COUPLING CANONICAL BOUNDARY ELEMENT METHOD WITH FEM TO SOLVE HARMONIC PROBLEM OVER CRACKED DOMAIN*

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Abstract

Using the canonical boundary reduction, suggested by Feng Kang [1,2], coupled with the finite element method, this paper gives the numerical solution of the harmonic boundary-value problem over the domain with crack or concave angle. When the coupling is conforming, convergence and error estimates are obtained. This coupling removes the limitation of the canonical boundary reduction to some typical domains, and avoids the shortcoming of the classical finite element method, because of which the accuracy is damaged seriously and the approximate solution does not reflect the behaviour of the solution near the singularity. Numerical calculations have verified those conclusions.

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It is known that elliptic boundary-value problems can be reduced to integral equations on the boundary in different ways. The canonical reduction, suggested by Feng Kang in recent years [1,2], is a natural and direct reduction, which preserves the essential characteristics of the original problem. Unfortunately, it is only applicable to some typical domains. The classical finite element method can be applied to relatively arbitrary domains, but except cracked domains. Therefore it is only natural to couple the canonical boundary element method with the finite element method.

1. The Method

In [3], a numerical method by canonical boundary reduction with error estimates is given for solving two kinds of boundary-value problems of harmonic equation over sector with crack and concave angle. For the harmonic boundary-value problem over general domain with crack and concave angle, we can couple the canonical boundary element method with the finite element method as follows.

Let Ω be a domain bounded by two sides Γ_1 and Γ_2 of a concave angle α ($\pi<\alpha\leqslant 2\pi$) and a smooth curve Γ. When $\alpha=2\pi$, the domain contains a crack. Consider the boundary-value problem

$$\begin{cases} \Delta u=0, \text{ in } \Omega, \\ \partial_n u=0, \text{ on } \Gamma_1\cup\Gamma_2, \qquad \partial_n u=f, \text{ on } \Gamma, \end{cases} \tag{1}$$

where $f\in H^{-\frac{1}{2}}(\Gamma)$ satisfies the consistency condition

$$\int_\Gamma f ds=0.$$

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It is well-known, that problem (1) has a solution unique up to a constant. Let

$$D(u, v)=\iint_{\Omega}\nabla u\cdot\nabla v\,dp.$$

Then problem (1) is equivalent to the variational problem

$$\begin{cases}\text{Find } u\in H^1(\Omega)/P_0 & \text{such that}\\ D(u, v)=\int_{\Gamma}vf\,ds, & \forall v\in H^1(\Omega),\end{cases}\tag{2}$$

where P_0 is all constants. Using the Lax-Milgram lemma, we can easily prove that problem (2) has one and only one solution in $H^1(\Omega)/P_0$.

Now take the vertex of angle α as origin and place Γ_1 on x axis. Then draw in Ω an arc $\Gamma'=\{(R, \theta)\,|_{0<\theta<\alpha}\}$ dividing Ω into Ω_1 and Ω_2, where Ω_2 is a sector. We have

$$\iint_{\Omega}\nabla u\cdot\nabla v dp=\iint_{\Omega_1}\nabla u\cdot\nabla v\,dp+\int_{\Gamma'}v\partial_n u\,ds,$$

and

$$u_n(\theta)=-\frac{\pi}{4\alpha^2R}\int_0^\alpha\Big(\frac{1}{\sin^2\frac{\theta-\theta'}{2\alpha}\pi}+\frac{1}{\sin^2\frac{\theta+\theta'}{2\alpha}\pi}\Big)u(R, \theta')d\theta', \quad 0<\theta<\alpha,$$

which is the canonical integral equation of Γ' obtained in [3]. It contains a singular kernel and can be defined in the sense of distributions. Then problem (1) is equivalent to the variational problem

$$\begin{cases}\text{Find } u\in H^1(\Omega_1)/P_0 \quad \text{such that}\\ D_1(u, v)+\overline{D}_2(\gamma'u, \gamma'v)=\int_{\Gamma}vf\,ds, \quad \forall v\in H^1(\Omega_1),\end{cases}\tag{3}$$

where $D_1(u, v)=\iint_{\Omega_1}\nabla u\cdot\nabla v\,dp,$

$$\overline{D}_2(u_0, v_0)=-\frac{\pi}{4\alpha^2}\int_0^\alpha\Big(\frac{1}{\sin^2\frac{\theta-\theta'}{2\alpha}\pi}+\frac{1}{\sin^2\frac{\theta+\theta'}{2\alpha}\pi}\Big)u_0(\theta')v_0(\theta)d\theta'\,d\theta,$$

γ' is the trace operator maping $H^1(\Omega_1)$ onto $H^{\frac{1}{2}}(\Gamma')$.

