LOCAL EXPLICIT MANY-KNOT SPLINE HERMITE APPROXIMATION SCHEMES*

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Abstract

If $f^{(i)}(\alpha)$ ($\alpha=a$, b, $i=0, 1, \cdots, k-2$) are given, then we get a class of the Hermite approximation operator $Qf=F$ satisfying $F^{(i)}(\alpha)=f^{(i)}(\alpha)$, where F is the many-knot spline function whose knots are at points $y_i: a=y_0<y_1<\cdots<y_{k-1}=b$, and $F\in P_k$ on $[y_{i-1}, y_i]$. The operator is of the form $Qf:=\sum_{i=0}^{k-2}[f^{(i)}(a)\phi_i+f^{(i)}(b)\psi_i]$. We give an explicit representation of ϕ_i and ψ_i in terms of B-splines $N_{i,k}$. We show that Q reproduces appropriate classes of polynomials.

1. Introduction

Some authors considered operators of the form $Qf=\sum\lambda_i f N_{i,k}$, where $\{N_{i,k}\}$ is a sequence of B-splines and $\{\lambda_i\}$ is a sequence of linear functionals. The variation diminishing method of Schoenberg ([9], [5], [6]) and the quasi-interpolant of de Boor and Fix are well-known. Such approximation schemes have several important advantages over spline interpolation. They can be constructed directly without matrix inversion, and local error bounds can be obtained naturally. Qi considered the so-called many-knot splines which have many more knots than degrees of freedom and constructed the cardinal spline $Qf=\sum f(x_i)q_{i,k}$, where $q_{i,k}$ is made up of B-splines on a uniform partition, has small support and satisfies $q_{i,k}(x_j)=\delta_{ij}$[7]. Such an approximation operator reproduces appropriate classes of polynomials[8].

The purpose of this paper is to construct a class of many-knot explicit local polynomial spline approximation operators for Hermite interpolation of real-valued functions defined on some interval $[a, b]$.

Let P_k be a set of polynomials of degree less than k, and let

$$a=y_0<y_1<\cdots<y_{k-1}=b. \tag{1.0}$$

We define

$$\hat{S}_k:=\{g: g|_{(y_i,y_{i+1})}\in P_k,\quad i=0, 1, \cdots, k-2\}.$$

$\hat{S}_k$ is the familiar class of polynomial splines of order k with knots at the points $y_i (i=0, 1, \cdots, k-1)$.

Let $\mathscr{F}$ be a linear space of real valued functions on $[a, b]$, and suppose $\mathscr{F}$ contains the class of polynomials P_k. Given $f\in\mathscr{F}$, we construct an approximation $F(\cdot)=Qf(\cdot)$ such that

$$\mathscr{F}^{(l)}(a)=f^{(l)}(a),\ \mathscr{F}^{(l)}(b)=f^{(l)}(b),\quad l=0, 1, \cdots, k-2. \tag{1.1}$$

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In other words, set

$$Qf := \sum_{j=0}^{k-2} f^{(j)}(a)\phi_j(x) + \sum_{j=0}^{k-2} f^{(j)}(b)\psi_j(x); \tag{1.2}$$

suppose ϕ_j, ψ_j satisfying

$$\phi_j^{(i)}(a)=\delta_{ij},\quad \phi_j^{(i)}(b)=0, \tag{1.3}$$

$$\psi_j^{(i)}(a)=0,\quad \psi_j^{(i)}(b)=\delta_{ij},\qquad i,\ j=0,\ 1,\ \cdots,\ k-2. \tag{1.4}$$

If ϕ_j and ψ_j are chosen in P_{2k-2}, then the problem above has been considered (see, for instance, [1], [3], [4]), and in this case $F\in P_{2k-2}$ on $[a, b]$.

We will find a many-knot spline $F\in \hat{S}_k$ satisfying (1.1). Such many-knot cardinal splines $\{\phi_j\}$ and $\{\psi_j\}$ are of degree less than k; therefore F is also of degree less than k. We present ϕ_j and ψ_j as explicit representations.

This paper proves that the many-knot spline Hermite approximation operator Q reproduces appropriate classes of polynomials on $[a, b]$.

