DIFFERENCE SCHEMES OF DEGENERATE PARABOLIC EQUATIONS*

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§1

Consider the partial differential equation of second order

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\sigma\frac{\partial u}{\partial x}+b\frac{\partial u}{\partial x}+du+f,\qquad (x,\ t)\in I\times(0,\ T],\tag{1}$$

where the unknown function u and coefficients σ, b, d, f are functions of x and t. Denote the interval $0<x<l$ by I. Let $Z\subset\bar{I}\times(0,\ T]$ be the point set, on which $\sigma=0$. If $\sigma(x,\ t)\geqslant 0$ on the domain $\bar{I}\times(0,\ T]$, and Z is not an empty set, then equation (1) is known as a degenerate parabolic equation. In order for the initial and boundary-value problem of the equation (1) to be properly posed, the initial and boundary conditions must be appropriate. The boundary conditions to be posed depend on the behaviour of coefficients $\sigma(x,\ t)$ and $b(x,\ t)$ on $x=0$ and $x=l$. If $\sigma(0,t)=0$, $b(0,t)<0$ simultaneously, or $\sigma(0,\ t)>0$, then on $x=0$, a boundary condition should be given; otherwise (i. e. if $\sigma(0,\ t)=0$ and $b(0,\ t)\geqslant 0$ simultaneously), no boundary condition on $x=0$ is needed. On $x=l$, when $\sigma(l,\ t)=0$, $b(l,\ t)>0$ simultaneously, or $\sigma(l,\ t)>0$, a boundary condition should be given; otherwise, it is not needed. Moreover, for equation (1), the initial condition

$$u(x,\ 0)=g_0(x),\qquad x\in\bar{I}\tag{2}$$

is always needed[1].

In this section we suppose

$$\sigma(0,\ t)=\sigma(l,\ t)=0,\ b(0,\ t)\geqslant 0,\ b(l,\ t)\leqslant 0,\ t\in(0,\ T].\tag{3}$$

In addition, we assume that the coefficients of the equation (1) are sufficiently smooth and that there exists a unique sufficiently smooth solution of the equation (1) with initial condition (2).

We solve the problem (1), (2) by a difference method. Divide the interval $[0,\ l]$ and $[0,\ T]$ into J and N parts respectively. The space step is $h=l/J$ and the time step is $\tau=T/N$. Let $\omega_h=\{x_j=jh\,|\,j=0,\ 1,\ \cdots,\ J\}$ and $\omega_\tau=\{t^n=n\tau\,|\,n=0,\ 1,\ \cdots,\ N\}$. The set of all net points on the domain $\bar{I}\times[0,\ T]$ is denoted by $\omega_h\times\omega_\tau$.

Let $y(x,\ t)$ and $z(x,\ t)$ be functions, defined on the set $\omega_h\times\omega_\tau$. Introduce the following notations

$$y_j^n=y(jh,\ n\tau)$$

$$y_{x,j}^n=\frac{1}{h}(y_{j+1}^n-y_j^n),\qquad y_{\bar{x},j}^n=\frac{1}{h}(y_j^n-y_{j-1}^n).$$

Define the inner products

* Received September 14, 1982.

$$(y^n, z^n) = \sum_{j=1}^{J-1} \alpha_j y_j^n z_j^n h, \qquad [y^n, z^n) = \sum_{j=0}^{J-1} \alpha_j y_j^n z_j^n h,$$

$$(y^n, z^n] = \sum_{j=1}^{J} \alpha_j y_j^n z_j^n h, \qquad [y^n, z^n] = \sum_{j=0}^{J} \alpha_j y_j^n z_j^n h,$$

where $\alpha_0 = \alpha_J = \frac{1}{2}$, $\alpha_1 = \alpha_2 = \cdots = \alpha_{J-1} = 1$, and consequently, the norms

$$\|y^n\| = (y^n, y^n)^{\frac{1}{2}}, \qquad |[y^n\| = [y^n, y^n)^{\frac{1}{2}},$$
$$\|y^n]| = (y^n, y^n]^{\frac{1}{2}}, \qquad |[y^n]| = [y^n, y^n]^{\frac{1}{2}}. \tag{4}$$

Because the boundary condition is given neither on $x=0$ nor on $x=l$, the difference scheme should be constructed on each point of the set $\omega_h \times (\omega_\tau \backslash t^0)$. On $x=0$ and $x=l$, the equation (1) can be reduced to the following form

$$\frac{\partial u}{\partial t} = (\sigma' + b)\frac{\partial u}{\partial x} + du + f,$$

where σ' denotes $\partial\sigma/\partial x$. Since the function $\sigma(x, t)$ is non-negative on the whole domain $\bar{I} \times (0, T]$, it is clear that

$$\sigma'(0, t) \geqslant 0, \quad \sigma'(l, t) \leqslant 0. \tag{5}$$

Let $y(x, t)$ be a function defined on the set $\omega_h \times \omega_\tau$. The Crank-Nicholson scheme approximating the differential equation (1) is

$$\frac{y_0^{n+1} - y_0^n}{\tau} = (\sigma_0'^{\,n+\frac{1}{2}} + b_0^{n+\frac{1}{2}})\frac{1}{2}(y_{x,0}^{n+1} + y_{x,0}^n) + d_0^{n+\frac{1}{2}}\frac{1}{2}(y_0^{n+1} + y_0^n) + f_0^{n+\frac{1}{2}}, \tag{6}$$

$$\frac{y_j^{n+1} - y_j^n}{\tau} = \left(a^{n+\frac{1}{2}}\frac{1}{2}(y_{\bar{x}}^{n+1} + y_{\bar{x}}^n)\right)_{x,j} + b_j^{n+\frac{1}{2}}\frac{1}{4}(y_{\bar{x},j}^{n+1} + y_{\bar{x},j}^n + y_{x,j}^{n+1} + y_{x,j}^n)$$
$$+ d_j^{n+\frac{1}{2}}\frac{1}{2}(y_j^{n+1} + y_j^n) + f_j^{n+\frac{1}{2}}, \quad j = 1, 2, \cdots, J-1, \tag{7}$$

$$\frac{y_J^{n+1} - y_J^n}{\tau} = (\sigma_J'^{\,n+\frac{1}{2}} + b_J^{n+\frac{1}{2}})\frac{1}{2}(y_{\bar{x},J}^{n+1} + y_{\bar{x},J}^n) + d_J^{n+\frac{1}{2}}\frac{1}{2}(y_J^{n+1} + y_J^n) + f_J^{n+\frac{1}{2}}, \tag{8}$$

where $a_j^{n+\frac{1}{2}} = \sigma_{j-\frac{1}{2}}^{n+\frac{1}{2}}$, and for any function $\phi(x, t)$, we have $\phi_\alpha^\beta = \phi\ (\alpha h, \beta\tau)$. The initial condition (2) is approximated by

$$y_j^0 = g_0(jh), \quad j = 0, 1, \cdots, J. \tag{9}$$

Equations (6)—(8) are the system of linear equations with unknowns y_0^{n+1}, y_1^{n+1}, $\cdots$, y_J^{n+1}. Let $C_d = \sup_{I\times[0,T]} \frac{1}{2}(d + |d|)$. When $C_d \neq 0$, the coefficient matrix of equation (6)—(8) is diagonally dominant for $\tau < 2/C_d$, and when $C_d = 0$, for arbitrary τ. Then, the system of difference equations (6)—(8) is solvable.