From the existence and uniqueness of the solution of the variational problem (2) the following is immediate.

Proposition 1. The variational problem (3) has one and only one solution in $H^1(\Omega_1)/P_0$.

Now divide arc Γ' into N_1 and subdivide Ω_1 into triangles such that its nodes on Γ' coincide with the dividing points of Γ'. Let $\{L_i(x, y)\}_{i=0}^{N_1+N_2}\subset H^1(\Omega_1)$ be basis functions, for example, piecewise linear; then their restrictions on Γ' are approximately piecewise linear on Γ'. Let

$$u\approx U(x, y)=\sum_{i=0}^{N_1+N_2}U_iL_i(x, y),$$

where the subscripts $i=0, 1, \cdots, N_1$ correspond to the nodes on Γ'. We have

$$\sum_{j=0}^{N_1+N_2}D_1(L_j, L_i)U_j+\sum_{j=0}^{N_1}\overline{D}_2(\gamma'L_j, \gamma'L_i)U_j=\int_{\Gamma}fL_i\,ds, \quad i=0, 1, \cdots, N_1+N_2,$$

or, for simplicity,

$$QU=b.\tag{4}$$

The system matrix is

$$Q=[D_1(L_j, L_i)]_{(N_1+N_2+1)\times(N_1+N_2+1)}+\begin{bmatrix}[\overline{D}_2(\gamma' L_j, \gamma' L_i)]_{(N_1+1)\times(N_1+1)} & 0 \\ 0 & 0_{N_2\times N_2}\end{bmatrix}$$
$$\equiv[q_{ij}^{(1)}]+[q_{ij}^{(2)}].$$

Its first part can be obtained by the finite element method, and its second part is given by the following formulae[2,3]

$$q_{00}^{(2)}=q_{N_1N_1}^{(2)}=\frac{1}{2}a_0, \qquad q_{0N_1}^{(2)}=q_{N_10}^{(2)}=\frac{1}{2}a_{N_1},$$

$$q_{i0}^{(2)}=q_{0i}^{(2)}=a_i, \qquad q_{iN_1}^{(2)}=q_{N_1i}^{(2)}=a_{N_1-i}, \qquad i=1, \cdots, N_1-1,$$

$$q_{ij}^{(2)}=q_{ji}^{(2)}=a_{i-j}+a_{i+j}, \quad i, j=1, \cdots, N_1-1,$$

where $a_k=\frac{16N_1^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin^4\frac{n\pi}{2N_1}\cos\frac{n}{N_1}k\pi,\ k=0, 1, \cdots.$

Then we obtain

Proposition 2. Q is a semi-positive definite symmetric matrix with rank N_1+N_2, and the system of linear equations (4) has a solution unique up to a constant vector $C=(c, \cdots, c)^T$, where c is a constant.

Proof. From the positive definite symmetry of bilinear form $D_1(u, v)+\overline{D}_2(\gamma' u, \gamma' v)$ on $(H^1(\Omega_1)/P_0)\times(H^1(\Omega_1)/P_0)$ it is immediate that Q is a semi-positive definite symmetric matrix with rank N_1+N_2. Moreover, we have the consistency condition

$$\sum_{i=0}^{N_1+N_2} b_i=\sum_{i=0}^{N_1+N_2}\int_\Gamma fL_i ds=\sum_{i\in N(\Gamma)}\int_\Gamma fL_i ds=\int_\Gamma f(\sum_{i\in N(\Gamma)} L_i)\, ds=\int_\Gamma f ds=0,$$

where $N(\Gamma)$ is the set of subscripts corresponding to the nodes on Γ. Thus the proof is complete.

In order to guarantee the uniqueness of the solution of (4), some condition is generally added. For instance, in the numerical example of this paper, we assume $U_{N_1/2}=0$.

