FINITE DIFFERENCE SOLUTIONS OF THE BOUNDARY PROBLEMS FOR THE SYSTEMS OF FERRO-MAGNETIC CHAIN*

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§ 1

In the classical study of one-dimnesional motion of ferro-magnetic chain, the so-called Landau-Lifschitz equation for the isotropic Heisenberg chain is of the form

$$s_t = s\times s_{xx} + s\times h, \tag{1}$$

where $s=(s_1, s_2, s_3)$ is a 3-dimensional vector valued unknown function, $h=(0, 0, h(t))$ and $h(t)$ is a constant or a function of t, "$\times$" denotes the cross-product operator of two 3-dimensional vectors.

Recently, a lot of works contributed to the study on the soliton solutions for Landau-Lifschitz equation, on the interactions of the soliton waves, on the properties of the infinite conservative laws and others[1-4]. The equation with the diffusion term

$$s_t = s\times s_{xx} + \nu s_{xx} \tag{2}$$

is called the spin equation. These systems also appear in the investigation of the problems of physics of the condensation state of medium. In [5] the periodic boundary problem and the initial problem for somewhat more general systems of ferro-magnetic chain

$$z_t = z\times z_{xx} + f(x, t, z) \tag{3}$$

are discussed, where $z=(u, v, w)$ and f are 3-dimensional vector valued functions. In [6], the boundary problems in rectangular domain $Q_T=\{0\leqslant x\leqslant l;\ 0\leqslant t\leqslant T\}$ for the system (3) are considered with one of the following boundary conditions (*): the first boundary condition

$$z(0, t) = z(l, t) = 0; \tag{4}$$

the second boundary condition

$$z_x(0, t) = z_x(l, t) = 0; \tag{5}$$

and the mixed boundary condition

$$z(0, t) = z_x(l, t) = 0 \tag{6}$$

or

$$z_x(0, t) = z(l, t) = 0 \tag{7}$$

and the initial condition

$$z(x, 0) = \varphi(x), \tag{8}$$

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where $\varphi(x)$ is a 3-dimensional vector valued initial function. The existence of the weak solutions of the appropriate problems for the system (3) of ferro-magnetic chain are established in [5, 6] by means of the method of vanishing of diffusion term in the corresponding spin system

$$z_t = \varepsilon z_{xx} + z \times z_{xx} + f(x, t, z). \tag{9}$$

It can be seen that the coefficient matrix of the terms of second order derivatives of the system (3) is zero-definite and is singular at $uvw=0$. So the system (3) can be regarded as a strongly degenerate parabolic system. The system (9) is a non-degenerate quasilinear parabolic system.

The purpose of this paper is to prove the solvability of the boundary problems (*), (8) for the system (3) of ferro-magnetic chain by the finite difference method. The symbol (*) denotes any given one of the boundary conditions (4), (5), (6) and (7).

Let us divide the rectangular domain Q_T into small grids by the parallel lines $x=x_j (j=0, 1, \cdots, J)$ and $t=t_n (n=0, 1, \cdots, N)$, where $x_j=jh$, $t_n=nk$ and $Jh=l$, $Nk=T$. We take the finite difference system

$$\frac{z_j^{n+1}-z_j^n}{k} = z_j^{n+1} \times \frac{\Delta_+\Delta_- z_j^{n+1}}{h^2} + f(x_j, t_{n+1}, z_j^{n+1}), \tag{3$_h$}$$

where $\Delta_+ u_j = u_{j+1} - u_j$ and $\Delta_- u_j = u_j - u_{j-1}$. The finite difference boundary conditions are as follows

$$z_0^n = z_J^n = 0; \tag{4$_h$}$$

$$z_1^n - z_0^n = z_J^n - z_{J-1}^n = 0; \tag{5$_h$}$$

$$z_0^n = z_J^n - z_{J-1}^n = 0; \tag{6$_h$}$$

$$z_1^n - z_0^n = z_J^n = 0, \tag{7$_h$}$$

where $n=1, 2, \cdots, N$. The finite difference initial condition is

$$z_j^0 = \bar{\varphi}_j, \ (j=0, 1, \cdots, J), \tag{8$_h$}$$

where $\bar{\varphi}_j = \varphi(x_j)$ $(j=0, 1, \cdots, J)$ and $\bar{\varphi}_1 = \varphi(0)$ (or $\bar{\varphi}_{J-1} = \varphi(l)$) in the case of the boundary condition $z_1^n - z_0^n = 0$ (or $z_J^n - z_{J-1}^n = 0$).

Now we make the following assumptions for the system (3) of ferro-magnetic chain and the initial 3-dimensional vector valued function $\varphi(x)$.