Let $z(x, t)$ be the difference between the solution of difference equations (6)—(9) and that of the differential equation (1) with initial condition (2), i. e.

$$z(x, t) = y(x, t) - u(x, t), \ (x, t) \in \omega_h \times \omega_\tau. \tag{10}$$

Putting $y = z + u$ in the difference equations (6)—(9), we obtain

$$\frac{z_0^{n+1} - z_0^n}{\tau} - (\sigma_0'^{\,n+\frac{1}{2}} + b_0^{n+\frac{1}{2}})\frac{1}{2}(z_{x,0}^{n+1} + z_{x,0}^n) - d_0^{n+\frac{1}{2}}\frac{1}{2}(z_0^{n+1} + z_0^n) = \psi_0^{n+\frac{1}{2}}, \tag{11}$$

$$\frac{z_j^{n+1} - z_j^n}{\tau} - \left(a^{n+\frac{1}{2}}\frac{1}{2}(z_{\bar{x}}^{n+1} + z_{\bar{x}}^n)\right)_{x,j} - b_j^{n+\frac{1}{2}}\frac{1}{4}(z_{x,j}^{n+1} + z_{x,j}^n + z_{\bar{x},j}^{n+1} + z_{\bar{x},j}^n)$$

$$-d_j^{n+\frac{1}{2}}\frac{1}{2}(z_j^{n+1}+z_j^n)=\psi_j^{n+\frac{1}{2}},\quad j=1,2,\cdots,J-1,\tag{12}$$

$$\frac{z_J^{n+1}-z_J^n}{\tau}-(\sigma_J'^{n+\frac{1}{2}}+b_J^{n+\frac{1}{2}})\frac{1}{2}(z_{\bar{x},J}^{n+1}+z_{\bar{x},J}^n)-d_J^{n+\frac{1}{2}}\frac{1}{2}(z_J^{n+1}+z_J^n)=\psi_J^{n+\frac{1}{2}},\tag{13}$$

$$z_j^0=0,\quad j=0,1,\cdots,J,\tag{14}$$

where the right members $\psi_j^{n+\frac{1}{2}}$ are truncation errors. Obviously, if the solution of the differential equation (1) with initial condition (2) is sufficiently smooth, we have

$$\begin{aligned}&\psi_0^{n+\frac{1}{2}}=O(\tau^2)+O(h),\qquad \psi_J^{n+\frac{1}{2}}=O(\tau^2)+O(h),\\&\psi_j^{n+\frac{1}{2}}=O(\tau^2)+O(h^2),\quad j=1,2,\cdots,J-1.\end{aligned}\tag{15}$$

Now, we estimate the norm of error function $z(x, t)$. For simplicity, we write $z_j^{n+\frac{1}{2}}=\frac{1}{2}(z_j^{n+1}+z_j^n)$. Multiplying equations (11), (12), (13) by $\frac{1}{2}z_0^{n+\frac{1}{2}}h$, $z_j^{n+\frac{1}{2}}h$, $\frac{1}{2}z_J^{n+\frac{1}{2}}h$ respectively and summing over $j=0, 1, \cdots, J$, we get

$$\begin{aligned}&\left[z^{n+\frac{1}{2}},\ \frac{z^{n+1}-z^n}{\tau}\right]-(z^{n+\frac{1}{2}},\ (a^{n+\frac{1}{2}}z_{\bar{x}}^{n+\frac{1}{2}})_x)-\frac{1}{2}\sum_{j=0}^{J-1}b_j^{n+\frac{1}{2}}z_j^{n+\frac{1}{2}}z_{x,j}^{n+\frac{1}{2}}h\\&-\frac{1}{2}\sum_{j=1}^{J}b_j^{n+\frac{1}{2}}z_j^{n+\frac{1}{2}}z_{\bar{x},j}^{n+\frac{1}{2}}h-\frac{1}{2}\sigma_0'^{n+\frac{1}{2}}z_0^{n+\frac{1}{2}}z_{x,0}^{n+\frac{1}{2}}h-\frac{1}{2}\sigma_J'^{n+\frac{1}{2}}z_J^{n+\frac{1}{2}}z_{\bar{x},J}^{n+\frac{1}{2}}h\\&-[z^{n+\frac{1}{2}},\ d^{n+\frac{1}{2}}z^{n+\frac{1}{2}}]=[z^{n+\frac{1}{2}},\ \psi^{n+\frac{1}{2}}].\end{aligned}\tag{16}$$

The first term on the left-hand side of (16) is

$$\left[z^{n+\frac{1}{2}},\ \frac{z^{n+1}-z^n}{\tau}\right]=\frac{1}{2\tau}(|[z^{n+1}]|^2-|[z^n]|^2).\tag{17}$$

Using the summation by parts formula, the second term gives

$$-(z^{n+\frac{1}{2}},(a^{n+\frac{1}{2}}z_{\bar{x}}^{n+\frac{1}{2}})_x)=\sum_{j=1}^{J}a_j^{n+\frac{1}{2}}(z_{\bar{x},j}^{n+\frac{1}{2}})^2h+a_1^{n+\frac{1}{2}}z_0^{n+\frac{1}{2}}z_{x,0}^{n+\frac{1}{2}}-a_J^{n+\frac{1}{2}}z_J^{n+\frac{1}{2}}z_{\bar{x},J}^{n+\frac{1}{2}}.\tag{18}$$

The third and fourth terms equal

$$\begin{aligned}&-\frac{1}{2}\sum_{j=0}^{J-1}b_j^{n+\frac{1}{2}}z_j^{n+\frac{1}{2}}z_{x,j}^{n+\frac{1}{2}}h-\frac{1}{2}\sum_{j=1}^{J}b_j^{n+\frac{1}{2}}z_j^{n+\frac{1}{2}}z_{\bar{x},j}^{n+\frac{1}{2}}h\\&=-\frac{1}{2}b_0^{n+\frac{1}{2}}z_0^{n+\frac{1}{2}}(z_1^{n+\frac{1}{2}}-z_0^{n+\frac{1}{2}})-\frac{1}{2}\sum_{j=1}^{J-1}b_j^{n+\frac{1}{2}}z_j^{n+\frac{1}{2}}(z_{j+1}^{n+\frac{1}{2}}-z_{j-1}^{n+\frac{1}{2}})\\&\quad-\frac{1}{2}b_J^{n+\frac{1}{2}}z_J^{n+\frac{1}{2}}(z_J^{n+\frac{1}{2}}-z_{J-1}^{n+\frac{1}{2}})\\&=\frac{1}{2}b_0^{n+\frac{1}{2}}(z_0^{n+\frac{1}{2}})^2-\frac{1}{2}b_J^{n+\frac{1}{2}}(z_J^{n+\frac{1}{2}})^2+\frac{1}{2}\sum_{j=0}^{J-1}b_{x,j}^{n+\frac{1}{2}}z_j^{n+\frac{1}{2}}z_{j+1}^{n+\frac{1}{2}}h.\end{aligned}\tag{19}$$

