AN L_1 MINIMIZATION PROBLEM BY GENERALIZED RATIONAL FUNCTIONS*[1)]

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Abstract

Let $P, Q \subset L_1(X, \Sigma, \mu)$ and $q(x)>0$ a. e. in X for all $q\in Q$. Define $R=\{p/q:p\in P, q\in Q\}$. In this paper we discuss an L_1 minimization problem of a nonnegative function $E(z, x)$, i. e. we wish to find a minimum of the functional $\phi(r)=\int_X qE(r, x)d\mu$ from $r=p/q\in R$. For such a problem we have established the complete characterizations of its minimum and of uniqueness of its minimum, when both P, Q are arbitrary convex subsets.

I. Introduction

Let (X, Σ, μ) be a σ-finite measure space and $L\equiv L_1(X, \Sigma, \mu)$ the linear normed space of all integrable functions on X with the norm

$$\|f\|=\int_X |f(x)|\,d\mu.$$

Assume that both P and Q are subsets in L and $q(x)>0$ almost everywhere in X for all $q\in Q$. Then we may construct the set of generalized rational functions

$$R=\{p/q:p\in P, q\in Q\}.$$

Suppose now that $E(z, x)$ is a nonnegative function from $(-\infty, \infty)\times X$ into $[0, \infty]$ such that $qE(r, \cdot)\in L$ for any element $r=p/q\in R$, where $E(r, \cdot)=E(r(\cdot),\cdot)$.

Our minimization problem then is to find an element $r_0=p_0/q_0\in R$ such that

$$\|q_0E(r_0, \cdot)\|=\inf_{r\in R}\|qE(r, \cdot)\|, \tag{1}$$

such an r_0 (if any) is called a minimum to E from R.

For a solution of the equation

$$\|E(r_0, \cdot)\|=\inf_{r\in R}\|E(r, \cdot)\|$$

we have not found, to the author's knowledge, its complete characterization and the complete characterization of its uniqueness. For a solution of equation (1), however, we can give all of them, provided that both P and Q are arbitrary convex subsets.

The minimization problem includes as special cases a number of ordinary and simultaneous approximation problems, such as

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$$E(z,x)=|f(x)-z|^s, \qquad 1\leqslant s<\infty,$$
$$E(z,x)=\sum_i |f_i(x)-z|,$$
$$E(z,x)=\max_i |f_i(x)-z|,$$

etc.

II. Main Results

Suppose both P and Q are convex subsets in L. For $r=p/q$, $r_0=p_0/q_0\in R$ and $t\in[0,1]$ write

$$p_t=p_0+t(p-p_0),$$
$$q_t=q_0+t(q-q_0),$$
$$r_t=p_t/q_t.$$

Our main results require several lemmas.

Lemma 1. *Let $f(x)$ be a convex function. Then for any $r=p/q$, $r_0=p_0/q_0\in R$*

$$\phi(t)\equiv(q_tf(r_t)-q_0f(r_0))/t$$

is increasing with respect to t in $(0,1]$.

Proof. Let $t\in(0,1]$. Since

$$\phi(t)=\frac{q_tf(r_t)-q_0f(r_0)}{t}=\frac{q_t(f(r_t)-f(r_0))}{t}+\frac{(q_t-q_0)f(r_0)}{t}$$
$$=q_t\cdot\frac{f(r_t)-f(r_0)}{r_t-r_0}\cdot\frac{r_t-r_0}{t}+(q-q_0)f(r_0)$$
$$=q(r-r_0)\cdot\frac{f(r_t)-f(r_0)}{r_t-r_0}+(q-q_0)f(r_0)$$

and

$$r_t-r_0=\frac{tq}{q_0+t(q-q_0)}(r-r_0),$$

for fixed x if $r(x)-r_0(x)>(<)0$, $r_t(x)-r_0(x)>(<)0$ and $r_t(x)$ is increasing (decreasing) with respect to t, which by the convexity of f implies that $(f(r_t(x))-f(r_0(x)))/(r_t(x)-r_0(x))$ is increasing (decreasing) with respect to t [2, p. 6]. Thus in both the cases $\phi(t)$ is increasing with respect to t.

From $\phi(t)\leqslant\phi(1)$, $t\in(0,1]$, we obtain the following lemma.