2. Construction of ϕ_j and ψ_j

Without loss of generality, we assume $a=0$ and $b=1$. First of all we set $k=3$ as an example.

Let ϕ_0, ϕ_1, ψ_0, ψ_1 be piecewise polynomials of degree 2 with knots $0, \frac{1}{2}, 1$ satisfying the following conditions

$$\phi_0(0)=1, \qquad \phi_1'(0)=1,$$

$$\phi_0'(0)=\phi_0(1)=\phi_0'(1)=0,\quad \phi_1(0)=\phi_1(1)=\phi_1'(1)=0,$$

$$\phi_0\left(\frac{1}{2}+0\right)=\phi_0\left(\frac{1}{2}-0\right),\quad \phi_1\left(\frac{1}{2}+0\right)=\phi_1\left(\frac{1}{2}-0\right),$$

$$\phi_0'\left(\frac{1}{2}+0\right)=\phi_0'\left(\frac{1}{2}-0\right),\quad \phi_1'\left(\frac{1}{2}+0\right)=\phi_1'\left(\frac{1}{2}-0\right),$$

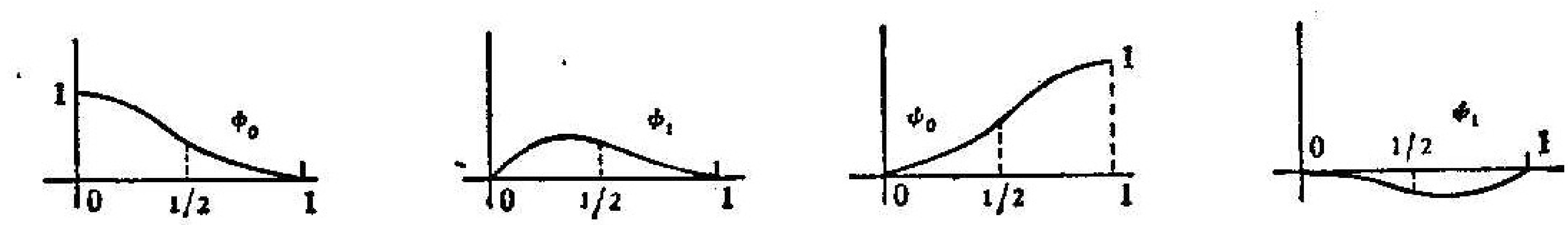
and $\psi_0(x):=\phi_0(1-x)$, $\psi_1(x):=-\phi_1(1-x)$.

Easily one gets

$$\phi_0(x)=\begin{cases}-2x^2+1, & x\in\left[0,\frac{1}{2}\right],\\ 2(x-1)^2, & x\in\left[\frac{1}{2},1\right];\end{cases}$$

$$\phi_1(x)=\begin{cases}-\frac{3}{2}x^2+x, & x\in\left[0,\frac{1}{2}\right],\\ \frac{1}{2}(x-1)^2, & x\in\left[\frac{1}{2},1\right].\end{cases}$$

Their graphs are sketched as follows



In order to consider the general case, denote

$$I_n:=\{0,\ 1,\ \cdots,\ n\}$$

$$\phi_j(x):=\sum_{\mu\in I_{k-1}} a_{j,\mu}x^\mu,\quad x\in[x_j,\ x_{j+1}],\ j\in I_{k-2}$$

(the partition is $0=x_0<x_1<x_2<\cdots<x_{k-1}=1$), and

$$\phi_j^{(l)}(x_i-0)=\phi_j^{(l)}(x_i+0),\quad i\in I_{k-2}\setminus\{0\},\ l\in I_{k-2},$$

$$\phi_0^{(i)}(0)=\delta_{i0},\quad \phi_{k-2}^{(i)}(1)=0,\quad i\in I_{k-2}.$$

Since we have $k(k-1)$ unknown coefficients $a_{j,\mu}$ with $k(k-1)$ conditions, so it seems possible to find $a_{j,\mu}$. But, it is difficult to get the explicit representations for $a_{j,\mu}$. Below we will directly present the explicit formulas for ϕ_j and ψ_j.

Here are the notations used in our discussion.