2. Convergence and Error Estimates

Let $S_h(\Omega_1)\subset H^1(\Omega_1)$ be the finite element space in Ω_1 associated with triangles and let $S_{0h}(\Gamma')=\gamma' S_h(\Omega_1)\subset H^{\frac{1}{2}}(\Gamma')$ be the boundary element space on Γ'. $\Pi: H^1(\Omega_1)\to S_h(\Omega_1)$ and Π_0: $H^{\frac{1}{2}}(\Gamma')\to S_{0h}(\Gamma')$ are interpolation operators. Suppose they are conforming:

$$\gamma'\Pi v=\Pi_0\gamma' v, \quad \forall v\in H^1(\Omega_1).$$

Let u be the solution of (3) and U_h the solution of (4), $\|\cdot\|_{D_1}$ and $\|\cdot\|_{\overline{D}_2}$ are energy norms on $H^1(\Omega_1)/P_0$ and $H^{1/2}(\Gamma')/P_0$ derived from $D_1(\cdot, \cdot)$ and $\overline{D}_2(\cdot, \cdot)$ respectively.

Lemma 1. $D_1(u-U_h, V)+\overline{D}_2(\gamma'(u-U_h), \gamma' V)=0,\ \forall V\in S_h(\Omega_1),$

$$\|u-U_h\|_{D_1}^2+\|\gamma' u-\gamma' U_h\|_{\overline{D}_2}^2=\min_{V\in S_h(\Omega_1)}(\|u-V\|_{D_1}^2+\|\gamma' u-\gamma' V\|_{\overline{D}_2}^2).$$

Proof. Since

$$D_1(u, v)+\overline{D}_2(\gamma' u, \gamma' v)=\int_\Gamma vf ds, \quad \forall v\in H^1(\Omega_1),$$

$$D_1(U_h, V)+\overline{D}_2(\gamma' U_h, \gamma' V)=\int_{\Gamma} Vf\,ds, \quad \forall V\in S_h(\Omega_1),$$

taking $v=V\in S_h(\Omega_1)$ and subtracting, we obtain

$$D_1(u-U_h, V)+\overline{D}_2(\gamma'(u-U_h), \gamma' V)=0, \quad \forall V\in S_h(\Omega_1).$$

Moreover, from the semi-positive definite symmetry of $D_1(u, v)+\overline{D}_2(\gamma' u, \gamma' v)$, we have

$$\begin{aligned} &D_1(u-U_h, u-U_h)+\overline{D}_2(\gamma'(u-U_h), \gamma'(u-U_h))\\ &=D_1(u-V, u-V)+\overline{D}_2(\gamma'(u-V), \gamma'(u-V))\\ &\quad -D_1(U_h-V, U_h-V)-\overline{D}_2(\gamma'(U_h-V), \gamma'(U_h-V))\\ &\leqslant D_1(u-V, u-V)+\overline{D}_2(\gamma'(u-V), \gamma'(u-V)),\\ &\qquad \forall V\in S_h(\Omega_1). \end{aligned}$$

The proof is thus completed.

Theorem 1. (Convergence) *If Π satisfies*

$$\|v-\Pi v\|_{H^1(\Omega_1)} \xrightarrow{h\to 0} 0, \quad \forall v\in H^1(\Omega_1),$$

then

$$\lim_{h\to 0}\{\|u-U_h\|^2_{D_1}+\|\gamma' u-\gamma' U_h\|^2_{D_2}\}^{\frac{1}{2}}=0.$$

Proof. Since the norms $\|\cdot\|_{D_1}$ and $\|\cdot\|_{H^1(\Omega_1)/P_0}$ are equivalent, and the norms $\|\cdot\|_{D_2}$ and $\|\cdot\|_{H^{1/2}(\Gamma')/P_0}$ are equivalent as well, there exist constants K and K_0 such that

$$\|v\|_{D_1}\leqslant K\|v\|_{H^1(\Omega_1)}, \quad \forall v\in H^1(\Omega_1),$$

$$\|v_0\|_{D_2}\leqslant K_0\|v_0\|_{H^{1/2}(\Gamma')}, \quad \forall v_0\in H^{1/2}(\Gamma').$$