(I) $f(x, t, z)$ is a 3-dimensional vector valued continuous function for $(x, t, z) \in Q_T \times \mathbb{R}^3$ and satisfies the condition of semiboundedness

$$(u-v) \cdot (f(x, t, u) - f(x, t, v)) \leqslant b|u-v|^2, \tag{10}$$

where $(x, t) \in Q_T$, $u, v \in \mathbb{R}^3$ and b is a constant.

(II) For $(x, t, z) \in Q_T \times \mathbb{R}^3$, there is

$$|f(x, t, z) - f(y, t, z)| \leqslant (A|z|^3 + B)|x-y| \tag{11}$$

for $x, y \in [0, l]$, $z \in \mathbb{R}^3$, $0 \leqslant t \leqslant T$, where $A \geqslant 0$ and $B \geqslant 0$ are constants.

(III) $\varphi(x)$ is a 3-dimensional vector valued continuously differentiable function in $[0, l]$ and satisfies the appropriate boundary condition (*).

The scalar product of two 3-dimensional vectors u and v is denoted by $u \cdot v$ and $|u|^2 = u \cdot u$. For the discrete vector valued functions $\{u_j\}$ and $\{v_j\}$, we take the notations, $(u \cdot v)_h = \sum_{j=0}^{J} (u_j \cdot v_j) h$ and $\|u\|_h^2 = (u \cdot u)_h$.

§ 2

Now we are going to prove the existence of the solution $z_j^{n+1}(j=0, 1, \cdots, J)$ for the nonlinear system $(3)_h$ and $(*)_h$, where $z_j^n(j=0, 1, \cdots, J)$ are known vectors.

Lemma 1. *For the discrete functions $\{u_j\}$ and $\{v_j\}$, there is the identity*

$$\sum_{j=1}^{J-1} u_j \Delta_+ \Delta_- v_j = -\sum_{j=0}^{J-1} (\Delta_+ u_j)(\Delta_+ v_j) - u_0(v_1 - v_0) + u_J(v_J - v_{J-1}). \tag{12}$$

This lemma can be easily verified by direct calculation.

Lemma 2. *Under the condition* (I) *and* $1-2bk>0$, *the finite difference system* $(3)_h$ *and* $(*)_h$ *has at least one solution* $z_j^{n+1}(j=0, 1, \cdots, J)$, *where* z_j^n *are known.*

Proof. For any 3-dimensional vectors u_j $(j=0, 1, \cdots, J)$, we define the 3-dimensional vectors z_j $(j=0, 1, \cdots, J)$ by

$$z_j = z_j^n + \lambda \frac{k}{h^2}(u_j \times \Delta_+ \Delta_- u_j) + \lambda k f(x_j, t_{n+1}, u_j), \quad j=1, \cdots, J-1 \tag{13}$$

and z_0 and z_J are given by the boundary condition $(*)_h$, where $0 \leqslant \lambda \leqslant 1$ is a parameter. This gives a mapping $z = T_\lambda u$ of $3(J+1)$-dimensional Euclidean space into itself, where $z = \{z_j\}$ and $u = \{u_j\}$. For the existence, we need to obtain the uniform bound for all possible solutions of system

$$z_j = z_j^n + \lambda \frac{k}{h} z_j \times \Delta_+ \Delta_- z_j + \lambda k f(x_j, t_{n+1}, z_j), \quad j=1, \cdots, J-1 \tag{14}$$

and the boundary condition $(*)_h$ with respect to $0 \leqslant \lambda \leqslant 1$.

Hence taking the scalar product of z_j and (14) and summing up for $j=1, \cdots, J-1$, we have

$$\sum_{j=1}^{J-1} |z_j|^2 = \sum_{j=1}^{J-1} z_j^n \cdot z_j + \lambda k \sum_{j=1}^{J-1} z_j \cdot f(x_j, t_{n+1}, z_j), \tag{15}$$

where $z_j \cdot (z_j \times \Delta_+ \Delta_- z_j) = 0$ and $j=1, 2, \cdots, J-1$. Here

$$\begin{aligned} z_j \cdot f(x_j, t_{n+1}, z_j) &= z_j \cdot [f(x_j, t_{n+1}, z_j) - f(x_j, t_{n+1}, 0)] + z_j \cdot f(x_j, t_{n+1}, 0) \\ &\leqslant (b+\delta)|z_j|^2 + \frac{1}{4\delta}|f(x_j, t_{n+1}, 0)|^2 \end{aligned}$$

follows from the property (I), where $\delta > 0$. Then the relation (15) becomes

$$(1 - 2\lambda(b+\delta)k) \sum_{j=1}^{J-1} |z_j|^2 \leqslant \sum_{j=1}^{J-1} |z_j^n|^2 + \frac{\lambda k}{2\delta} \sum_{j=1}^{J-1} |f(x_j, t_{n+1}, 0)|^2.$$

When k is small such that $1-2bk>0$, then $z_j(j=0, 1, \cdots, J)$ are uniformly bounded with respect to the parameter $0 \leqslant \lambda \leqslant 1$. This proves the existence of the solution of the finite difference system $(3)_h$ and $(*)_h$.

