ON THE CONVERGENCE RATE OF THE BOUNDARY PENALTY METHOD*1)

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Abstract

The convergence rate of the boundary penalty finite element method is discussed for a model Poisson equation with inhomogeneous Dirichlet boundary conditions and a sufficiently smooth solution. It is proved that an optimal convergence rate can be achieved which agrees with the rate obtained recently in the numerical experiments by Utku and Carey.

1. Introduction

The boundary penalty finite element method has been developed to approximate Dirichlet boundary conditions in the solution of elliptic boundary value problems. (see, for example, references [2] to [5].) In finite element programs there exists another technique for approximating Dirichlet data by adding a large number to certain diagonal entries in the stiffness matrix and by scaling the load vector. Utku and Carey [1] have recently discussed the relationship between these two techniques for treating Dirichlet data and derived an abstract error estimate for the boundary penalty method. They observed that the rates achieved in the numerical experiments are better than those obtained in the theoretical analysis. In this note we show, using the regularity of the solution of the given problem, that the theoretical result of Utku and Carey can be improved, and we present an error estimate which fully agrees with the numerical experiments in [1].

2. A Model Problem

As a model problem, consider the solution of Poisson's equation

$$-\Delta u=f \quad \text{in } \Omega \tag{1}$$

with inhomogeneous Dirichlet data

$$u=g \quad \text{on } \partial\Omega. \tag{2}$$

Suppose that $g\in H^{\frac{1}{2}}(\partial\Omega)$. Then, by the trace theorem (see [6, p. 143]), there exists a function $\tilde{g}\in H^1(\Omega)$ such that $|\tilde{g}|_{\partial\Omega}=g$. The inhomogeneous Dirichlet problem (1), (2) is equivalent to the following variational equation: find $u\in H^1(\Omega)$ such that

$$u-\tilde{g}\in H_0^1(\Omega), \quad a(u, v)=(f, v), \quad v\in H_0^1(\Omega), \tag{3}$$

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where

$$a(u, v)=\int_{\Omega}\nabla u\cdot\nabla v\,dx\,dy, \qquad (f, v)=\int_{\Omega}fv\,dx\,dy. \tag{4}$$

Assuming the given domain Ω to be a convex polygon, we decompose it into triangular elements K satisfying the usual regularity conditions for triangulations. For simplicity, consider the continuous piecewise linear function spaces $V_h\in H^1(\Omega)$ and let Π_1 be the interpolation operator of functions $u\in H^2(\Omega)$ at the vertices of the triangles K in $\bar{\Omega}$, such that

$$\Pi_1 u\in V_h, \ u\in H^2(\Omega). \tag{5}$$

From interpolation theory the following estimates are obtained:

$$\|u-\Pi_1 u\|_{0,K}\leqslant Ch^2|u|_{2,K}, \tag{6}$$

$$|u-\Pi_1 u|_{1,K}\leqslant Ch|u|_{2,K}, \tag{7}$$

for $u\in H^2(\Omega)$, and

$$\int_{\partial K}w^2\,ds\leqslant C_1 h|w|^2_{1,K}+\frac{C_2}{h}\|w\|^2_{0,K} \tag{8}$$

for $w\in H^1(\Omega)$, uniformly for all elements K. Here and in the following all C_i and C denote generic constants independent of h. As direct consequences of the inequalities (6), (7), (8), we have for $u\in H^2(\Omega)$:

$$\|u-\Pi_1 u\|^2_1\leqslant Ch^2|u|^2_2, \tag{9}$$

$$\int_{\partial\Omega}(u-\Pi_1 u)^2\,ds\leqslant Ch^3|u|^2_2. \tag{10}$$

The penalty method for approximating the solution of the variational equation (3) by the finite element spaces V_h consists in finding $u_h\in V_h$ such that

$$a(u_h, v_h)+h^{-\sigma}\int_{\partial\Omega}(u_h-g)v_h\,ds=(f, v_h), \quad v_h\in V_h, \tag{11}$$

where $\sigma>0$ is a penalty parameter to be determined in order to maximize the rate of convergence.

3. H^2-Solution

Lemma 1. *Let u_0, u_h be the solutions of equations (3), (11), respectively, and suppose that $u_0\in H^2(\Omega)$. Then the inequality*

$$\begin{aligned}&|u_0-u_h|^2_1+h^{-\sigma}\int_{\partial\Omega}\left(\frac{\partial u_0}{\partial n}h^\sigma+u_h-g\right)^2 ds\\ &\leqslant|u_0-v_h|^2_1+h^{-\sigma}\int_{\partial\Omega}\left(\frac{\partial u_0}{\partial n}h^\sigma+v_h-g\right)^2 ds\end{aligned} \tag{12}$$

holds for all $v_h\in V_h$.