The last term on the left-hand side of (16) is

$$-[z^{n+\frac{1}{2}},\ d^{n+\frac{1}{2}}z^{n+\frac{1}{2}}]=-|[\sqrt{\overset{+}{d}{}^{n+\frac{1}{2}}}z^{n+\frac{1}{2}}]|^2+|[\sqrt{-\bar{d}^{n+\frac{1}{2}}}z^{n+\frac{1}{2}}]|^2,\tag{20}$$

where

$$\overset{+}{d}=\frac{1}{2}(d+|d|),\quad \bar{d}=\frac{1}{2}(d-|d|).\tag{21}$$

Considering (17)—(20), (16) becomes

$$\frac{1}{2\tau}(|[z^{n+1}]|^2-|[z^n]|^2)+\sum_{j=1}^{J}a_j^{n+\frac{1}{2}}(z_{\bar{x},j}^{n+\frac{1}{2}})^2h+a_1^{n+\frac{1}{2}}z_0^{n+\frac{1}{2}}z_{x,0}^{n+\frac{1}{2}}-a_J^{n+\frac{1}{2}}z_J^{n+\frac{1}{2}}z_{\bar{x},J}^{n+\frac{1}{2}}$$

$$+\frac{1}{2}b_0^{n+\frac{1}{2}}(z_0^{n+\frac{1}{2}})^2-\frac{1}{2}b_J^{n+\frac{1}{2}}(z_J^{n+\frac{1}{2}})^2+\frac{1}{2}\sum_{j=0}^{J-1}b_{x,j}^{n+\frac{1}{2}}z_j^{n+\frac{1}{2}}z_{j+1}^{n+\frac{1}{2}}h$$

$$-\frac{1}{2}\sigma_0'^{n+\frac{1}{2}}z_0^{n+\frac{1}{2}}z_{x,0}^{n+\frac{1}{2}}h-\frac{1}{2}\sigma_J'^{n+\frac{1}{2}}z_J^{n+\frac{1}{2}}z_{\bar{x},J}^{n+\frac{1}{2}}h$$

$$-|[\sqrt{\overset{+}{d}^{n+\frac{1}{2}}}z^{n+\frac{1}{2}}]|^2+|[\sqrt{-\bar{d}^{n+\frac{1}{2}}}z^{n+\frac{1}{2}}]|^2$$

$$\leqslant\frac{1}{2}|[z^{n+\frac{1}{2}}]|^2+\frac{1}{2}|[\psi^{n+\frac{1}{2}}]|^2. \tag{22}$$

By substracting certain non-negative terms from the left-hand side of inequality (22), we get

$$\frac{1}{2\tau}(|[z^{n+1}]|^2-|[z^n]|^2)+\Phi_0^{n+\frac{1}{2}}+\Phi_J^{n+\frac{1}{2}}\leqslant-\frac{1}{2}\sum_{j=0}^{J-1}b_{x,j}^{n+\frac{1}{2}}z_j^{n+\frac{1}{2}}z_{j+1}^{n+\frac{1}{2}}h$$

$$+|[\sqrt{\overset{+}{d}^{n+\frac{1}{2}}}z^{n+\frac{1}{2}}]|^2+\frac{1}{2}|[z^{n+\frac{1}{2}}]|^2+\frac{1}{2}|[\psi^{n+\frac{1}{2}}]|^2, \tag{23}$$

where $\Phi_0^{n+\frac{1}{2}}$, $\Phi_J^{n+\frac{1}{2}}$ are abbreviations

$$\Phi_0^{n+\frac{1}{2}}=a_1^{n+\frac{1}{2}}z_0^{n+\frac{1}{2}}z_{x,0}^{n+\frac{1}{2}}-\frac{1}{2}\sigma_0'^{n+\frac{1}{2}}z_0^{n+\frac{1}{2}}z_{x,0}^{n+\frac{1}{2}}h, \tag{24}$$

$$\Phi_J^{n+\frac{1}{2}}=-a_J^{n+\frac{1}{2}}z_J^{n+\frac{1}{2}}z_{\bar{x},J}^{n+\frac{1}{2}}-\frac{1}{2}\sigma_J'^{n+\frac{1}{2}}z_J^{n+\frac{1}{2}}z_{\bar{x},J}^{n+\frac{1}{2}}h. \tag{25}$$

Since σ is a sufficiently smooth function, we can expand $a_1^{n+\frac{1}{2}}$ and $a_J^{n+\frac{1}{2}}$ at $x=0$ and $x=l$ respectively. Then we have

$$\Phi_0^{n+\frac{1}{2}}=\left[\sigma_0'^{n+\frac{1}{2}}\frac{h}{2}+\frac{1}{2}\sigma_{\theta_1}''^{n+\frac{1}{2}}\left(\frac{h}{2}\right)^2\right]z_0^{n+\frac{1}{2}}z_{x,0}^{n+\frac{1}{2}}-\frac{1}{2}\sigma_0'^{n+\frac{1}{2}}z_0^{n+\frac{1}{2}}z_{x,0}^{n+\frac{1}{2}}h$$

$$=\frac{1}{8}\sigma_{\theta_1}''^{n+\frac{1}{2}}z_0^{n+\frac{1}{2}}z_{x,0}^{n+\frac{1}{2}}h^2,\qquad 0<\theta_1<\frac{1}{2}, \tag{26}$$

$$\Phi_J^{n+\frac{1}{2}}=-\left[-\sigma_J'^{n+\frac{1}{2}}\frac{h}{2}+\frac{1}{2}\sigma_{J-\theta_2}''^{n+\frac{1}{2}}\left(\frac{h}{2}\right)^2\right]z_J^{n+\frac{1}{2}}z_{\bar{x},J}^{n+\frac{1}{2}}-\frac{1}{2}\sigma_J'^{n+\frac{1}{2}}z_J^{n+\frac{1}{2}}z_{\bar{x},J}^{n+\frac{1}{2}}h$$

$$=-\frac{1}{8}\sigma_{J-\theta_2}''^{n+\frac{1}{2}}z_J^{n+\frac{1}{2}}z_{\bar{x},J}^{n+\frac{1}{2}}h^2,\qquad 0<\theta_2<\frac{1}{2}, \tag{27}$$

where σ'' denotes $\partial^2\sigma/\partial x^2$. Hence, the inequality (23) implies

$$\frac{1}{2\tau}(|[z^{n+1}]|^2-|[z^n]|^2)\leqslant-\frac{1}{8}\sigma_{\theta_1}''^{n+\frac{1}{2}}z_0^{n+\frac{1}{2}}z_{x,0}^{n+\frac{1}{2}}h^2+\frac{1}{8}\sigma_{J-\theta_2}''^{n+\frac{1}{2}}z_J^{n+\frac{1}{2}}z_{\bar{x},J}^{n+\frac{1}{2}}h^2$$

$$-\frac{1}{2}\sum_{j=0}^{J-1}b_{x,j}^{n+\frac{1}{2}}z_j^{n+\frac{1}{2}}z_{j+1}^{n+\frac{1}{2}}h+|[\sqrt{\overset{+}{d}^{n+\frac{1}{2}}}z^{n+\frac{1}{2}}]|^2+\frac{1}{2}|[z^{n+\frac{1}{2}}]|^2+\frac{1}{2}|[\psi^{n+\frac{1}{2}}]|^2. \tag{28}$$

Since σ is sufficiently smooth on $\bar{I}\times[0,T]$, σ'' is bounded,

$$|\sigma''(x,t)|\leqslant C_\sigma,\qquad (x,t)\in\bar{I}\times[0,T].$$

Then the absolute value of the sum of the first two terms on the right-hand side of the inequality (28) is dominated by

$$\frac{1}{8}C_\sigma\{|z_0^{n+\frac{1}{2}}(z_1^{n+\frac{1}{2}}-z_0^{n+\frac{1}{2}})|+|z_J^{n+\frac{1}{2}}(z_J^{n+\frac{1}{2}}-z_{J-1}^{n+\frac{1}{2}})|\}h$$
$$\leqslant\frac{3}{8}C_\sigma\frac{1}{2}(|[z^{n+1}]|^2+|[z^n]|^2). \tag{29}$$

Since $b(x, t)$ is also assumed to be sufficiently smooth, it certainly satisfies the Lipschitz condition

$$|b(x, t)-b(\tilde{x}, t)|\leqslant C_b|x-\tilde{x}|, \quad x, \tilde{x}\in\bar{I}, \quad t\in[0, T].$$