Lemma 2. *Let $f(x)$ be a convex function. Then for any $r=p/q$, $r_0=p_0/q_0\in R$*

$$(q_tf(r_t)-q_0f(r_0))/t\leqslant qf(r)-q_0f(r_0), \qquad t\in(0,1]. \tag{2}$$

In order to state the following basic lemma we need to generalize the notion of the directional derivative to be applicable to our case. To this end for $r_i=p_i/q_i\in R$, $i=0,1,2$, define

$$e(r_0,x;r_1,r_2)=\lim_{t\to0+}\left[(q_0+t(q_1-q_2))\,E\left(\frac{p_0+t(p_1-p_2)}{q_0+t(q_1-q_2)},x\right)-q_0E(r_0,x)\right]\Big/t$$

if the limit exists. Hence

$$e(r_0,x;r,r_0)=\lim_{t\to0+}(q_tE(r_t,x)-q_0E(r_0,x))/t.$$

Lemma 3. *Let P and Q be convex sets in L. Suppose that $E(z,x)$ is convex with respect to z for each $x\in X$. Then for any $r=p/q$, $r_0=p_0/q_0\in R$*

$$\int_X e(r_0, x; r, r_0)d\mu \leqslant \|qE(r, \cdot)\| - \|q_0E(r_0, \cdot)\|$$
$$\leqslant -\int_X e(r, x; r_0, r)d\mu. \tag{3}$$

Proof. By Lemma 2 for $t \in (0, 1]$

$$(q_tE(r_t, x) - q_0E(r_0, x))/t \leqslant qE(r, x) - q_0E(r_0, x). \tag{4}$$

The left expression of the inequality is increasing with respect to t in $(0, 1]$ by Lemma 1 and always possesses a limit $e(r_0, x; r, r_0)$ as $t \to 0+$. Furthermore we have [1, Chap. 5, Exercise 17, p. 177]

$$\int_X e(r_0, x; r, r_0)\, d\mu = \lim_{t\to 0+} \int_X (q_tE(r_t, x) - q_0E(r_0, x))/t\, d\mu. \tag{5}$$

Then from (4) it follows that

$$\int_X e(r_0, x; r, r_0)d\mu \leqslant \|qE(r, \cdot)\| - \|q_0E(r_0, \cdot)\|,$$

which is the left inequality in (3). And the right inequality in (3) follows from interchanging r and r_0 in the above inequality.

The main results are as follows.

Theorem 1. *Let P, $Q \subset L$ be convex sets and $r_0 \in R$. Suppose that $E(z, x)$ is convex with respect to z for each $x \in X$. Then r_0 is a minimum to E from R if and only if*

$$\int_X e(r_0, x; r, r_0)d\mu \geqslant 0, \quad \forall r \in R. \tag{6}$$

Proof. Necessity. Let $r = p/q \in R$ and $r_0 = p_0/q_0$. Since $r_t \in R$ for $t \in [0, 1]$,

$$\|q_tE(r_t, \cdot)\| \geqslant \|q_0E(r_0, \cdot)\|. \tag{7}$$

Then (6) follows from (5) and (7).

Sufficiency. By Lemma 3

$$\|qE(r, \cdot)\| \geqslant \|q_0E(r_0, \cdot)\|, \quad \forall r \in R,$$

which means that r_0 is a minimum to E from R.

Theorem 2. *Under the assumptions of Theorem 1 if there exists a minimum to E from R, then the following statements are equivalent:*

(a) $\|q_0E(r_0, \cdot)\| < \|qE(r, \cdot)\|, \quad \forall r \in R \setminus \{r_0\};$

(b) $\int_X e(r, x; r_0, r)d\mu < 0, \quad \forall r \in R \setminus \{r_0\};$

(c) $\int_X e(r, x; r_0, r)d\mu < \int_X e(r_0, x; r, r_0)d\mu, \quad \forall r \in R \setminus \{r_0\}.$

Proof. (a) ⇒ (b). Statement (b) follows directly from (a) by Lemma 3.

(a) ⇒ (c). Statement (a) implies (6) by Theorem 1. From (6) and (b) it follows that (c) is valid.

(b) ⇒ (a) and (c) ⇒ (a). Suppose not and let $r \in R \setminus \{r_0\}$ be a minimum to E from R. Thus for such an r we have by Theorem 1 that

$$\int_X e(r, x; r_0, r)d\mu \geqslant 0 \tag{8}$$

and by Lemma 3 that

$$\int_X e(r_0, x; r, r_0)d\mu \leqslant 0. \tag{9}$$

But (8) contradicts (b), and (8) and (9) together contradict (c).

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References

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