Proof. Since u_h is the solution of the variational equation (11),

$$\begin{aligned}&a(u_h, u_h)+h^{-\sigma}\int_{\partial\Omega}(u_h-g)^2\,ds-2(f, u_h)\\ &\leqslant a(v_h, v_h)+h^{-\sigma}\int_{\partial\Omega}(v_h-g)^2\,ds-2(f, v_h)\end{aligned} \tag{13}$$

for $v_h\in V_h$. On the other hand, it follows from (1), (2) that

$$a(u_0,\ v_h)-\int_{\partial\Omega}\frac{\partial u_0}{\partial n}v_h\,ds=(f,\ v_h). \tag{14}$$

Hence, by (12) and (13), we have

$$\begin{aligned}|u_0-u_h|_1^2+h^{-\sigma}\int_{\partial\Omega}\left(\frac{\partial u_0}{\partial n}h^\sigma+u_h-g\right)^2 ds &= a(u_0,\ u_0)+h^\sigma\int_{\partial\Omega}\left(\frac{\partial u_0}{\partial n}\right)^2 ds\\ &\quad -2\int_{\partial\Omega}\frac{\partial u_0}{\partial n}g\,ds+a(u_h,\ u_h)+h^{-\sigma}\int_{\partial\Omega}(u_h-g)^2\,ds\\ &\quad -2\left(a(u_0,\ u_h)-\int_{\partial\Omega}\frac{\partial u_0}{\partial n}u_h\,ds\right)\\ &\leqslant a(u_0,\ u_0)+h^{+\sigma}\int_{\partial\Omega}\left(\frac{\partial u_0}{\partial n}\right)^2 ds-2\int_{\partial\Omega}\frac{\partial u_0}{\partial n}g\,ds+a(v_h,\ v_h)\\ &\quad +h^{-\sigma}\int_{\partial\Omega}(v_h-g)^2 ds-2\left(a(u_0,\ v_h)-\int_{\partial\Omega}\frac{\partial u_0}{\partial n}v_h\,ds\right)\\ &=|u_0-v_h|_1^2+h^{-\sigma}\int_{\partial\Omega}\left(\frac{\partial u_0}{\partial n}h^\sigma+v_h-g\right)^2 ds.\end{aligned}$$

Theorem 1. *Suppose that $u_0\in H^2(\Omega)$ and let u_h be the solution of (11). Then*

$$\|u_0-u_h\|_1\leqslant Ch^{\frac{3}{4}}\|u_0\|_2. \tag{15}$$

Proof. Taking $v_h=\Pi_1 u_0\in V_h$ the Lemma 1 gives

$$\begin{aligned}|u_0-u_h|_1^2+h^{-\sigma}\int_{\partial\Omega}\left(\frac{\partial u_0}{\partial n}h^\sigma+u_h-g\right)^2 ds\\ \leqslant|u_0-\Pi_1u_0|_1^2+h^{-\sigma}\int_{\partial\Omega}\left(\frac{\partial u_0}{\partial n}h^\sigma+\Pi_1u_0-g\right)^2 ds.\end{aligned} \tag{16}$$

Consider now the second term on the right. Using the trace theorem and the inequality (10), we have

$$\begin{aligned}h^{-\sigma}\int_{\partial\Omega}\left(\frac{\partial u_0}{\partial n}h^\sigma+\Pi_1u_0-g\right)^2 ds &\leqslant 2\left(h^\sigma\int_{\partial\Omega}\left(\frac{\partial u_0}{\partial n}\right)^2 ds+h^{-\sigma}\int_{\partial\Omega}(\Pi_1u_0-g)^2\,ds\right)\\ &\leqslant C(h^\sigma\|u_0\|_2^2+h^{3-\sigma}|u_0|_2^2)\leqslant C(h^\sigma+h^{3-\sigma})\|u_0\|_2^2.\end{aligned} \tag{17}$$

Application of the inequality (9) to the first term on the right side of (16), in combination with (17), yields

$$|u_0-u_h|_1^2\leqslant Ch^{2\mu}\|u_0\|_2^2, \tag{18}$$

$$h^{-\sigma}\int_{\partial\Omega}\left(\frac{\partial u_0}{\partial n}h^\sigma+u_h-g\right)^2 ds\leqslant Ch^{2\mu}\|u_0\|_2^2, \tag{19}$$

where

$$\mu=\min\left(1,\ \frac{\sigma}{2},\ \frac{3-\sigma}{2}\right). \tag{20}$$

Now, it follows from (19) and the trace theorem that

$$\begin{aligned}\int_{\partial\Omega}(u_h-g)^2 ds &\leqslant 2\left(\int_{\partial\Omega}\left(\frac{\partial u_0}{\partial n}h^\sigma+u_h-g\right)^2 ds+h^{2\sigma}\int_{\partial\Omega}\left(\frac{\partial u_0}{\partial n}\right)^2 ds\right)\\ &\leqslant C(h^{2\mu+\sigma}+h^{2\sigma})\|u_0\|_2^2,\end{aligned} \tag{21}$$

and, therefore, using Poincaré's inequality and (18), (21), we have

$$\|u_0-u_h\|_1^2 \leqslant C_3(|u_0-u_h|_1^2+\int_{\partial\Omega}(g-u_h)^2\,ds)$$

$$\leqslant C_4(h^{2\mu}+h^{2\mu+\sigma}+h^{2\sigma})\|u_0\|_2^2 \leqslant Ch^{2\mu}\|u_0\|_2^2 \tag{22}$$

since $\sigma>0$ and $2\mu\leqslant\sigma$.