Then, we have

$$\left|-\frac{1}{2}\sum_{j=0}^{J-1}b_{x,j}^{n+\frac{1}{2}}z_j^{n+\frac{1}{2}}z_{j+1}^{n+\frac{1}{2}}h\right|\leqslant\frac{1}{2}C_b\cdot\frac{1}{2}(|[z^{n+1}]|^2+|[z^n]|^2). \tag{30}$$

Therefore, from (28), it follows that

$$\frac{1}{2\tau}(|[z^{n+1}]|^2-|[z^n]|^2)$$
$$\leqslant\left(\frac{3}{8}C_\sigma+\frac{1}{2}C_b+C_d+\frac{1}{2}\right)\frac{1}{2}(|[z^{n+1}]|^2+|[z^n]|^2)+\frac{1}{2}|[\psi^{n+\frac{1}{2}}]|^2. \tag{31}$$

Let $\frac{3}{8}C_\sigma+\frac{1}{2}C_b+C_d+\frac{1}{2}=M_1$. If $\tau<1/2M_1$, we have $\frac{1}{1-M_1\tau}<1+2M_1\tau$, $\frac{1+M_1\tau}{1-M_1\tau}<1+4M_1\tau$, and the inequality (31) can be written as

$$(1-M_1\tau)|[z^{n+1}]|^2\leqslant(1+M_1\tau)|[z^n]|^2+\tau|[\psi^{n+\frac{1}{2}}]|^2. \tag{32}$$

Dividing both sides of (32) by $1-M_1\tau$, we have

$$|[z^{n+1}]|^2\leqslant(1+4M_1\tau)|[z^n]|^2+(1+2M_1\tau)\tau|[\psi^{n+\frac{1}{2}}]|^2. \tag{33}$$

By summing up inequalities (33) from $n=0$ to N_0-1 $(0<N_0\leqslant N)$, we obtain

$$|[z^{N_0}]|^2\leqslant|[z^0]|^2+4M_1\sum_{n=0}^{N_0-1}|[z^n]|^2\tau+(1+2M_1\tau)\sum_{n=0}^{N_0-1}|[\psi^{n+\frac{1}{2}}]|^2\tau.$$

Because the function z satisfies (14), the above inequality becomes

$$|[z^n]|^2\leqslant4M_1\sum_{m=1}^{n-1}|[z^m]|^2\tau+(1+2M_1\tau)\sum_{m=0}^{n-1}|[\psi^{m+\frac{1}{2}}]|^2\tau, \quad n=1, 2, \cdots, N. \tag{34}$$

This implies[2]

$$|[z^n]|^2\leqslant2Te^{4M_1T}\max_{0\leqslant m\leqslant n-1}|[\psi^{m+\frac{1}{2}}]|^2.$$

Using (15), we have

$$|[\psi^{n+\frac{1}{2}}]|^2=\frac{1}{2}(\psi_0^{n+\frac{1}{2}})^2h+\sum_{j=1}^{J-1}(\psi_j^{n+\frac{1}{2}})^2h+\frac{1}{2}(\psi_J^{n+\frac{1}{2}})^2h=O(\tau^4)+O(h^3). \tag{35}$$

From (35), it follows that

$$|[z^n]|=O(\tau^2)+O(h^{\frac{3}{2}}), \quad n=1, 2, \cdots, N. \tag{36}$$

Then we obtain the following theorem about the convergence of the difference scheme (6)—(9).

Theorem 1. *Suppose that the coefficients of the degenerate parabolic equation* (1) *are sufficiently smooth and satisfy* (3). *Moreover, there exists a unique sufficiently smooth solution* $u(x, t)$ *of the equation* (1) *with initial condition* (2). *Then the solution* $y(x, t)$ *of the difference scheme* (6)—(9) *unconditionally converges to* $u(x,t)$ *as* $\tau\to0$, $h\to0$, *and the rate of convergence is* $O(\tau^2)+O(h^{3/2})$, *i. e.*

$$|[y(\cdot,t)-u(\cdot,t)]|=O(\tau^2)+O(h^{\frac{3}{2}}),\quad t\in\omega_\tau. \tag{36'}$$

Furthermore, the convergence of the fully implicit scheme

$$\frac{y_0^{n+1}-y_0^n}{\tau}=(\sigma_0'^{n+1}+b_0^{n+1})y_{x,0}^{n+1}+d_0^{n+1}y_0^{n+1}+f_0^{n+1}, \tag{37}$$

$$\frac{y_j^{n+1}-y_j^n}{\tau}=(a^{n+1}y_{\bar{x}}^{n+1})_{x,j}+\overset{+}{b}{}_j^{n+1}y_{x,j}^{n+1}+\bar{b}_j^{n+1}y_{\bar{x},j}^{n+1}+d_j^{n+1}y_j^{n+1}+f_j^{n+1},$$
$$j=1,2,\cdots,J-1, \tag{38}$$

$$\frac{y_J^{n+1}-y_J^n}{\tau}=(\sigma_J'^{n+1}+b_J^{n+1})y_{\bar{x},J}^{n+1}+d_J^{n+1}y_J^{n+1}+f_J^{n+1} \tag{39}$$

can also be easily proved. In (37)—(39) the definition of $\overset{+}{b}, \bar{b}$ is the same as that of $\overset{+}{d}, \bar{d}$ in (21). Now, the error function $z(x,t)$ satisfies the following equations

$$\left[1+\frac{\tau}{h}(\sigma_0'^{n+1}+b_0^{n+1})-d_0\tau\right]z_0^{n+1}=z_0^n+\frac{\tau}{h}(\sigma_0'^{n+1}+b_0^{n+1})z_1^{n+1}+\tau\psi_0^{n+1}, \tag{40}$$

$$\left[1+\frac{\tau}{h^2}\left(\sigma_{j+\frac{1}{2}}^{n+1}+\sigma_{j-\frac{1}{2}}^{n+1}\right)+\frac{\tau}{h}\left(\overset{+}{b}{}_j^{n+1}-\bar{b}_j^{n+1}\right)-d_j^{n+1}\tau\right]z_j^{n+1}$$
$$=z_j^n+\left[\frac{\tau}{h^2}\sigma_{j+\frac{1}{2}}^{n+1}+\frac{\tau}{h}\overset{+}{b}{}_j^{n+1}\right]z_{j+1}^{n+1}+\left[\frac{\tau}{h^2}\sigma_{j-\frac{1}{2}}^{n+1}-\frac{\tau}{h}\bar{b}_j^{n+1}\right]z_{j-1}^{n+1}+\tau\psi_j^{n+1},$$
$$j=1,2,\cdots,J-1, \tag{41}$$

$$\left[1-\frac{\tau}{h}(\sigma_J'^{n+1}+b_J^{n+1})-d_J^{n+1}\tau\right]z_J^{n+1}=z_J^n-\frac{\tau}{h}(\sigma_J'^{n+1}+b_J^{n+1})z_{J-1}^{n+1}+\tau\psi_J^{n+1}, \tag{42}$$

where the truncation error

$$\psi_j^{n+1}=O(\tau)+O(h),\quad j=0,1,\cdots,J.$$

Without loss of generality, we assume that $d(x,t)\leqslant-\delta<0$, then by means of maximum principle of difference equations (see [3], Chapter IV), we have

$$\max_{j,n}|z_j^n|\leqslant\frac{1}{\delta}\max_{j,n}|\psi_j^n|.$$

Therefore, the solution of the difference scheme (37)—(39) and (9) converges unconditionally to the solution of the equation (1) with initial condition (2) in the maximum norm, and its rate of convergence is $O(\tau)+O(h)$.

