GENERALIZED BERNSTEIN-BÉZIER POLYNOMIALS*

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In Computer Aided Geometric Design, the following functions

$$f_{n,0}(x)=1,$$
$$f_{n,i}(x)=\frac{(-x)^i}{(i-1)!}\frac{d^{i-1}}{dx^{i-1}}\left[\frac{(1-x)^n-1}{x}\right],\quad i=1,\ 2,\ \cdots,\ n \tag{1}$$

are known as the nth Bézier basis functions [1],[2]. The analytical properties of these functions have been studied by many authors. It is proved in [3] that

$$f_{n,i}(x)=J_{n,i}(x)+J_{n,i+1}(x)+\cdots+J_{n,n}(x), \tag{2}$$

where $J_{n,i}$ stands for $\binom{n}{i}x^i(1-x)^{n-i}$, $i=0,\ 1,\ \cdots,\ n$, the nth Bernstein basis function. Simple calculations show that

$$f_{n,i}(x)-f_{n,i+1}(x)=J_{n,i}(x), \tag{3}$$
$$f'_{n,i}(x)=nJ_{n-1,i-1}(x),\quad i=1,\ 2,\ \cdots,\ n. \tag{4}$$

It is clear from (3) and (4) that

$$f_{n,1}(x)>f_{n,2}(x)>\cdots>f_{n,n}(x),\quad x\in(0,\ 1), \tag{5}$$

and that $f_{n,i}(x)$, $i=1,\ 2,\ \cdots,\ n$, increases strictly from 0 to 1 on [0, 1].

For each function $\varphi(x)$ defined on [0, 1] and each real number $\alpha>0$, we define

$$B_{n,\alpha}(\varphi;x)=\varphi(0)+\sum_{i=1}^{n}\left[\varphi\left(\frac{i}{n}\right)-\varphi\left(\frac{i-1}{n}\right)\right]f^{\alpha}_{n,i}(x), \tag{6}$$

or equivalently

$$B_{n,\alpha}(\varphi;x)=\sum_{i=0}^{n}\varphi\left(\frac{i}{n}\right)[f^{\alpha}_{n,i}(x)-f^{\alpha}_{n,i+1}(x)], \tag{7}$$

where $f_{n,n+1}=0$. In the case $\alpha=1$, we see from (7) and (3) that $B_{n,1}(\varphi;x)$ is just the nth Bernstein polynomial of $\varphi(x)$. (6) and (7) are called the generalized Bernstein-Bézier polynomial of $\varphi(x)$, although they may fail to be polynomials when α is not a positive integer.

In this paper, the uniform convergence

$$\lim_{n\to\infty}B_{n,\alpha}(\varphi;x)=\varphi(x)$$

is established for $\varphi(x)$ continuous on [0, 1] and for each $\alpha>0$. And a theorem similar to that of Kelisky and Rivlin for the iterates of Bernstein operators is proved.

A proof of the uniform convergence of $B_{n,\alpha}(\varphi)$ is also given, which is elementary but rather tedious. Professor Chen Xiru points out that $f_{n,i}(x)$ represents the probability that an event A occurs i or more than i times in n independent trials, where x is the probability that A occurs in a given trial, as shown by (2). He also

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indicates that by the Tchebichev inequality ([6], p. 11) we have for arbitrarily given $\varepsilon>0$

$$\lim_{n\to\infty} f_{n,i}(x)=\begin{cases}1, & \text{for } i\leqslant n(x-\varepsilon),\\ 0, & \text{for } i\geqslant n(x+\varepsilon)\end{cases} \tag{8}$$

uniformly for $x\in[0, 1]$, and that the first lemma of this paper follows immediately from the fact that $0\leqslant f_{n,i}(x)\leqslant 1$.

The following two identities are useful in the sequel:

$$\frac{1}{n}\sum_{i=1}^{n} f_{n,i}(x)=x, \tag{9}$$

$$\frac{1}{n^2}\sum_{i=1}^{n} i f_{n,i}(x)=\frac{x}{n}+\left(1-\frac{1}{n}\right)\frac{x^2}{2}. \tag{10}$$

Since by (4) we have

$$\sum_{i=1}^{n} f'_{n,i}(x)=n\sum_{i=1}^{n} J_{n-1,i-1}(x)=n$$

and $f_{n,i}(0)=0$, (9) is proved. Similarly we have

$$\sum_{i=1}^{n} i f'_{n,i}(x)=n\sum_{i=1}^{n} iJ_{n-1,i-1}(x)=n\sum_{i=1}^{n} i[f_{n-1,i-1}(x)-f_{n-1,i}(x)]$$

$$=n\sum_{i=0}^{n-1} f_{n-1,i}(x)=n[1+(n-1)x].$$

Hence (10) follows. We are going to prove the following

Lemma 1. *For each real number $\alpha>0$, we have*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} f^{\alpha}_{n,i}(x)=x \tag{11}$$

uniformly in $[0, 1]$.