And from $\Gamma'\subset\partial\Omega_1$ and the trace theorem, we have a constant T such that

$$\|\gamma' v\|_{H^{1/2}(\Gamma')}\leqslant T\|v\|_{H^1(\Omega_1)}, \quad \forall v\in H^1(\Omega_1).$$

Since $u\in H^1(\Omega_1)$, for any given $\varepsilon>0$, there exists $\tilde{u}\in C^\infty(\overline{\Omega})$ such that

$$\|u-\tilde{u}\|_{H^1(\Omega_1)}<\frac{\varepsilon}{2\sqrt{K^2+K_0^2T^2}}.$$

Moreover, for fixed $\tilde{u}$, there exists h_0 such that

$$\|\tilde{u}-\Pi\tilde{u}\|_{H^1(\Omega_1)}<\frac{\varepsilon}{2\sqrt{K^2+K_0^2T^2}}$$

when $h<h_0$. Using Lemma 1, we obtain

$$\begin{aligned} \|u-U_h\|^2_{D_1}+\|\gamma' u-\gamma' U_h\|^2_{D_2}&=\inf_{V\in S_h(\Omega_1)}(\|u-V\|^2_{D_1}+\|\gamma' u-\gamma' V\|^2_{D_2})\\ &\leqslant(K^2+K_0^2T^2)(\|u-\tilde{u}\|_{H^1(\Omega_1)}+\|\tilde{u}-\Pi\tilde{u}\|_{H^1(\Omega_1)})^2<\varepsilon^2, \end{aligned}$$

i. e.

$$(\|u-U_h\|^2_{D_1}+\|\gamma' u-\gamma' U_h\|^2_{D_2})^{\frac{1}{2}}<\varepsilon.$$

The proof is complete.

Theorem 2. *If $u\in H^{k+1}(\Omega_1)$, $k\geqslant 1$, Π satisfies*

$$\|v-\Pi v\|_{H^1(\Omega_1)}\leqslant Ch^j\|v\|_{j+1,\Omega_1}, \quad \forall v\in H^{j+1}(\Omega_1), \quad j=1, \cdots, k,$$

then

$$(\|u-U_h\|^2_{D_1}+\|\gamma' u-\gamma' U_h\|^2_{D_2})^{\frac{1}{2}}\leqslant Ch^k\|u\|_{k+1,\Omega_1}$$

where C is a constant independent of u and h.

Proof. From Lemma 1 we have

$$\|u-U_h\|^2_{D_1}+\|\gamma'u-\gamma'U_h\|^2_{\bar{D}_2}\leqslant\inf_{V\in S_h(\Omega_1)}C\|u-V\|^2_{H^1(\Omega_1)}$$
$$\leqslant C\|u-\Pi u\|^2_{H^1(\Omega_1)}\leqslant Ch^{2k}\|u\|^2_{k+1,\Omega_1},$$

i. e.

$$(\|u-U_h\|^2_{D_1}+\|\gamma'u-\gamma'U_h\|^2_{\bar{D}_2})^{\frac{1}{2}}\leqslant Ch^k\|u\|_{k+1,\Omega_1}.$$

Theorem 3. *If $u\in H^{k+1}(\Omega_1)$, $k\geqslant1$, Π satisfies the condition of Theorem 2,*

$$\iint_{\Omega_1}(u-U_h)dp=0,$$

then

$$\|u-U_h\|_{L_2(\Omega_1)}\leqslant Ch^{k+1}\|u\|_{k+1,\Omega_1}.$$

Proof. Let w be the solution of the following boundary-value problem

$$\begin{cases}-\Delta w=g, & \text{in } \Omega,\\ \dfrac{\partial w}{\partial n}=0, & \text{on } \partial\Omega,\end{cases}$$

where $g=\begin{cases}u-U_h, & \text{in } \Omega_1,\\ 0, & \text{in } \Omega_2.\end{cases}$

The solution evidently exists. Since

$$u-U_h\in H^1(\Omega_1)\subset L_2(\Omega_1),$$

so $g\in L_2(\Omega)$. Then from the differentiability of the solution of the Neumann problem of harmonic equation, we have $w\in H^2(\Omega)$, and

$$\|w\|_{H^2(\Omega)/P_0}\leqslant C\|g\|_{L_2(\Omega)}=C\|u-U_h\|_{L_2(\Omega_1)}.$$