§2

Consider the simplest two-dimensional degenerate parabolic equation

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x_1}\sigma\frac{\partial u}{\partial x_1}+b_1\frac{\partial u}{\partial x_1}+b_2\frac{\partial u}{\partial x_2}+du+f,\qquad(x_1,x_2)\in\Omega,\ t\in(0,T]. \tag{43}$$

Let Ω be a rectangular domain $\{0<x_1<l_1,\ 0<x_2<l_2\}$, $\partial\Omega$ be its boundary, $\bar{\Omega}$ be its closure. $\partial_1\Omega$ denotes the point set $\{0<x_1<l_1,\ x_2=0\}$, and $\partial_2\Omega$, $\partial_3\Omega$, $\partial_4\Omega$ denote $\{x_1=l_1, 0<x_2<l_2\}$, $\{0<x_1<l_1,\ x_2=l_2\}$, $\{x_1=0,\ 0<x_2<l_2\}$ respectively. Let $\bar{\partial}_2\Omega$ and $\bar{\partial}_4\Omega$ denote the closure of $\partial_2\Omega$ and $\partial_4\Omega$. We assume

$$\sigma(x_1,x_2,t)>0,\quad(x_1,x_2,t)\in\bar{\Omega}\times(0,T], \tag{44}$$

$$b_2(x_1,0,t)\geqslant0,\quad 0<x_1<l_1,\ 0<t\leqslant T,$$
$$b_2(x_1,l_2,t)\leqslant0,\quad 0<x_1<l_1,\ 0<t\leqslant T. \tag{45}$$

Then, the initial and boundary-value problem of the equation (43) is properly posed, when the initial condition

$$u(x_1, x_2, 0) = g_0(x_1, x_2), \quad (x_1, x_2) \in \bar{\Omega} \tag{46}$$

and the boundary condition

$$u(x_1, x_2, t) = g_1(x_1, x_2, t), \quad (x_1, x_2) \in \bar{\partial}_2\Omega \cup \bar{\partial}_4\Omega, \quad t \in (0, T] \tag{47}$$

are given. We suppose that the coefficients in the equation (43) are sufficiently smooth and that there exists a unique sufficiently smooth solution for initial and boundary-value problem (43), (46), (47) on the domain $\bar{\Omega} \times [0, T]$.

Solve the equations (43), (46), (47) by difference methods. Divide the time interval $[0, T]$ into N parts, and the time step is $\tau = T/N$. Denote the set $\left\{ t^n = n\tau \,\middle|\, n = 0, \frac{1}{2}, 1, \cdots, N - \frac{1}{2}, N \right\}$ by $\bar{\omega}_\tau$. For the rectangular domain $\bar{\Omega}$, take the space steps $h_1 = l_1/J$, $h_2 = l_2/K$. The net points are represented by $P_{jk}(jh_1, kh_2)$. Let $\bar{\omega} = \{P_{jk} \mid j = 0, 1, \cdots, J; k = 0, 1, \cdots, K\}$ and $\omega = \bar{\omega} \cap \Omega$, $\partial\omega = \bar{\omega} \cap \partial\Omega$, $\gamma_i = \bar{\omega} \cap \partial_i\Omega$ $(i = 1, 2, 3, 4)$, $\bar{\gamma}_i = \bar{\omega} \cap \bar{\partial}_i\Omega$ $(i = 2, 4)$.

Let $v(x_1, x_2, t)$, $w(x_1, x_2, t)$ be functions, defined on the set $\bar{\omega} \times \bar{\omega}_\tau$. Introduce the following notations

$$v^n = v^n_{jk} = v^n(P_{jk}) = v(jh_1, kh_2, n\tau),$$

$$v^n_{x_1} = v^n_{x_1, jk} = \frac{1}{h_1}(v^n_{j+1,k} - v^n_{j,k}), \quad v^n_{\bar{x}_1} = v^n_{\bar{x}_1, jk} = \frac{1}{h_1}(v^n_{jk} - v^n_{j-1,k}),$$

$$v^n_{x_2} = v^n_{x_2, jk} = \frac{1}{h_2}(v^n_{j,k+1} - v^n_{jk}), \quad v^n_{\bar{x}_2} = v^n_{\bar{x}_2, jk} = \frac{1}{h_2}(v^n_{jk} - v^n_{j,k-1}).$$

Define the inner product on the set $\mathscr{S} \subseteq \bar{\omega}$

$$(v^n, w^n)_{\mathscr{S}} = \sum_{P_{jk} \in \mathscr{S}} \alpha(P_{jk}) v^n(P_{jk}) w^n(P_{jk}) h_1 h_2,$$

where $\alpha(P_{jk}) = \frac{1}{2}$ for $P_{jk} \in \partial\omega$, and $\alpha(P_{jk}) = 1$ for $P_{jk} \in \omega$. Consequently, define the norm

$$\|v^n\|_{\mathscr{S}} = (v^n, v^n)^{\frac{1}{2}}_{\mathscr{S}}.$$

Since the boundary condition is given only on $\bar{\gamma}_2 \cup \bar{\gamma}_4$, we must set up the difference scheme on the set $\omega_0 = \omega \cup \gamma_1 \cup \gamma_3$. Construct the fractional step difference scheme

$$\frac{y^{n+\frac{1}{2}} - y^n}{\tau} = (a^{n+\alpha} y^{n+\frac{1}{2}}_{\bar{x}_1})_{x_1} + \frac{1}{2} b_1^{n+\beta} (y^{n+\frac{1}{2}}_{\bar{x}_1} + y^{n+\frac{1}{2}}_{x_1}) + \theta d^{n+\beta} y^{n+\frac{1}{2}} + \eta f^{n+\beta}, \ (x_1, x_2) \in \omega_0, \tag{48}$$

$$\frac{y^{n+1} - y^{n+\frac{1}{2}}}{\tau} = b_2^{n+\beta} y^{n+1}_{x_2} + (1-\theta) d^{n+\beta} y^{n+1} + (1-\eta) f^{n+\beta}, \quad (x_1, x_2) \in \gamma_1, \tag{49$_1$}$$

$$\frac{y^{n+1} - y^{n+\frac{1}{2}}}{\tau} = \frac{1}{2} b_2^{n+\beta} (y^{n+1}_{\bar{x}_2} + y^{n+1}_{x_2}) + (1-\theta) d^{n+\beta} y^{n+1} + (1-\eta) f^{n+\beta}, \quad (x_1, x_2) \in \omega, \tag{49$_2$}$$

$$\frac{y^{n+1} - y^{n+\frac{1}{2}}}{\tau} = b_2^{n+\beta} y^{n+1}_{\bar{x}_2} + (1-\theta) d^{n+\beta} y^{n+1} + (1-\eta) f^{n+\beta}, \quad (x_1, x_2) \in \gamma_3, \tag{49$_3$}$$

together with the initial and boundary conditions

$$y(x_1, x_2, 0)=g_0(x_1, x_2), \qquad (x_1, x_2)\in\bar{\omega} \tag{50}$$

$$y(x_1, x_2, t)=g_1(x_1, x_2, t), \qquad (x_1, x_2)\in\bar{\gamma}_2\cup\bar{\gamma}_4,\ t\in\bar{\omega}_\tau\backslash t^\circ, \tag{51}$$

where $a_{jk}^{n+\alpha}=\sigma\left(\left(j-\frac{1}{2}\right)h_1,\ kh_2,\ (n+\alpha)\tau\right)$ and α, θ, η are constants, lying in the interval [0, 1]. The function with superscript $n+\beta$ represents its value at $t=(n+\beta)\tau$. The constant β also takes values between 0 and 1, but for different functions the values of β may not necessarily be the same.