Proof. Assume $\alpha\geqslant 1$. For arbitrarily given real numbers $\varepsilon>0$ and $\delta>0$, there exists a positive integer $N=N(\varepsilon, \delta)$ by (8) such that

$$0\leqslant 1-f^{\alpha-1}_{n,i}(x)<\delta, \quad \text{if } i\leqslant n(x-\varepsilon),$$

$$0\leqslant f_{n,i}(x)<\delta, \quad \text{if } i\geqslant n(x+\varepsilon),$$

for $x\in[0, 1]$ and $n>N$. Hence we have

$$0\leqslant x-\frac{1}{n}\sum_{i=1}^{n} f^{\alpha}_{n,i}(x)=\frac{1}{n}\sum_{i=1}^{n} f_{n,i}(x)[1-f^{\alpha-1}_{n,i}(x)]$$

$$=\frac{1}{n}\Big[\sum_{i\leqslant n(x-\varepsilon)}+\sum_{i\geqslant n(x+\varepsilon)}+\sum_{n(x-\varepsilon)<i<n(x+\varepsilon)}\Big].$$

With the last three terms denoted by Σ_1, Σ_2, Σ_3 respectively, the following estimates are easily obtained

$$0\leqslant\Sigma_1\leqslant\frac{\delta}{n}\sum_{i\leqslant n(x-\varepsilon)} f_{n,i}(x)\leqslant\frac{\delta}{n}\sum_{i=1}^{n} f_{n,i}(x)\leqslant\delta,$$

$$0\leqslant\Sigma_2\leqslant\frac{\delta}{n}\sum_{i\geqslant n(x+\varepsilon)}[1-f^{\alpha-1}_{n,i}(x)]\leqslant\frac{\delta}{n}\sum_{i=1}^{n} 1=\delta,$$

$$0\leqslant\Sigma_3\leqslant\frac{1}{n}\sum_{n(x-\varepsilon)<i<n(x+\varepsilon)} 1\leqslant 2\varepsilon.$$

Hence the lemma is proved for $\alpha\geqslant 1$. It remains to consider the case $0<\alpha<1$. Since we have

$$0\leqslant\frac{1}{n}\sum_{i=1}^{n}f_{n,i}^{\alpha}(x)-x=\frac{1}{n}\sum_{i=1}^{n}f_{n,i}^{\alpha}(x)[1-f_{n,i}^{1-\alpha}(x)]$$

by the same reasoning we see that lemma 1 is still valid for $0<\alpha<1$.

Lemma 2. *For each real number* $\alpha>0$, *we have*

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{i=1}^{n}if_{n,i}^{\alpha}(x)=x^2/2 \tag{12}$$

uniformly on [0, 1].

Proof. Since

$$\frac{1}{n^2}\left|\sum_{i=1}^{n}if_{n,i}(x)-\sum_{i=1}^{n}if_{n,i}^{\alpha}(x)\right|\leqslant\frac{1}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)|f_{n,i}(x)-f_{n,i}^{\alpha}(x)|$$
$$\leqslant\frac{1}{n}\sum_{i=1}^{n}|f_{n,i}(x)-f_{n,i}^{\alpha}(x)|=\left|x-\frac{1}{n}\sum_{i=1}^{n}f_{n,i}^{\alpha}(x)\right|,$$

lemma 2 follows directly by (10) and lemma 1.

It is clear that $B_{n,\alpha}(1;x)=1$ by the definition (6). Lemma 1 and lemma 2 imply respectively that

$$\lim_{n\to\infty}B_{n,\alpha}(x;x)=x$$

and

$$\lim_{n\to\infty}B_{n,\alpha}(x^2;x)=x^2$$

uniformly on [0, 1]. By (5) and (7) we see that $B_{n,\alpha}$ is a linear and positive operator. Invoking Korovkin's theorem[4] we conclude

Theorem 1. *For any function* $\varphi(x)$ *continuous in* [0, 1] *and any real number* $\alpha>0$, *we have*

$$\lim_{n\to\infty}B_{n,\alpha}(\varphi;x)=\varphi(x)$$

uniformly on [0, 1].