§ 3

In order to establish the weak solution of the boundary problem $(*)$, (8) for the system (3) of ferro-magnetic chain, we want to estimate the solution $z_j^n(j=0, 1, \cdots, J; n=0, 1, \cdots, N)$ of the finite difference system $(3)_h$, $(*)_h$ and $(8)_h$ and its difference quotients.

Taking the scalar product of the 3-dimensional vector $z_j^{n+1}kh$ and the finite difference system

$$\frac{z_j^{n+1}-z_j^n}{k}=\frac{1}{h^2}z_j^{n+1}\times\Delta_+\Delta_-z_j^{n+1}+f(x_j,\ t_{n+1},\ z_j^{n+1}) \tag{3$_h$}$$

and summing up the resulting relations for $j=1,\ 2,\ \cdots,\ J-1$, we get

$$\|z^{n+1}\|_h=(z^{n+1}\cdot z^n)_h+k(z^{n+1}\cdot f^{n+1})_h,$$

where $f_j^{n+1}=f(x_j,\ t_{n+1},\ z_j^{n+1})$. This equality can be written in the following iterative form

$$\|z^{n+1}\|_h^2\leqslant\frac{\|z^n\|_h^2+\frac{k}{2\delta}\|f(\cdot,\ t_{n+1},\ 0)\|_h^2}{1-2(b+\delta)k}.$$

Hence we have

$$\|z^n\|_h^2\leqslant(1-2(b+\delta)k)^{-n}\left\{\|z^0\|_h^2+\frac{1}{4\delta(b+\delta)}\max_{n=0,1,\cdots,N}\|f(\cdot,\ t_{n+1},\ 0)\|_h^2\right\}.$$

Lemma 3. *Under the condition* (I) *and* $\varphi(x)\in C([0,\ l])$, $\|z^n\|_h$ *is uniformly bounded with respect to h and k for $nk\leqslant T$ and $1-2bk>0$, i. e.,*

$$\|z^n\|_h\leqslant k_1,\quad n=0,\ 1,\ \cdots,\ N, \tag{16}$$

where k_1 is independent of h and k.

Making the scalar product of $\Delta_+\Delta_-z_j^{n+1}\frac{k}{h}$ with the system (3)$_h$ and then summing up for $j=1,\ \cdots,\ J-1$, we obtain

$$\frac{1}{h}\sum_{j=1}^{J-1}(\Delta_+\Delta_-z_j^{n+1}\cdot(z_j^{n+1}-z_j^n))=\frac{k}{h}\sum_{j=1}^{J-1}(\Delta_+\Delta_-z_j^{n+1}\cdot f_j^{n+1}),\ n=0,\ 1,\ \cdots,\ N-1, \tag{17}$$

where $\Delta_+\Delta_-z_j^{n+1}\cdot(z_j^{n+1}\times\Delta_+\Delta_-z_j^{n+1})=0$. For the left hand part of this equality

$$\frac{1}{h}\sum_{j=1}^{J-1}\Delta_+\Delta_-z_j^{n+1}\cdot(z_j^{n+1}-z_j^n)=-\|W^{n+1}\|_h^2+(W^{n+1}\cdot W^n)_h, \tag{18}$$

where $W_j=\frac{\Delta_+z_j}{h}\ (j=0,\ 1,\ \cdots,\ J-1)$.

Now we turn to estimate the right hand part of the equality (17). This part can be written as follows

$$\frac{k}{h}\sum_{j=1}^{J-1}(\Delta_+\Delta_-z_j^{n+1}\cdot f_j^{n+1})=-\frac{k}{h}\sum_{j=0}^{J-1}(\Delta_+z_j^{n+1}\cdot\Delta_+f_j^{n+1})$$

$$-\frac{k}{h}(\Delta_+z_0^{n+1}\cdot f(0,\ t_{n+1},\ z_0^{n+1}))+\frac{k}{h}(\Delta_-z_J^{n+1}\cdot f(l,\ t_{n+1},z_J^{n+1})). \tag{19}$$

In the case of the second finite difference boundary condition (5)$_h$, $\Delta_+v_0^{n+1}=\Delta_-v_J^{n+1}=0$, the relation (19) becomes

$$\frac{k}{h}\sum_{j=1}^{J-1}(\Delta_+\Delta_-z_j^{n+1}\cdot f_j^{n+1})=-\frac{k}{h}\sum_{j=0}^{J-1}(\Delta_+z_j^{n+1},\ \Delta_+f_j^{n+1}). \tag{20}$$

In the case of the first finite difference boundary condition (4)$_h$ or the mixed finite difference boundary condition (6)$_h$ or (7)$_h$, the last two terms of (19) take the form $\Delta_+z_0^{n+1}\cdot f(0,\ t_{n+1},\ 0)$ or $\Delta_-z_J^{n+1}\cdot f(l,\ t_{n+1},\ 0)$. If the system (3) is homogeneous, i. e., simply $f(x,\ t,\ 0)\equiv0$, then (20) is also valid.