From (22) it is seen that the optimal choice of the penalty parameter, maximizing the order of convergence, is $\sigma=\frac{3}{2}$. We thus obtain the error estimate (15).

Theorem 1 shows that the order of convergence is $\mu=\frac{3}{4}$ in the H^1-norm, which is much better than $\frac{1}{2}$ in the H^1-seminorm as indicated in [1]. However, we should mention here that (15) can be obtained from a general theorem of Babuska[3].

4. H^3-Solution

In the following we shall show that Theorem 1 may be substantially improved to achieve an optimal order of convergence, provided that the solution u_0 of the given problem is smooth enough.

Theorem 2. *Suppose that $u_0\in H^3(\Omega)$ and let u_h be the solution of* (11). *Then*

$$\|u_0-u_h\|_1\leqslant Ch\|u_0\|_3. \tag{23}$$

Proof. Consider the auxiliary Dirichlet problem:

$$\Delta v=0 \quad \text{in } \Omega \tag{24}$$

with

$$v=\frac{\partial u_0}{\partial n} \quad \text{on } \partial\Omega. \tag{25}$$

Since $u_0\in H^3(\Omega)$, the trace theorem gives $\frac{\partial u_0}{\partial n}\in H^{\frac{3}{2}}(\partial\Omega)$ and

$$\left\|\frac{\partial u_0}{\partial n}\right\|_{H^{1/2}(\partial\Omega)}\leqslant C\|u_0\|_2, \tag{26}$$

$$\left\|\frac{\partial u_0}{\partial n}\right\|_{H^{3/2}(\partial\Omega)}\leqslant C\|u_0\|_3. \tag{27}$$

Hence the problem (24), (25) has a unique solution v, which satisfies the following estimates (see [6, p. 181])

$$\|v\|_1\leqslant C_5\left\|\frac{\partial u_0}{\partial n}\right\|_{H^{1/2}(\partial\Omega)}\leqslant C\|u_0\|_2, \tag{28}$$

$$\|v\|_2\leqslant C_6\left\|\frac{\partial u_0}{\partial n}\right\|_{H^{3/2}(\partial\Omega)}\leqslant C\|u_0\|_3. \tag{29}$$

Applying the inequalities (9), (10), (29) to v, we get

$$\|v-\Pi_1 v\|_1^2\leqslant C_7h^2|v|_2^2\leqslant Ch^2\|u_0\|_3^2, \tag{30}$$

$$\int_{\partial\Omega}(v-\Pi_1 v)^2\,ds\leqslant C_8h^3|v|_2^2\leqslant Ch^3\|u_0\|_3^2. \tag{31}$$

Therefore, setting $v_h=\Pi_1u_0-h^\sigma\Pi_1v\in V_h$ in Lemma 1 and using the inequalities (9), (10), (28), (30), (31), we have

$$\begin{aligned}|u_0-u_h|_1^2+h^{-\sigma}\int_{\partial\Omega}\left(\frac{\partial u_0}{\partial n}h^\sigma+u_h-g\right)^2 ds\\ \leqslant 3(|u_0-\Pi_1 u_0|_1^2+h^{2\sigma}|\Pi_1 v-v|_1^2+h^{2\sigma}|v|_1^2)\\ +2\left(h^\sigma\int_{\partial\Omega}\left(\frac{\partial u_0}{\partial n}-\Pi_1 v\right)^2 ds+h^{-\sigma}\int_{\partial\Omega}(\Pi_1 u_0-g)^2 ds\right)\\ \leqslant C_9(h^2|u_0|_2^2+h^{2\sigma+2}\|u_0\|_3^2+h^{2\sigma}\|u_0\|_2^2+h^{3+\sigma}\|u_0\|_3^2+h^{3-\sigma}|u_0|_2^2)\\ \leqslant Ch^{2\mu}\|u_0\|_3^2, \qquad \mu=\min\left(1,\ \sigma,\ \frac{3-\sigma}{2}\right).\end{aligned} \tag{32}$$

By the same argument used in the proof of Theorem 1, we conclude from (32) that

$$\|u_0-u_h\|_1\leqslant Ch^\mu\|u_0\|_3. \tag{33}$$

From (33) it follows that the optimal penalty parameter is $\sigma=1$, giving the order of convergence $\mu=1$ in the H^1-norm.

This theorem shows that the theoretical order of convergence fully agrees with the experiments of Utku and Carey in [1].

Note that Theorem 2 cannot be derived from the similar result of Babuska-Aziz in [4, p. 253], according to which one obtains

$$\|u_0-u_h\|_1\leqslant Ch\|f\|_2 \tag{34}$$

for the case of homogeneous Dirichlet boundary conditions, i. e. $g=0$ on $\partial\Omega$. In that case, from Theorem 2 we have

$$\|u_0-u_h\|_1\leqslant Ch\|u_0\|_3\leqslant Ch\|f\|_1, \tag{35}$$

giving a better error bound than (34).

The above Theorems 1 and 2 were formulated only for linear elements. It is easy to generalize the results to higher order elements.

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