Let $z(x_1, x_2, t)$ denote the difference between the solution of the difference equations and that of the differential equation. Therefore, the error function satisfies the following equations

$$\frac{z^{n+\frac{1}{2}}-z^n}{\tau}-(a^{n+\alpha}z_{\bar{x}_1}^{n+\frac{1}{2}})_{x_1}-\frac{1}{2}b_1^{n+\beta}(z_{\bar{x}_1}^{n+\frac{1}{2}}+z_{x_1}^{n+\frac{1}{2}})-\theta d^{n+\beta}z^{n+\frac{1}{2}}=\psi_1^{n+1},\ (x_1, x_2)\in\omega_0, \tag{52}$$

$$\frac{z^{n+1}-z^{n+\frac{1}{2}}}{\tau}-b_2^{n+\beta}z_{x_2}^{n+1}-(1-\theta)d^{n+\beta}z^{n+1}=\psi_2^{n+1},\ (x_1, x_2)\in\gamma_1, \tag{53$_1$}$$

$$\frac{z^{n+1}-z^{n+\frac{1}{2}}}{\tau}-\frac{1}{2}b_2^{n+\beta}(z_{\bar{x}_2}^{n+1}+z_{x_2}^{n+1})-(1-\theta)d^{n+\beta}z^{n+1}=\psi_2^{n+1},\ (x_1, x_2)\in\omega, \tag{53$_2$}$$

$$\frac{z^{n+1}-z^{n+\frac{1}{2}}}{\tau}-b_2^{n+\beta}z_{\bar{x}_2}^{n+1}-(1-\theta)d^{n+\beta}z^{n+1}=\psi_2^{n+1},\ (x_1, x_2)\in\gamma_3 \tag{53$_3$}$$

and homogeneous initial and boundary conditions

$$z(x_1, x_2, 0)=0, \qquad (x_1, x_2)\in\bar{\omega}, \tag{54}$$

$$z(x_1, x_2, t)=0, \qquad (x_1, x_2)\in\bar{\gamma}_2\cup\bar{\gamma}_4,\ t\in\bar{\omega}_\tau\backslash t^\circ. \tag{55}$$

The right-hand members of (52), (53) are the truncation errors

$$\begin{aligned}\psi_1^{n+1}=&\left[\frac{\partial}{\partial x_1}\sigma\frac{\partial u}{\partial x_1}+b_1\frac{\partial u}{\partial x_1}+\theta du+\eta f-\frac{1}{2}\frac{\partial u}{\partial t}\right]^{n+1}-\left[\frac{u^{n+\frac{1}{2}}-u^n}{\tau}-\frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^{n+1}\right]\\&+\left[(a^{n+\alpha}u_{\bar{x}_1}^{n+\frac{1}{2}})_{x_1}-\left(\frac{\partial}{\partial x_1}\sigma\frac{\partial u}{\partial x_1}\right)^{n+1}\right]+\left[\frac{1}{2}b_1^{n+\beta}(u_{\bar{x}_1}^{n+\frac{1}{2}}+u_{x_1}^{n+\frac{1}{2}})-\left(b_1\frac{\partial u}{\partial x_1}\right)^{n+1}\right]\\&+\theta\,[d^{n+\beta}u^{n+\frac{1}{2}}-(du)^{n+1}]+\eta[f^{n+\beta}-f^{n+1}],\ (x_1, x_2)\in\omega_0,\end{aligned}$$

$$\begin{aligned}\psi_2^{n+1}=&\left[b_2\frac{\partial u}{\partial x_2}+(1-\theta)du+(1-\eta)f-\frac{1}{2}\frac{\partial u}{\partial t}\right]^{n+1}-\left[\frac{u^{n+1}-u^{n+\frac{1}{2}}}{\tau}-\frac{1}{2}\left(\frac{\partial u}{\partial t}\right)^{n+1}\right]\\&+\psi_{2,b}^{n+1}+(1-\theta)\,[d^{n+\beta}u^{n+1}-(du)^{n+1}]+(1-\eta)\,[f^{n+\beta}-f^{n+1}],\ (x_1, x_2)\in\omega_0,\end{aligned}$$

where

$$\psi_{2,b}^{n+1}=\begin{cases}b_2^{n+\beta}u_{x_2}^{n+1}-\left(b_2\frac{\partial u}{\partial x_2}\right)^{n+1}, & (x_1, x_2)\in\gamma_1,\\ \frac{1}{2}b_2^{n+\beta}(u_{\bar{x}_2}^{n+1}+u_{x_2}^{n+1})-\left(b_2\frac{\partial u}{\partial x_2}\right)^{n+1}, & (x_1, x_2)\in\omega,\\ b_2^{n+\beta}u_{\bar{x}_2}^{n+1}-\left(b_2\frac{\partial u}{\partial x_2}\right)^{n+1}, & (x_1, x_2)\in\gamma_3.\end{cases}$$

Let

$$\mathring{\psi}_1^{n+1}=\left[\frac{\partial}{\partial x_1}\sigma\frac{\partial u}{\partial x_1}+b_1\frac{\partial u}{\partial x_1}+\theta\,du+\eta f-\frac{1}{2}\frac{\partial u}{\partial t}\right]^{n+1},$$

$$\mathring{\psi}_2^{n+1}=\left[b_2\frac{\partial u}{\partial x_2}+(1-\theta)du+(1-\eta)f-\frac{1}{2}\frac{\partial u}{\partial t}\right]^{n+1},$$

$$\tilde{\psi}_1^{n+1}=\psi_1^{n+1}-\mathring{\psi}_1^{n+1},$$

$$\tilde{\psi}_2^{n+1}=\psi_2^{n+1}-\mathring{\psi}_2^{n+1},$$

we have

$$\mathring{\psi}_1^{n+1}+\mathring{\psi}_2^{n+1}=0,\qquad (x_1,\ x_2)\in\omega_0,\ n=0,\ 1,\ \cdots,\ N-1. \tag{56}$$

If the solution of (43), (46), (47) is sufficiently smooth, it is clear that

$$\begin{aligned}&\tilde{\psi}_1^{n+1}=O(\tau)+O(h_1^2), && (x_1,\ x_2)\in\omega_0,\\ &\tilde{\psi}_2^{n+1}=O(\tau)+O(h_2), && (x_1,\ x_2)\in\gamma_1\cup\gamma_3,\\ &\tilde{\psi}_2^{n+1}=O(\tau)+O(h_2^2), && (x_1,\ x_2)\in\omega.\end{aligned} \tag{57}$$

Let $z(x_1,\ x_2,\ t)=w(x_1,\ x_2,\ t)+v(x_1,\ x_2,\ t)$, where w satisfies equations

$$\frac{w^{n+\frac{1}{2}}-w^n}{\tau}=\mathring{\psi}_1^{n+1}, \tag{58}$$

$$(x_1,\ x_2)\in\omega_0,\ n=0,\ 1,\ \cdots,\ N-1.$$

$$\frac{w^{n+1}-w^{n+\frac{1}{2}}}{\tau}=\mathring{\psi}_2^{n+1}, \tag{59}$$

Therefore, $v(x_1,\ x_2,\ t)$ satisfies equations

$$\frac{v^{n+\frac{1}{2}}-v^n}{\tau}-(a^{n+\alpha}v_{\bar{x}_1}^{n+\frac{1}{2}})_{x_1}-\frac{1}{2}b_1^{n+\beta}(v_{\bar{x}_1}^{n+\frac{1}{2}}+v_{x_1}^{n+\frac{1}{2}})-\theta d^{n+\beta}v^{n+\frac{1}{2}}$$
$$=\tilde{\psi}_1^{n+1}+(a^{n+\alpha}w_{\bar{x}_1}^{n+\frac{1}{2}})_{x_1}+\frac{1}{2}b_1^{n+\beta}(w_{\bar{x}_1}^{n+\frac{1}{2}}+w_{x_1}^{n+\frac{1}{2}})+\theta d^{n+\beta}w^{n+\frac{1}{2}},$$
$$(x_1,\ x_2)\in\omega_0,\ n=0,\ 1,\ \cdots,\ N-1, \tag{60}$$