Let $X:=(x_i)$ be a nondecreasing sequence. The i-th B-spline of order k for the knot sequence (x_i) is denoted by

$$N_{i,k}(x):=(x_{i+k}-x_i)[x_i,\ \cdots,\ x_{i+k}](\cdot-x)_+^{k-1}$$

for all $x\in R$, where the symbol $[x_i,\cdots,x_{i+k}]$ denotes the k-th order divided-difference functional

$$\mathrm{sym}_\mu(a_1,\ a_2,\ \cdots,\ a_{n-1}):=\sum_{(\nu_1,\cdots,\nu_\mu)} a_{\nu_1}a_{\nu_2}\cdots a_{\nu_\mu},$$

$$\nu_j\in I_{n-1},\ \nu_i\neq\nu_j(i\neq j),$$

$$\xi_i^{(\mu)}:=\mathrm{sym}_{\mu-1}(x_{i+1},\ x_{i+2},\ \cdots,\ x_{i+k-1})\Big/\binom{k-1}{\mu-1},$$

$$\xi_i^{(1)}:=\mathrm{sym}_0(\cdots):=1.$$

From (1.0), we define

$$x_i:=y_i-1,\ x_{k-1+i}:=y_i,\quad \text{for}\quad i\in I_{k-1}. \tag{2.2}$$

Thus we get a partition on $[-1,\ 1]$ from $[0,\ 1]$:

$$-1=x_0<x_1<\cdots<x_{k-2}<x_{k-1}=0<x_k<x_{k+1}<\cdots<x_{2(k-1)}=1. \tag{2.3}$$

We construct the following functions on $[0,\ 1]$ as a special kind of combination of B-splines

$$\phi_j(x):=\frac{1}{j!}\sum_{i\in I_{k-2}}\xi_i^{(j+1)}N_{i,k}(x),\ \text{for}\ x\in[0,\ 1],\ j\in I_{k-2}. \tag{2.4}$$

Theorem 1. *The functions $\phi_j(x)$ defined in (2.4) satisfy*

$$\phi_j^{(l)}(0)=\delta_{lj}, \tag{2.5}$$

$$\phi_j^{(l)}(x)=0\ \textit{for}\ |x|\geqslant 1,\quad l,\ j\in I_{k-2}. \tag{2.6}$$

Proof. If $i\in I_{k-2}$ and $|x|\geqslant 1$, then $N_{i,k}^{(l)}(x)=0$. Therefore $\phi_j^{(l)}(x)=0$ for all $l,j\in I_{k-2}$ and $|x|\geqslant 1$. If $i\overline{\in} I_{k-2}$, then $N_{i,k}^{(l)}(0)=0$ since

$$I_{k-2}=\{i|i\in\{\cdots,\ -2,\ -1,\ 0,\ 1,\ 2,\ \cdots\},\ N_{i,k}(0)\neq 0\}.$$

By Marsden's Identity[6], for $x\in[0,\ 1]$,

$$x^{\mu-1}=\sum_{i\in I_{2k-3}}\xi_i^{(\mu)}N_{i,k}(x),\quad \mu=1,\ 2,\ \cdots,\ k. \tag{2.7}$$

Thus

$$\phi_j^{(l)}(x)\big|_{x=0}=\Big(\frac{1}{j!}\sum_{i\in I_{k-2}}\xi_i^{(j+1)}N_{i,k}(x)\Big)^{(l)}\Big|_{x=0}=\frac{1}{j!}\Big[\Big(\sum_{i\in I_{k-2}}+\sum_{i=k-1}^{2k-3}\Big)\xi_i^{(j+1)}N_{i,k}(x)\Big]^{(l)}\Big|_{x=0}$$

$$=\frac{1}{j!}(x^j)^{(l)}\Big|_{x=0}=\delta_{lj},\quad \text{for } l,\ j\in I_{k-2}.$$

Let

$$\psi_j(x):=\phi_j(x-1).$$

From (2.2), (2.3), we easily see

$$\psi_j(x)=\frac{1}{j!}\sum_{i\in I_{k-2}}\xi_i^{(j+1)}N_{i+k-1,k}(x). \tag{2.8}$$

By (2.5) we get

$$\psi_j^{(l)}(0)=0.$$

$$\psi_j^{(l)}(1)=\delta_{lj},\ \text{for } l,\ j\in I_{k-2}.$$

Examples.