We now turn to the discussion of the limit of iterates of the operator $B_{n,\alpha}$. If we introduce the following matrix notations

$$\left[\Delta\varphi\left(\frac{1}{n}\right)\right]=\left[\varphi\left(\frac{1}{n}\right)-\varphi(0),\ \varphi\left(\frac{2}{n}\right)-\varphi\left(\frac{1}{n}\right),\ \cdots,\ \varphi(1)-\varphi\left(\frac{n-1}{n}\right)\right],$$

$$F_n^{\alpha}(x)=\begin{bmatrix}f_{n,1}^{\alpha}(x)\\ \vdots\\ f_{n,n}^{\alpha}(x)\end{bmatrix},$$

$$K_{n,\alpha}=\left[f_{n,i}^{\alpha}\left(\frac{j}{n}\right)-f_{n,i}^{\alpha}\left(\frac{j-1}{n}\right)\right]_{i,j=1,2,\cdots,n},$$

then (6) can be rewritten as

$$B_{n,\alpha}(\varphi;x)=\varphi(0)+\left[\Delta\varphi\left(\frac{1}{n}\right)\right]F_n^{\alpha}(x)$$

and

$$B_{n,\alpha}^2(\varphi;x)=B_{n,\alpha}[B_{n,\alpha}(\varphi;x);x]=\varphi(0)+\left[\Delta\varphi\left(\frac{1}{n}\right)\right]K_{n,\alpha}F_n^{\alpha}(x).$$

In general we have

$$B_{n,\alpha}^{m+1}(\varphi;x)=\varphi(0)+\left[\Delta\varphi\left(\frac{1}{n}\right)\right]K_{n,\alpha}^{m}F_{n}^{\alpha}(x),\qquad m=0,\,1,\,2,\,\cdots.\tag{13}$$

From (13) we see that the investigation of the limit of iterates of the operator $B_{n,\alpha}$ is shifted to that of the limit of the matrix sequence $K_{n,\alpha}^{m}$ as $m\to\infty$. If we put

$$P=\begin{bmatrix}1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1\\ & & & & 1\end{bmatrix},$$

then

$$P^{-1}=\begin{bmatrix}1 & 1 & \cdots & 1\\ 0 & 1 & \cdots & 1\\ \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & \cdots & 1\end{bmatrix}.$$

Since

$$K_{n,\alpha}=\left[f_{n,i}^{\alpha}\left(\frac{j}{n}\right)\right]P=P^{-1}P\left[f_{n,i}^{\alpha}\left(\frac{j}{n}\right)\right]P,$$

we can say that the matrix $K_{n,\alpha}$ is similar to the matrix

$$P\left[f_{n,i}^{\alpha}\left(\frac{j}{n}\right)\right]=P\begin{bmatrix}f_{n,1}^{\alpha}\left(\frac{1}{n}\right), & f_{n,1}^{\alpha}\left(\frac{2}{n}\right), & \cdots, & f_{n,1}^{\alpha}\left(\frac{n-1}{n}\right), & 1\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ f_{n,n}^{\alpha}\left(\frac{1}{n}\right), & f_{n,n}^{\alpha}\left(\frac{2}{n}\right), & \cdots, & f_{n,n}^{\alpha}\left(\frac{n-1}{n}\right), & 1\end{bmatrix}$$

$$=\begin{bmatrix}f_{n,1}^{\alpha}\left(\frac{1}{n}\right)-f_{n,2}^{\alpha}\left(\frac{1}{n}\right), & \cdots, & f_{n,1}^{\alpha}\left(\frac{n-1}{n}\right)-f_{n,2}^{\alpha}\left(\frac{n-1}{n}\right), & 0\\ \cdots & \cdots & \cdots & \cdots\\ f_{n,n-1}^{\alpha}\left(\frac{1}{n}\right)-f_{n,n}^{\alpha}\left(\frac{1}{n}\right), & \cdots, & f_{n,n-1}^{\alpha}\left(\frac{n-1}{n}\right)-f_{n,n}^{\alpha}\left(\frac{n-1}{n}\right), & 0\\ f_{n,n}^{\alpha}\left(\frac{1}{n}\right), & \cdots, & f_{n,n}^{\alpha}\left(\frac{n-1}{n}\right), & 1\end{bmatrix}$$