$$\frac{v^{n+1}-v^{n+\frac{1}{2}}}{\tau}-b_2^{n+\beta}v_{x_2}^{n+1}-(1-\theta)d^{n+\beta}v^{n+1}=\tilde{\psi}_2^{n+1}+b_2^{n+\beta}w_{x_2}^{n+1}+(1-\theta)d^{n+\beta}w^{n+1},$$
$$(x_1,\ x_2)\in\gamma_1,\ n=0,\ 1,\ \cdots,\ N-1, \tag{61$_1$}$$

$$\frac{v^{n+1}-v^{n+\frac{1}{2}}}{\tau}-\frac{1}{2}b_2^{n+\beta}(v_{\bar{x}_2}^{n+1}+v_{x_2}^{n+1})-(1-\theta)d^{n+\beta}v^{n+1}$$
$$=\tilde{\psi}_2^{n+1}+\frac{1}{2}b_2^{n+\beta}(w_{\bar{x}_2}^{n+1}+w_{x_2}^{n+1})+(1-\theta)d^{n+\beta}w^{n+1},$$
$$(x_1,\ x_2)\in\omega,\ n=0,\ 1,\ \cdots,\ N-1, \tag{61$_2$}$$

$$\frac{v^{n+1}-v^{n+\frac{1}{2}}}{\tau}-b_2^{n+\beta}v_{\bar{x}_2}^{n+1}-(1-\theta)d^{n+\beta}v^{n+1}=\tilde{\psi}_2^{n+1}+b_2^{n+\beta}w_{\bar{x}_2}^{n+1}+(1-\theta)d^{n+\beta}w^{n+1},$$
$$(x_1,\ x_2)\in\gamma_3,\ n=0,\ 1,\ \cdots,\ N-1. \tag{61$_3$}$$

Meanwhile, w and v satisfy the homogeneous initial and boundary conditions (54), (55) as z.

By adding (58) and (59), we get

$$\frac{w^{n+1}-w^n}{\tau}=0,\qquad (x_1,\ x_2)\in\omega_0,\ n=0,\ 1,\ \cdots,\ N-1,$$

so $w^{n+1}=w^n$. Since $w^0=0$, we obtain $w^n=0$. Hence

$$z^n=v^n, \qquad (x_1,\ x_2)\in\omega_0,\ n=0,\ 1,\ \cdots,\ N. \tag{62}$$

On the other hand, from (58) follows $w^{n+1}=\tau\mathring{\psi}_1^{n+1}$. Denote the right-hand member of (60) by Φ^{n+1}. Therefore, we have

$$\Phi^{n+1}=O(\tau)+O(h_1^2) \tag{63}$$

if u is sufficiently smooth.

The inner product of the equation (60) and $v^{n+\frac{1}{2}}$ over the set ω_0 is

$$\left(v^{n+\frac{1}{2}},\ \frac{v^{n+\frac{1}{2}}-v^n}{\tau}\right)_{\omega_0}-(v^{n+\frac{1}{2}},\ (a^{n+\alpha}v_{\bar{x}_1}^{n+\frac{1}{2}})_{x_1})_{\omega_0}-\frac{1}{2}(v^{n+\frac{1}{2}},\ b_1^{n+\beta}\,(v_{\bar{x}_1}^{n+\frac{1}{2}}+v_{x_1}^{n+\frac{1}{2}}))_{\omega_0}$$
$$-\theta(v^{n+\frac{1}{2}},\ d^{n+\beta}v^{n+\frac{1}{2}})_{\omega_0}=(v^{n+\frac{1}{2}},\ \Phi^{n+1})_{\omega_0}. \tag{64}$$

The first term of the left-hand side of (64) can be written as

$$\left(v^{n+\frac{1}{2}},\ \frac{v^{n+\frac{1}{2}}-v^n}{\tau}\right)_{\omega_0}=\frac{1}{2\tau}(\|v^{n+\frac{1}{2}}\|^2_{\omega_0}-\|v^n\|^2_{\omega_0})+\frac{\tau}{2}\left\|\frac{v^{n+\frac{1}{2}}-v^n}{\tau}\right\|^2_{\omega_0}.$$

Using the summation by parts formula, the second term is

$$-(v^{n+\frac{1}{2}},\ (a^{n+\alpha}v_{\bar{x}_1}^{n+\frac{1}{2}})_{x_1})_{\omega_0}=\frac{1}{2}\sum_{j=1}^{J}a_{j0}^{n+\alpha}(v_{\bar{x}_1,j0}^{n+\frac{1}{2}})^2h_1h_2+\sum_{k=1}^{K-1}\sum_{j=1}^{J}a_{jk}^{n+\alpha}(v_{\bar{x}_1,jk}^{n+\frac{1}{2}})^2h_1h_2$$
$$+\frac{1}{2}\sum_{j=1}^{J}a_{jK}^{n+\alpha}(v_{\bar{x}_1,jK}^{n+\frac{1}{2}})^2h_1h_2\geq 0.$$

The third term equals

$$-\frac{1}{2}(v^{n+\frac{1}{2}},\ b_1^{n+\beta}(v_{\bar{x}_1}^{n+\frac{1}{2}}+v_{x_1}^{n+\frac{1}{2}}))_{\omega_0}=\frac{1}{4}\sum_{j=1}^{J-1}b_{1,\bar{x}_1,j0}^{n+\beta}v_{j-1,0}^{n+\frac{1}{2}}v_{j0}^{n+\frac{1}{2}}h_1h_2$$
$$+\frac{1}{2}\sum_{k=1}^{K-1}\sum_{j=1}^{J-1}b_{1,\bar{x}_1,jk}^{n+\beta}v_{j-1,k}^{n+\frac{1}{2}}v_{jk}^{n+\frac{1}{2}}h_1h_2+\frac{1}{4}\sum_{j=1}^{J-1}b_{1,\bar{x}_1,jK}^{n+\beta}v_{j-1,K}^{n+\frac{1}{2}}v_{jK}^{n+\frac{1}{2}}h_1h_2.$$

Then, (64) takes the form

$$\frac{1}{2\tau}(\|v^{n+\frac{1}{2}}\|^2_{\omega_0}-\|v^n\|^2_{\omega_0})+\frac{\tau}{2}\left\|\frac{v^{n+\frac{1}{2}}-v^n}{\tau}\right\|^2_{\omega_0}-(v^{n+\frac{1}{2}},\ (a^{n+\alpha}v_{\bar{x}_1}^{n+\frac{1}{2}})_{x_1})_{\omega_0}$$
$$+\frac{1}{4}\sum_{j=1}^{J-1}b_{1,\bar{x}_1,j0}^{n+\beta}v_{j-1,0}^{n+\frac{1}{2}}v_{j0}^{n+\frac{1}{2}}h_1h_2+\frac{1}{2}\sum_{k=1}^{K-1}\sum_{j=1}^{J-1}b_{1,\bar{x}_1,jk}^{n+\beta}v_{j-1,k}^{n+\frac{1}{2}}v_{jk}^{n+\frac{1}{2}}h_1h_2$$
$$+\frac{1}{4}\sum_{j=1}^{J-1}b_{1,\bar{x}_1,jK}^{n+\beta}v_{j-1,K}^{n+\frac{1}{2}}v_{jK}^{n+\frac{1}{2}}h_1h_2+\theta\|\sqrt{-\bar{d}^{n+\beta}}\,v^{n+\frac{1}{2}}\|^2_{\omega_0}-\theta\|\sqrt{\overset{+}{d}{}^{n+\beta}}\,v^{n+\frac{1}{2}}\|^2_{\omega_0}$$
$$=(v^{n+\frac{1}{2}},\ \Phi^{n+1})_{\omega_0}.$$