The right part of (20) can be written in the form

$$\frac{k}{h}\sum_{j=0}^{J-1}(\Delta_+z_j^{n+1}\cdot\Delta_+f_j^{n+1})=\frac{k}{h}\sum_{j=0}^{J-1}\Delta_+z_j^{n+1}\cdot[f(x_{j+1},t_{n+1},z_{j+1}^{n+1})-f(x_{j+1},t_{n+1},z_j^{n+1})]$$
$$+\frac{k}{h}\sum_{j=0}^{J-1}\Delta_+z_j^{n+1}\cdot[f(x_{j+1},t_{n+1},z_j^{n+1})-f(x_j,t_{n+1},z_j^{n+1})]. \quad (21)$$

From the property (I), we have

$$\frac{1}{h}\sum_{j=0}^{J-1}\Delta_+z_j^{n+1}\cdot[f(x_{j+1},t_{n+1},z_{j+1}^{n+1})-f(x_{j+1},t_{n+1},z_j^{n+1})]\leqslant b\|W^{n+1}\|_h^2. \quad (22)$$

According to the assumption (II), there is

$$\left|\frac{1}{h}\sum_{j=0}^{J-1}\Delta_+z_j^{n+1}\cdot[f(x_{j+1},t_{n+1},z_j^{n+1})-f(x_j,t_{n+1},z_j^{n+1})]\right|$$
$$\leqslant\sum_{j=0}^{J-1}|\Delta_+z_j^{n+1}|\cdot(A|z_j^{n+1}|^3+B)\leqslant\frac{1}{2}\|W^{n+1}\|_h^2+A^2\sum_{j=0}^{J-1}|z_j^{n+1}|^6h+B^2l. \quad (23)$$

For the second term of the right hand side of the above inequality, there is the estimation

$$\sum_{j=1}^{J-1}|z_j^{n+1}|^6h\leqslant C_1\|z^{n+1}\|_h^4(\|W^{n+1}\|_h^2+\|z^{n+1}\|_h^2). \quad (24)$$

In fact, for any discrete function $\{u_j\}$ $(j=0,1,\cdots,J)$, there is relation

$$\max_{j=0,1,\cdots,J}|u_j|\leqslant C_2\|u\|_h^{\frac{1}{2}}\left(\left\|\frac{\Delta_+u}{h}\right\|_h+\|u\|_h\right)^{\frac{1}{2}} \quad (25)$$

(see Lemma 8 in [7]). Then

$$\sum_{j=0}^{J}|u_j|^6h\leqslant\max_{j=0,1,\cdots,J}(|u_j|)^4\|u\|_h^2\leqslant C_2^4\|u\|_h^4\left(\left\|\frac{\Delta_+u}{h}\right\|_h+\|u\|_h\right)^2.$$

Finally, the relation (20) can be replaced by the following

$$\|W^{n+1}\|_h^2=(W^{n+1}\cdot W^n)_h+C_3k\|W^{n+1}\|_h^2+C_4k, \quad (26)$$

where the constants

$$C_3=\left(b+\frac{1}{2}\right)+A^2C_1\|z^{n+1}\|_h^4 \quad \text{and} \quad C_4=B^2l+A^2C_1\|z^{n+1}\|_h^6 \quad (27)$$

are independent of h and k. From (26) we have

$$(1-2C_3k)\|W^{n+1}\|_h^2\leqslant\|W^n\|_h^2+2C_4k.$$

When k is sufficient small, such that $1-2C_3k>0$, $\|W^n\|_h$ is uniformly bounded with respect to h and k. Hence we have the following lemmas.

Lemma 4. *Under the conditions* (I), (II) *and* (III), *the solution* z_j^n $(j=0,1,\cdots,J;n=0,1,\cdots,N)$ *of the finite difference system* $(3)_h$, $(5)_h$ *and* $(8)_h$, *corresponding to the second boundary problem* (5), (8) *for the system* (3) *of ferro-magnetic chain has the estimation relation*

$$\left\|\frac{\Delta_+z^n}{h}\right\|_h\leqslant K_2, \quad (28)$$

where $nk\leqslant T$, K *is sufficient small and* K_2 *is independent of* h *and* k.

Lemma 5. *Suppose that the conditions* (I), (II) *and* (III) *are satisfied and suppose that the system* (3) *is homogeneous, i. e.,* $f(x,t,0)\equiv0$. *The solutions of the finite difference system* $(3)_h$, $(4)_h$, $(8)_h$; $(3)_h$, $(6)_h$, $(8)_h$ *and* $(3)_h$, $(7)_h$, $(8)_h$, *corresponding to the first boundary prolem* (4), (8) *and the mixed boundary problem* (6), (8) *and* (7), (8) *for the system* (3) *of ferro-magnetic chain respectively have the estimation relation* (28) *for* $nk\leqslant T$ *and sufficient small* k.

