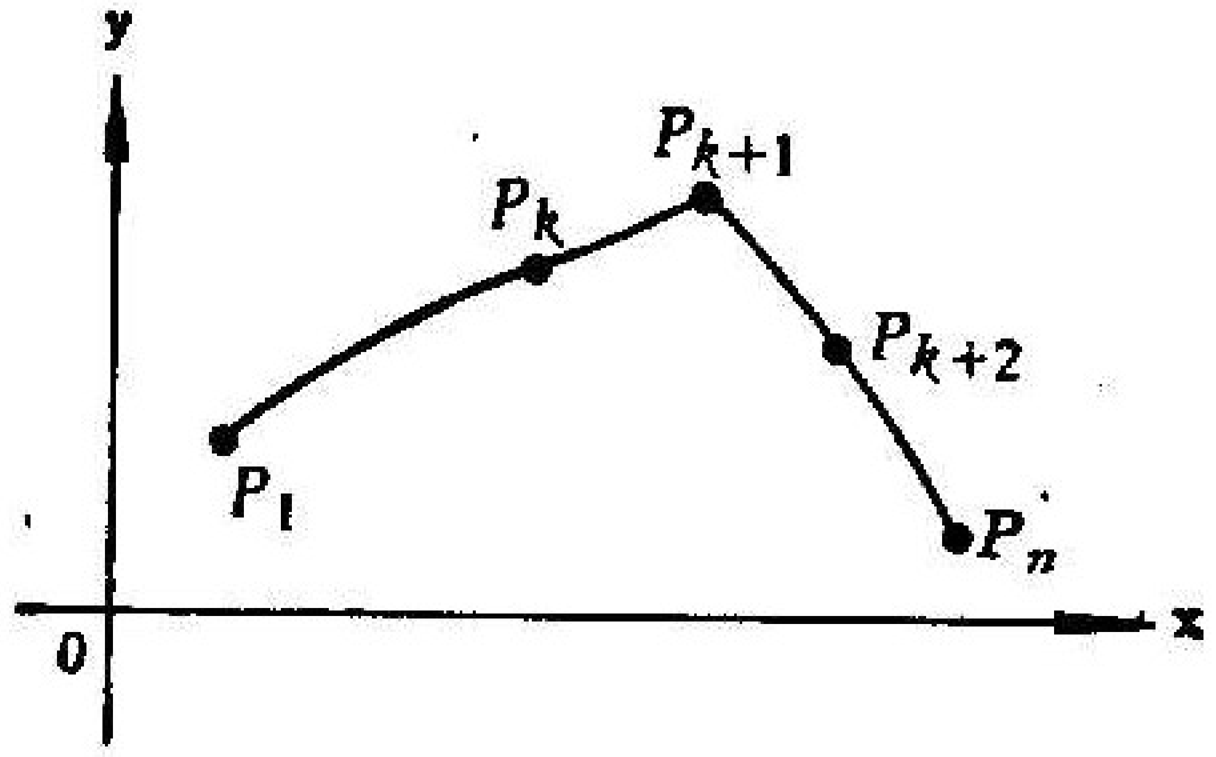

Fig. 3

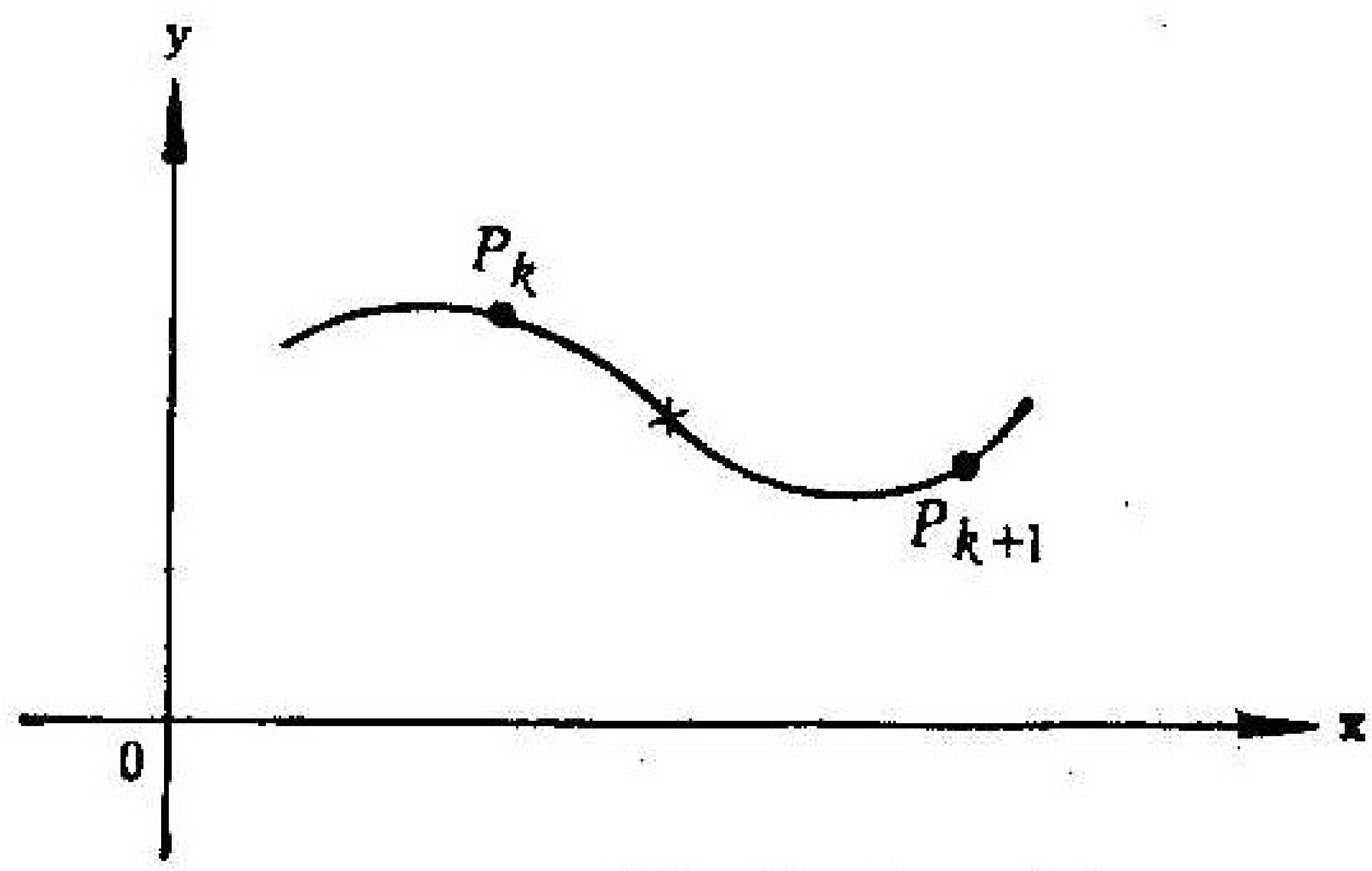

Fig. 4

$$\lambda=\mu=\bar{\mu} \quad \text{as} \quad j \neq \begin{cases} k, \\ k+1. \end{cases}$$

The above six cases can be justified by substituting proper λ, μ into (6). Figs. 1—4 are plotted out by an automatic drafting machine.

There is no difficulty in extending the treatment to the cases of several straight line segments, cusps and points of inflexion.

4. Extension to Surfaces

If a set of points in space is given by $P_{ij}(z_i, x_j, y_{ij})$, $i=1, 2, \cdots, m$; $j=1,2,\cdots,n$, extensions can be made on the curves to produce a surface in product form

$$\begin{aligned} y=s(z, x)=[l_1(z, i, \alpha) \quad l_2(z, i, \alpha, \beta) \quad l_3(z, i, \alpha, \beta) \quad l_4(z, i, \beta)]A \\ [l_1(x, j, \lambda) \quad l_2(x, j, \lambda, \mu) \quad l_3(x, j, \lambda, \mu) \quad l_4(x, j, \mu)]^T \end{aligned} \tag{13}$$

$$z_i \leqslant z \leqslant z_{i+1},\ i=1, 2, \cdots, m-1;\ x_j \leqslant x \leqslant x_{j+1},\ j=1, 2, \cdots, n-1,$$

where

$$A=\begin{bmatrix} y_{i-1,j-1} & y_{i-1,j} & y_{i-1,j+1} & y_{i-1,j+2} \\ y_{i,j-1} & y_{i,j} & y_{i,j+1} & y_{i,j+2} \\ y_{i+1,j-1} & y_{i+1,j} & y_{i+1,j+1} & y_{i+1,j+2} \\ y_{i+2,j-1} & y_{i+2,j} & y_{i+2,j+1} & y_{i+2,j+2} \end{bmatrix},$$