By substracting certain non-negative terms from the left-hand side of the above equation, we get

$$\frac{1}{2\tau}(\|v^{n+\frac{1}{2}}\|^2_{\omega_0}-\|v^n\|^2_{\omega_0})\leq\left(\frac{1}{2}C_b+\theta C_d+\frac{1}{2}\right)\|v^{n+\frac{1}{2}}\|^2_{\omega_0}+\frac{1}{2}\|\Phi^{n+1}\|^2_{\omega_0}, \tag{65}$$

where C_b is the Lipschitz constant for the coefficients $b_i(x_1,\ x_2,\ t)$ $(i=1,\ 2)$ and C_d is the upper bound of the function $\overset{+}{d}$ on the domain $\bar{\Omega}\times[0,\ T]$. Multiplying both sides of (65) by 2τ, we have

$$(1-M_1\tau)\|v^{n+\frac{1}{2}}\|^2_{\omega_0}\leqslant\|v^n\|^2_{\omega_0}+\tau\|\Phi^{n+1}\|^2_{\omega_0}, \tag{66}$$

where $M_1=C_b+2\theta C_d+1$.

Similarly, the inner product of equation (61) and v^{n+1} over the set ω_0 is

$$\left(v^{n+1}, \frac{v^{n+1}-v^{n+\frac{1}{2}}}{\tau}\right)_{\omega_0}-\frac{1}{2}\sum_{k=0}^{K-1}\sum_{j=1}^{J-1}b^{n+\beta}_{2,jk}v^{n+1}_{jk}v^{n+1}_{x_2,jk}h_1h_2$$
$$-\frac{1}{2}\sum_{k=1}^{K}\sum_{j=1}^{J-1}b^{n+\beta}_{2,jk}v^{n+1}_{jk}v^{n+1}_{\bar{x}_2,jk}h_1h_2$$
$$-(1-\theta)(v^{n+1}, d^{n+\beta}v^{n+1})_{\omega_0}=(v^{n+1}, \tilde{\psi}^{n+1}_2)_{\omega_0}. \tag{67}$$

Applying the formula (19) in § 1, (67) can be written as

$$\frac{1}{2\tau}(\|v^{n+1}\|^2_{\omega_0}-\|v^{n+\frac{1}{2}}\|^2_{\omega_0})+\frac{\tau}{2}\left\|\frac{v^{n+1}-v^{n+\frac{1}{2}}}{\tau}\right\|^2_{\omega_0}$$
$$+\frac{1}{2}\sum_{j=1}^{J-1}b^{n+\beta}_{2,j0}(v^{n+1}_{j,0})^2h_1-\frac{1}{2}\sum_{j=1}^{J-1}b^{n+\beta}_{2,jK}(v^{n+1}_{jK})^2h_1$$
$$+\frac{1}{2}\sum_{k=0}^{K-1}\sum_{j=1}^{J-1}b^{n+\beta}_{2,x_2,jk}v^{n+1}_{jk}v^{n+1}_{j,k+1}h_1h_2$$
$$-(1-\theta)\|\sqrt{\overset{+}{d}{}^{n+\beta}}\,v^{n+1}\|^2_{\omega_0}+(1-\theta)\|\sqrt{-\overset{-}{d}{}^{n+\beta}}\,v^{n+1}\|^2_{\omega_0}$$
$$=(v^{n+1}, \tilde{\psi}^{n+1}_2)_{\omega_0}. \tag{68}$$

Substracting certain non-negative terms from the left-hand side of (68), we obtain

$$\frac{1}{2\tau}(\|v^{n+1}\|^2_{\omega_0}-\|v^{n+\frac{1}{2}}\|^2_{\omega_0})\leqslant\left\{\frac{1}{2}C_b+(1-\theta)C_d+\frac{1}{2}\right\}\|v^{n+1}\|^2_{\omega_0}+\frac{1}{2}\|\tilde{\psi}^{n+1}_2\|^2_{\omega_0}.$$

Hence, we have

$$(1-M_2\tau)\|v^{n+1}\|^2_{\omega_0}\leqslant\|v^{n+\frac{1}{2}}\|^2_{\omega_0}+\tau\|\tilde{\psi}^{n+1}_2\|^2_{\omega_0}, \tag{69}$$

where $M_2=C_b+2(1-\theta)C_d+1$.

Let $\tau\leqslant\tau_0=\min\left\{\frac{1}{2M_1}, \frac{1}{2M_2}\right\}$. Multiply both sides of inequality (69) by $1-M_1\tau$, and by adding it to (66), we get

$$(1-M_1\tau)(1-M_2\tau)\|v^{n+1}\|^2_{\omega_0}\leqslant\|v^n\|^2_{\omega_0}+\tau\|\Phi^{n+1}\|^2_{\omega_0}+(1-M_1\tau)\tau\|\tilde{\psi}^{n+1}_2\|^2_{\omega_0}.$$

Let $M_0=3(M_1+M_2)$, then it is easy to see that $\frac{1}{(1-M_1\tau)(1-M_2\tau)}\leqslant 1+M_0\tau$. Hence, we have

$$\|v^{n+1}\|^2_{\omega_0}\leqslant(1+M_0\tau)\|v^n\|^2_{\omega_0}+(1+M_0\tau)\tau(\|\Phi^{n+1}\|^2_{\omega_0}+\|\tilde{\psi}^{n+1}_2\|^2_{\omega_0}). \tag{70}$$

This is an inequality similar to (33), and it implies

$$\|v^n\|^2_{\omega_0}\leqslant 4Te^{M_0T}\{\max_{1\leqslant m\leqslant n}(\|\Phi^m\|^2_{\omega_0}+\|\tilde{\psi}^m_2\|^2_{\omega_0})\},\quad n=1, 2, \cdots, N. \tag{71}$$

By (57) and (63), we obtain

$$\|v^n\|^2_{\omega_0}=O(\tau)+O(h_1^2)+O(h_2^{\frac{3}{2}}), \tag{72}$$

from which follows the convergence of the scheme (48)—(51).

Theorem 2. *Suppose that the coefficients of the equation* (43) *are sufficiently smooth, and satisfy* (44), (45). *Moreover, there exists a unique sufficiently smooth solution* $u(x_1, x_2, t)$ *of the problem* (43), (46), (47) *on the domain* $\bar{\Omega}\times[0, T]$. *Then the solution of the fractional step difference scheme* (48)—(51) *converges unconditionally to* $u(x_1, x_2, t)$ *as* $\tau\to 0$, $h_1\to 0$, $h_2\to 0$, *and its rate of convergence is* $O(\tau)+O(h_1^2)+O(h_2^{\frac{3}{2}})$.

Proof. From (62) and (72) it follows immediately

$$\|z^n\|_{\omega_0}=\|y(\cdot,\ \cdot,t^n)-u(\cdot,\ \cdot,\ t^n)\|_{\omega_0}=O(\tau)+O(h_1^2)+O(h_2^{\frac{3}{2}}),\quad n=1,\ 2,\ \cdots,\ N.$$

Acknowledgement. The author thanks Prof. Zhang Guan-quan for his valuable suggestion.

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