Now we turn to estimate certain difference quotient in the direction k. Let $s_j^{n+1}=\sum_{i=0}^{j} z_i^{n+1}h$. From the finite difference system $(3)_h$, we have

$$\frac{s_j^{n+1}-s_j^n}{k}=\sum_{i=0}^{j}\frac{1}{h}(z_i^{n+1}\times\Delta_+\Delta_-z_i^{n+1})+\sum_{i=0}^{j}hf(x_i,t_{n+1},z_i^{n+1}).$$

By direct calculation, we see

$$z_j^{n+1}\times\Delta_+\Delta_-z_j^{n+1}=\Delta_-(z_j^{n+1}\times\Delta_+z_j^{n+1}).$$

So
$$\sum_{i=0}^{j}\frac{1}{h}(z_i^{n+1}\times\Delta_+\Delta_-z_i^{n+1})=z_j^{n+1}\times\frac{\Delta_+z_j^{n+1}}{h}-z_0^{n+1}\times\frac{\Delta_+z_0^{n+1}}{h}.$$

Since the last term equals to zero due to the finite difference boundary condition at $x=0$, there is

$$\frac{s_j^{n+1}-s_j^n}{k}=z_j^{n+1}\times\frac{\Delta_+z_j^{n+1}}{h}+\sum_{i=0}^{j}hf(x_i,\ t_{n+1},\ z_i^{n+1}).$$

Directly from (25) and the estimations (16) and (28) of Lemmas 3, 4 and 5, we know that $z_j^n(j=0,1,\cdots,J;n=0,1,\cdots,N)$ is uniformly bounded for h and k, then $f(x_j,t_{n+1},z_j^{n+1})$ is also uniformly bounded for h and k.

Lemma 6. *Under the conditions of Lemmas 4 and 5, there is the estimation*

$$\left\|\frac{s^{n+1}-s^n}{k}\right\|_h\leqslant K_3, \tag{29}$$

where $s_j=\sum_{i=0}^{j}z_ih$ $(j=0,1,\cdots,J)$ *and* K_3 *is independent of* h *and* k.

From the definition of s_j^n $(j=0,1,\cdots,J;n=0,1,\cdots,N)$, it is obvious that

$$z_j^n=\frac{\Delta_-s_j^n}{h},\quad \frac{\Delta_+z_j^n}{h}=\frac{\Delta_+\Delta_-s_j^n}{h^2}.$$

Now we have the uniform estimations

$$\|s^n\|_h,\ \left\|\frac{\Delta_-s^n}{h}\right\|_h,\ \left\|\frac{\Delta_+\Delta_-s^n}{h^2}\right\|_h,\ \left\|\frac{s^{n+1}-s^n}{k}\right\|_h\leqslant K_4 \tag{30}$$

for $n=0,1,\cdots,N$, where K_4 is independent of h and k.

Using the results of Lemma 8 in [7], we have

$$\max_{j=1,2,\cdots,J-1}|\Delta_+\Delta_-s_j^n|\leqslant K_5h^{\frac{3}{2}},\quad n=0,1,\cdots,N;$$

$$\max_{n=0,1,\cdots,N-1}|\Delta_+s_j^{n+1}-\Delta_+s_j^n|\leqslant K_6hk^{\frac{1}{4}},\quad j=0,1,\cdots,J-1.$$

Lemma 7. *Under the conditions of Lemmas 4 and 5, the solution* $z_j^n(j=0,1,\cdots,J;n=0,1,\cdots,N)$ *of the finite difference system* $(3)_h$, $(*)_h$ *and* $(8)_h$ *has the estimations*

$$\max_{j=1,2,\cdots,J-1}|\Delta_+z_j^n|\leqslant K_5h^{\frac{1}{2}},\quad n=0,1,\cdots,N \tag{31}$$

and
$$\max_{n=0,1,\cdots,N-1}|z_j^{n+1}-z_j^n|\leqslant K_6k^{\frac{1}{4}},\quad j=0,1,\cdots,J-1. \tag{32}$$

§ 4

In this section we want to prove that the boundary problems $(*)$, (8) for the systems (3) of ferro-magnetic chain has at least one solution. At first we define the weak solution of the boundary problems $(*)$, (8) for the systems (3) as follows.