whose elements are all nonnegative by (5) and the sum of all the elements in the jth column is $f_{n,1}^{\alpha}\left(\frac{j}{n}\right)<1$, where $j=1,\,2,\,\cdots,\,n-1$. We have shown that the matrix $K_{n,\alpha}$ has 1 as its isolated eigenvalue and other eigenvalues are less than 1 by modulus. Since $K_{n,\alpha}$ is a row stochastic matrix, it has the column vector whose components are all 1 as an eigenvector corresponding to the eigenvalue 1. By the theory of canonical form we know that the Jordan block associated with the isolated eigenvalue 1 is the 1×1 submatrix (1). Hence there exists a nonsingular matrix M whose elements in the first column are all 1 such that

$$K_{n,\alpha}=M\begin{bmatrix}1 & 0\\ 0 & Q\end{bmatrix}M^{-1},$$

where Q is an $(n-1)\times(n-1)$ matrix with spectral radius less than 1, thus

$$\lim_{m\to\infty} K^m_{n,\alpha}=\lim_{m\to\infty} M\begin{bmatrix}1 & 0\\ 0 & Q^m\end{bmatrix}M^{-1}$$

$$=M\begin{bmatrix}1 & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & \cdots & 0\end{bmatrix}M^{-1}=\begin{bmatrix}\lambda_1 & \lambda_2 & \cdots & \lambda_n\\ \cdots & \cdots & \cdots & \cdots\\ \lambda_1 & \lambda_2 & \cdots & \lambda_n\end{bmatrix},$$

where $[\lambda_1, \lambda_2, \cdots, \lambda_n]$ is the first row of M^{-1}. Since the limit matrix should still be row stochastic, we have

$$\lambda_1+\lambda_2+\cdots+\lambda_n=1. \tag{14}$$

If λ_i is replaced by $\lambda_i(\alpha)$, $i=1, 2, \cdots, n$, we can say that $[\lambda_1(\alpha), \cdots, \lambda_n(\alpha)]^\tau$ is a normalized eigenvector associated with the maximal eigenvalue 1 of the matrix $K^\tau_{n,\alpha}$, where τ denotes the transpose operation of matrices. $[\lambda_1(\alpha), \cdots, \lambda_n(\alpha)]^\tau$ is called the normalized maximal positive eigenvector of $K^\tau_{n,\alpha}$. By normalization we mean that (14) is satisfied by the components of the eigenvector.

Let $m\to\infty$ in both sides of (13) we obtain

Theorem 2. *For any function $\varphi(x)$ defined in $[0, 1]$, we have*

$$\lim_{m\to\infty} B^m_{n,\alpha}(\varphi; x)=\varphi(0)+[\varphi(1)-\varphi(0)]\sum_{i=1}^{n}\lambda_i(\alpha)f^\alpha_{n,i}(x) \tag{15}$$

uniformly on $[0, 1]$, where $[\lambda_1(\alpha), \cdots, \lambda_n(\alpha)]^\tau$ is the normalized maximal positive eigenvector of the matrix $K^\tau_{n,\alpha}$.

Consider the special case $\alpha=1$. By (9), the sum of the jth column of $K_{n,1}$

$$\sum_{i=1}^{n}\left[f_{n,i}\left(\frac{j}{n}\right)-f_{n,i}\left(\frac{j-1}{n}\right)\right]=n\left(\frac{j}{n}\right)-n\left(\frac{j-1}{n}\right)=1, \quad j=1, 2, \cdots, n,$$

thus the matrix $K_{n,1}$ is doubly stochastic. Hence

$$\lambda_1(1)=\lambda_2(1)=\cdots=\lambda_n(1)=1/n$$

and by (9) again (15) becomes

$$\lim_{m\to\infty} B^m_{n,1}(\varphi; x)=\varphi(0)+[\varphi(1)-\varphi(0)]x.$$

This result was established for the Bernstein operators by Kelisky and Rivlin in 1967[5].

As a direct consequence of theorem 2, we can determine all fixed points of the operator $B_{n,\alpha}$.

Theorem 3. *The set of all fixed points of the operator $B_{n,\alpha}$ consists of the following functions*

$$C_1+C_2\sum_{i=1}^{n}\lambda_i(\alpha)\, f^\alpha_{n,i}(x),$$

where C_1 and C_2 are arbitrary constants.

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