$k=3$:

$$\begin{pmatrix}\phi_0\\ \phi_1\end{pmatrix}=\begin{pmatrix}1 & 1\\ \alpha-\frac{1}{2} & \alpha\end{pmatrix}\begin{pmatrix}N_{0,3}(x)\\ N_{1,3}(x)\end{pmatrix},$$

$$\alpha=\mathrm{sym}_1(y_0,\ y_1)/2=\frac{y_0+y_1}{2}=\frac{y_1}{2}.$$

When the partition is uniform, then

$$\phi_0=N_{0,3}(x)+N_{1,3}(x),$$

$$\phi_1=-\frac{1}{4}N_{0,3}(x)+\frac{1}{4}N_{1,3}(x),\quad x\in[0,\ 1].$$

$k=4$:

$$\begin{pmatrix}\phi_0\\ \phi_1\\ \phi_2\end{pmatrix}=\begin{pmatrix}1 & 1 & 1\\ \alpha_1-\frac{2}{3} & \alpha_1-\frac{1}{3} & \alpha_1\\ \Big(\alpha_2-\alpha_1+\frac{1-y_0}{3}\Big)\Big/2! & \Big(\alpha_2-\alpha_1+\frac{y_2}{3}\Big)\Big/2! & \alpha_2/2!\end{pmatrix}$$

$$\cdot\begin{pmatrix}N_{0,4}(x)\\ N_{1,4}(x)\\ N_{2,4}(x)\end{pmatrix},\quad x\in[0,\ 1],$$

where

$$\alpha_1=\mathrm{sym}_1(y_0,\ y_1,\ y_2)/3=\frac{y_0+y_1+y_2}{3},$$

$$\alpha_2=\mathrm{sym}_2(y_0,\ y_1,\ y_2)/3=\frac{y_0y_1+y_1y_2+y_2y_0}{3}.$$

In uniform case,

$$\phi_0=N_{0,4}+N_{1,4}+N_{2,4},$$

$$\phi_1=-\frac{1}{3}N_{0,4}+\frac{1}{3}N_{2,4},$$

$$\phi_2=\frac{2}{54}N_{0,4}-\frac{1}{54}N_{1,4}+\frac{2}{54}N_{2,4},\quad x\in[0,\ 1].$$

3. The Operator Q Reproduces Appropriate Classes of Polynomials

Using the functions ϕ_j and ψ_j, we have the following approximation operator

$$Qf(\cdot):=\sum_{j\in I_{k-1}}[f^{(j)}(0)\phi_j+f^{(j)}(1)\psi_j](\cdot).$$

Q defines a linear operator mapping $\mathscr{F}$ into $\hat{S}_k$.

Theorem 2. *$Qg=g$ for all $g\in P_k$.*

Proof. Let

$$\text{span}(N):=\text{span}(N_{i,k};\ i\in I_{2k-3}),$$
$$\text{span}(\phi,\ \psi):=\text{span}(\phi_j,\ \psi_j;\ j\in I_{k-2}),$$
$$S:=\{g:Qg=g\}.$$

The dimension of $\text{span}(\phi,\ \psi)$ is $2k-2$. Then both $\text{span}(N)$ and $\text{span}(\phi,\ \psi)$ are linear subspaces of $\mathscr{F}$ on $[0,\ 1]$ of dimension $2k-2$.

Obviously $$P_k\subseteq \text{span}(N),$$
i. e. $$P_k\subseteq \text{span}(\phi,\ \psi).$$

Now it is sufficient to prove that

$$S=\text{span}(\phi,\ \psi). \tag{3.1}$$

It follows from the definition of the set S and the operator Q that

$$S\subseteq \text{span}(\phi,\ \psi). \tag{3.2}$$

On the other hand, Theorem 1 implies that we have $Qf=f$ for any $f\in \text{span}(\phi,\ \psi)$. Hence

$$\text{span}(\phi,\ \psi)\subseteq S. \tag{3.3}$$

(3.2) and (3.3) mean that (3.1) is valid.

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