Definition. *The 3-dimensional vector valued function* $z(x,t)\in L_2((0,T);$

$W_2^{(1)}(0, l))\cap C(Q_T)$ *is called the weak solution of the boundary problem* $(*)$, (8) *for the system* (3) *of ferro-magnetic chain, if for any test function*

$$g(x, t)\in G=\{g\,|\,g\in C^{(1)}(Q_T),\ g(x, T)\equiv 0\},$$

the following integral relation holds:

$$\iint_{Q_T}[g_t z-g_x(z\times z_x)+gf(x, t, z)]\,dxdt+\int_0^l g(x, 0)\varphi(x)dx=0. \tag{33}$$

Let $z_{hk}(x, t)=z_j^{n+1}$ for $(x, t)\in Q_j^n=\{jh<x\leqslant(j+1)h;\ nk<t\leqslant(n+1)k\}$ ($j=0, 1, \cdots, J-1$; $n=0, 1, \cdots, N-1$). Then $z_{hk}(x, t)$ is a 3-dimensional vector valued piecewise constant function in the rectangular domain $Q_T=\{0\leqslant x\leqslant l;\ 0\leqslant t\leqslant T\}$. Similarly we define $\bar z_{hk}(x, t)=\frac{\Delta_+ z_j^{n+1}}{h}$ in Q_j^n ($j=0, 1, \cdots, J-1; n=0, 1, \cdots, N-1$), then $\bar z_{hk}(x, t)$ is also a 3-dimensional vector valued piecewise constant function in Q_T. From Lemmas 3, 4 and 5, we have directly that thus constructed 3-dimensional vector valued functions $z_{hk}(x, t)$ and $\bar z_{hk}(x, t)$ have the estimation

$$\sup_{0\leqslant t\leqslant T}\|z_{hk}(\cdot, t)\|_{L_2(0,l)}+\sup_{0\leqslant t\leqslant T}\|\bar z_{hk}(\cdot, t)\|_{L_2(0,l)}\leqslant K_7, \tag{34}$$

where K_7 is independent of h and k.

Now we take a sequence $\{h_i, k_i\}$, such that when $i\to\infty$, $h_i^2+k_i^2\to 0$ and also $z_{h_ik_i}(x, t)$ and $\bar z_{h_ik_i}(x, t)$ converge weakly to $z(x, t)$ and $\bar z(x, t)$ in $L_p((0, T); L_2(0, l))$ respectively, where $1\leqslant p<\infty$. The norms of $z(x, t)$ and $\bar z(x, t)$ are uniformly bounded for $1\leqslant p<\infty$. Hence we have

$$\sup_{0\leqslant t\leqslant T}\|z(\cdot, t)\|_{L_2(0,l)}+\sup_{0\leqslant t\leqslant T}\|\bar z(\cdot, t)\|_{L_2(0,l)}\leqslant K_7. \tag{35}$$

this means that $z(x, t)$ and $\bar z(x, t)$ are two 3-dimensional vector valued functions belonging to $L_\infty((0, T), L_2(0, l))$.

In order to prove that $\bar z(x, t)=z_x(x, t)$, we take a smooth test function $g(x, t)$ with finite support in the open rectangular domain $\{0<x<l;\ 0\leqslant t<T\}$. By direct calculation we have

$$\sum_{n=0}^{N-1}\sum_{j=0}^{J}\Big(g_j^{n+1}\frac{z_{j+1}^{n+1}-z_j^{n+1}}{h}+z_{j+1}^{n+1}\frac{g_{j+1}^{n+1}-g_j^{n+1}}{h}\Big)hk=\sum_{n=0}^{N-1}(g_J^{n+1}z_J^{n+1}-g_0^{n+1}z_0^{n+1})h=0.$$

We define similarly the piecewise constant function $g_{hk}(x, t)$ and $\bar g_{hk}(x, t)$, corresponding to the discrete function g_j^{n+1} and $\frac{g_{j+1}^{n+1}-g_j^{n+1}}{h}$ respectively as before. Then we have the integral relation

$$\iint_{Q_T}[g_{hk}(x, t)\bar z_{hk}(x, t)+\bar g_{hk}(x, t)z_{hk}(x, t)]\,dx\,dt=0.$$

Since $g_{hk}(x, t)$ and $\bar g_{hk}(x, t)$ are uniformly convergent to $g(x, t)$ and $g_x(x, t)$ respectively as $h^2+k^2\to 0$, we obtain

$$\iint_{Q_T}[g(x, t)\bar z(x, t)+g_x(x, t)z(x, t)]\,dx\,dt=0. \tag{36}$$

Hence $z(x, t)$ have the generalized derivative $z_x(x, t)=\bar z(x, t)$.

From the estimations in Lemma 7, we see that $z_{hk}(x, t)$ not only weakly converges to $z(x, t)$, but also uniformly converges to $z(x, t)$ in the rectangular domain Q_T. Furthermore the limiting 3-dimensional vector valued function $z(x, t)\in C^{(\frac{1}{2},\frac{1}{4})}(Q_T)$.

Therefore $z(x, t)$ satisfies the initial condition (8).