$$l_1(z, i, \alpha)=\alpha\frac{k_{i+1}}{k_i}\left(-\left(\frac{z-z_i}{k_{i+1}}\right)^3+\alpha\left(\frac{z-z_i}{k_{i+1}}\right)^2-\left(\frac{z-z_i}{k_{i+1}}\right)\right),$$

$$\begin{aligned} l_2(z, i, \alpha, \beta)=&\left(1+\alpha\frac{k_{i+1}}{k_i}+\alpha-\beta\right)\left(\frac{z-z_i}{k_{i+1}}\right)^3 \\ &+\left(-1-2\alpha-2\alpha\frac{k_{i+1}}{k_i}+\beta\right)\left(\frac{z-z_i}{k_{i+1}}\right)^2 \\ &+\left(\alpha+\alpha\frac{k_{i+1}}{k_i}-1\right)\frac{z-z_i}{k_{i+1}}+1, \end{aligned}$$

$$\begin{aligned} l_3(z, i, \alpha, \beta)=&\left(\beta\frac{k_{i+1}}{k_{i+2}}-\frac{k_{i+1}}{k_{i+2}}-1+\beta+\alpha\right)\left(\frac{z-z_i}{k_{i+1}}\right)^3 \\ &+\left(1-\beta\frac{k_{i+1}}{k_{i+2}}+\frac{k_{i+1}}{k_{i+2}}-\beta+2\alpha\right)\left(\frac{z-z_i}{k_{i+1}}\right)^2+(1-\alpha)\frac{z-z_i}{k_{i+1}}, \end{aligned}$$

$$l_4(z, i, \beta)=(1-\beta)\frac{k_{i+1}}{k_{i+2}}\left(\left(\frac{z-z_i}{k_{i+1}}\right)^3-\left(\frac{z-z_i}{k_{i+1}}\right)^2\right),$$

$$k_{i+1}=z_{i+1}-z_i, \quad i=1, 2, \cdots, m-1;$$

where α, β are also control coefficients satisfying $0 \leqslant \alpha \leqslant 1, 0 \leqslant \beta \leqslant 1$, while $l_1(x, j, \lambda)$, $l_2(x, j, \lambda, \mu)$, $l_3(x, j, \lambda, \mu)$, $l_4(x, j, \mu)$ remain unchanged as in (7).

Obviously, surfaces with corresponding characteristics can be readily developed, as in § 3, by giving λ, μ, α, β their proper values.

Fig. 5 shows a surface patch obtained by formula (13) and drawn by a NC drafting machine.

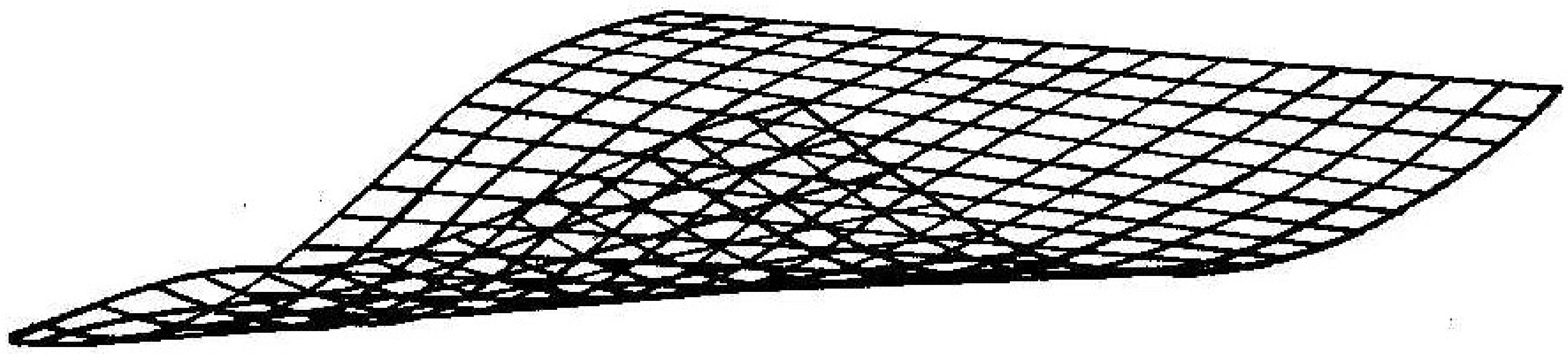

Fig. 5

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