Now we turn to prove that $z(x, t)$ is a weak solution of the boundary problem (*), (8) for the system (3) of ferro-magnetic chain. From the finite dfference system $(3)_h$, we can get

$$\sum_{j=1}^{J-1}\sum_{n=0}^{N-1} g_j^n \frac{z_j^{n+1}-z_j^n}{k} hk = \sum_{j=1}^{J-1}\sum_{n=0}^{N-1} g_j^n\left(z_j^{n+1}\times\frac{\Delta_+\Delta_- z_j^{n+1}}{h^2}\right)hk + \sum_{j=1}^{J-1}\sum_{n=0}^{N-1} g_j^n f_j^{n+1}hk, \tag{37}$$

where $f_j^{n+1}=f(x_j, t_{n+1}, z_j^{n+1})$. From the identities

$$g_j^n\frac{z_j^{n+1}-z_j^n}{k} = -z_j^{n+1}\frac{g_j^{n+1}-g_j^n}{k}+\frac{(g_j^{n+1}z_j^{n+1}-g_j^n z_j^n)}{k}$$

and

$$g_j^n\left(z_j^{n+1}\times\frac{\Delta_+\Delta_- z_j^{n+1}}{h^2}\right) = -\frac{\Delta_- g_j^n}{h}\left(z_{j-1}^{n+1}\times\frac{\Delta_+ z_{j-1}^{n+1}}{h}\right) + \frac{1}{h}\left[g_j^n\left(z_j^{n+1}\times\frac{\Delta_+ z_j^{n+1}}{h}\right)-g_{j-1}^n\left(z_{j-1}^{n+1}\times\frac{\Delta_+ z_{j-1}^{n+1}}{h}\right)\right],$$

we have

$$\sum_{j=1}^{J-1}\sum_{n=0}^{N-1} g_j^n\frac{z_j^{n+1}-z_j^n}{k}hk = -\sum_{j=1}^{J-1}\sum_{n=0}^{N-1}\frac{g_j^{n+1}-g_j^n}{k}z_j^{n+1}hk - \sum_{j=0}^{J-1} g_j^0 z_j^0 h + \sum_{j=0}^{J-1} g_j^N z_j^N h \tag{38}$$

and

$$\sum_{j=1}^{J-1}\sum_{n=0}^{N-1} g_j^n\left(z_j^{n+1}\times\frac{\Delta_+\Delta_- z_j^{n+1}}{h^2}\right)hk = -\sum_{j=1}^{J-1}\sum_{n=0}^{N-1}\frac{\Delta_- g_j^n}{h}\left(z_{j-1}^{n+1}\times\frac{\Delta_+ z_{j-1}^{n+1}}{h}\right) + \sum_{n=0}^{N-1} g_{J-1}^n\left(z_{J-1}^{n+1}\times\frac{\Delta_+ z_{J-1}^{n+1}}{h}\right)k - \sum_{n=0}^{N-1} g_0^n\left(z_0^{n+1}\times\frac{\Delta_+ z_0^{n+1}}{h}\right). \tag{39}$$

Since $g(x, T)\equiv 0$, the last term of (38) vanishes. On account of the boundary condition (*) at $x=0$, i. e., $z_0^{n+1}=0$ or $\Delta_+ z_0^{n+1}=0$ $(n=0, 1, \cdots, N-1)$, the last term of (39) is also equal to zero. If the finite difference boundary condition is $\Delta_+ z_{J-1}^{n+1}=0$ at the end point $x=l$, then

$$\sum_{n=0}^{N-1} g_{J-1}^n\left(z_{J-1}^{n+1}\times\frac{\Delta_+ z_{J-1}^{n+1}}{h}\right)k=0.$$

If the finite difference boundary condition is $z_J^{n+1}=0$, then

$$\sum_{n=0}^{N-1} g_{J-1}^n\left(z_{J-1}^{n+1}\times\frac{\Delta_- z_J^{n+1}}{h}\right)k = -\sum_{n=0}^{N-1} g_{J-1}^n\left(\Delta_- z_J^{n+1}\times\frac{\Delta_- z_J^{n+1}}{h}\right)k=0. \tag{40}$$

Hence (37) can be replaced by

$$\sum_{j=1}^{J-1}\sum_{n=0}^{N-1}\frac{g_j^{n+1}-g_j^n}{k}z_j^{n+1}hk - \sum_{j=1}^{J-1}\sum_{n=0}^{N-1}\frac{\Delta_- g_j^n}{h}\left(z_{j-1}^{n+1}\times\frac{\Delta_+ z_{j-1}^{n+1}}{h}\right)hk + \sum_{j=1}^{J-1}\sum_{n=0}^{N-1} g_j^n f_j^{n+1}hk + \sum_{j=0}^{J-1} g_j^0 z_j^0 h + \sum_{n=0}^{N-1} g_{J-1}^n\left(z_{J-1}^{n+1}\times\frac{\Delta_+ z_{J-1}^{n+1}}{h}\right)k=0.$$

It can also be expressed as

$$\iint_{Q_T}\tilde g_{hk}(x, t)z_{hk}(x, t)dx\,dt - \iint_{Q_T}\bar g_{hk}(x-h, t-k)\left[z_{hk}(x-h, t)\times\bar z_{hk}(x-k, t)\right]dx\,dt + \iint_{Q_T} g_{hk}(x, t-k)F_{hk}(x, t)dx\,dt + \int_0^l g_{hk}(0, 0)\bar\varphi_h(x)dx + O(h^{\frac{1}{2}})=0, \tag{41}$$

where $\tilde g_{hk}(x, t)$ is the appropriate piecewise constant function, corresponding to the discrete function $\frac{g_j^{n+1}-g_j^n}{k}$, defined as before; $F_{hk}(x, t)=f_j^{n+1}=f(x_j, t_{n+1}, z_j^{n+1})$ in $Q_j^n(j=0, 1, \cdots, J-1; n=0, 1, \cdots, N-1)$ and $\bar\varphi_h(x)=\bar\varphi_j$ in $(jh, (j+1)h]$ $(j=0, 1, \cdots,$

$J-1$) are two 3-dimensional vector valued piecewise constant functions. Since $g(x, t)$ is a smooth function, $g_{hk}(x, t)$, $\bar{g}_{hk}(x, t)$ and $\tilde{g}_{hk}(x, t)$ are uniformly convergent to $g(x, t), g_x(x, t)$ and $g_t(x, t)$ respectively in Q_T as $h^2+k^2\to 0$ and $g_{hk}(x, 0)$ is uniformly convergent to $g(x, 0)$ in $[0, l]$ as $h\to 0$. On the other hand $z_{hk}(x, t)$ and $F_{hk}(x, t)$ are uniformly convergent to $z(x, t)$ and $f(x, t, z(x, t))$ respectively in Q_T and $\bar{z}_{hk}(x, t)$ converges weakly to $z_x(x, t)$ as $h_i^2+k_i^2\to 0$. And obviously $\bar{\varphi}_h(x)$ converges uniformly to $\varphi(x)$ in $[0, l]$ as $h\to 0$. Hence as $h_i^2+k_i^2\to 0$, (41) tends to the integral identity (33). Therefore the 3-dimensional vector valued limiting function $z(x, t)$ is the weak solution of the boundary problem (*), (8) for the system (3) of ferro-magnetic chain.

Theorem 1. *Under the conditions* (I), (II) *and* (III), *the second boundary problem* (5), (8) *for the system* (3) *of ferro-magnetic chain has at least one weak solution* $z(x, t) \in L_\infty((0, T);\ W_2^{(1)}(0, l)) \cap C^{\left(\frac{1}{2}, \frac{1}{4}\right)}(Q_T)$.

Theorem 2. *Under the conditions* (I), (II), (III) *and* $f(x, t, 0)\equiv 0$, *the first boundary problem* (4), (8) *and the mixed boundary problems* (6), (8) *and* (7), (8) *for the homogeneous system* (3) *of ferro-magnetic chain have at least one weak solution* $z(x, t) \in L_\infty((0, T);\ W_2^{(1)}(0, l)) \cap C^{\left(\frac{1}{2}, \frac{1}{4}\right)}(Q_T)$.

Remark. The conditions for the existence of weak solution of boundary problems (*), (8) for the system (3) can be weakened. For example, the condition $\varphi(x) \in C^{(1)}([0, l])$ can be replaced by $\varphi(x) \in W_2^{(1)}(0, l)$.

Let $\{\varphi_i(x)\}$ $(i=1, 2, \cdots)$ be a sequence of 3-dimensional vector valued functions, such that for each i, $\varphi_i(x)\in C^{(1)}([0, l])$ and $\varphi_i(x)$ converges to $\varphi(x)$ in $W_2^{(1)}(0, l)$ as $i\to\infty$. For each i, we consider the problem for the system (3) with the boundary condition (*) and the initial condition

$$z(x, 0)=\varphi_i(x). \tag{8$_i$}$$

Denote the weak solution of the boundary problem (3), (*), $(8)_i$ by $z_i(x, t)$. It can be proved that

$$\sup_{0<t<T} \|z_i(\cdot, t)\|_{W_2^{(1)}(0,l)} + \sup_{0<t<T} \|s_{it}(\cdot, t)\|_{L_2(0,l)} \leqslant K_8, \tag{42}$$

where $s_i(x, t)=\int_0^x z_i(\xi, t)d\xi$ and K_8 is independent of $i=1, 2, \cdots$. There is a subsequence $\{z_i(x, t)\}$ convergent to $z(x, t)\in L_\infty((0, T);\ W_2^{(1)}(0, l)) \cap C^{\left(\frac{1}{2}, \frac{1}{4}\right)}(Q_T)$. Hence $z(x, t)$ is the weak solution of the boundary problem (*), (8) for the system (3).

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