Since

$$\iint_{\Omega_1}(u-U_h)v\,dp=\iint_{\Omega_1}(-\Delta w)v\,dp=\int_{\partial\Omega_1}v\frac{\partial w}{\partial n}ds+D_1(w,v)$$
$$=\int_{\Gamma'}v\frac{\partial w}{\partial n}ds+D_1(w,v)=D_1(w,v)+\bar{D}_2(\gamma'w,\gamma'v),$$

where n is the inward normal of Ω_1, taking $v=u-U_h$ and using Lemma 1 and the trace theorem, we obtain

$$\|u-U_h\|^2_{L_2(\Omega_1)}=D_1(w,u-U_h)+\bar{D}_2(\gamma'w,\gamma'(u-U_h))$$
$$=D_1(w-\Pi w,u-U_h)+\bar{D}_2(\gamma'w-\gamma'\Pi w,\gamma'u-\gamma'U_h)$$
$$\leqslant C\|w-\Pi w\|_{H^1(\Omega_1)/P_0}\|u-U_h\|_{D_1}\leqslant Ch\|w\|_{H^2(\Omega_1)/P_0}\|u-U_h\|_{D_1}.$$

By differentiability we have

$$\|u-U_h\|^2_{L_2(\Omega_1)}\leqslant Ch\|u-U_h\|_{L_2(\Omega_1)}\|u-U_h\|_{D_1},$$

i. e.

$$\|u-U_h\|_{L_2(\Omega_1)}\leqslant Ch\|u-U_h\|_{D_1}.$$

Then from Theorem 2 we get

$$\|u-U_h\|_{L_2(\Omega_1)}\leqslant Ch^{k+1}\|u\|_{k+1,\Omega_1}.$$

Moreover, in the classical theory of the finite element method there is the following

Lemma 2. *Let S_h be the space spanned by piecewise linear basis functions associated* *with triangle elements, and let Δ_e denote the area of triangle element e, $v\in S_h$; then*

$$\|v\|_{L_\infty(e)}\leqslant\sqrt{12}\,\Delta_e^{-\frac{1}{2}}\|v\|_{L_2(e)}.$$

From this Lemma and Theorem 3 we can obtain

Theorem 4. *Let Π: $H^1(\Omega_1)\to S_h(\Omega_1)$ be the piecewise linear interpolation operator associated with triangle elements, $u\in H^2(\Omega_1)$, $\iint_{\Omega_1}(u-U_h)dp=0$; then*

$$\|u-U_h\|_{L_\infty(\Omega_1)}\leqslant Ch\|u\|_{2,\Omega_1}.$$

This estimate is not the best. The proof is similar to that of the corresponding result in the classical theory of the finite element method and so is omitted.

3. Numerical Example

Let Ω be a cracked square

$$\Omega=\{(x,\ y)\,|_{|x|,\ |y|<1}\}\setminus\{(x,\ 0)\,|_{0<x<1}\},$$

and let Γ be its boundary. Solve the boundary-value problem

$$\begin{cases}\Delta u=0, & \text{in } \Omega,\\ \partial_n u(x,\ +0)=\partial_n u(x,\ -0)=0,\ 0<x<1,\\ \partial_n u|_\Gamma=f,\end{cases}$$

where

$$f(1,\ y)=\pm\sqrt{\frac{1+\sqrt{1+y^2}}{2(1+y^2)}},\quad y\gtrless 0,$$

$$f(-1,\ y)=\mp\sqrt{\frac{\sqrt{y^2+1}-1}{2(y^2+1)}},\quad y\gtrless 0,$$

$$f(x,\ 1)=\frac{1}{\sqrt{2(x^2+1)(x+\sqrt{x^2+1})}},$$

$$f(x,\ -1)=-\frac{1}{\sqrt{2(x^2+1)\ (x+\sqrt{x^2+1})}}.$$

We take $R=0.5$, 0.8 and 0.99 respectively to solve numerically the problem by coupling the canonical boundary element method with the finite element method, and compare the result with that obtained by the finite element method.

Table 1. The maximum error at nodes

Method		N_1	Number of nodes	Maximum error at nodes	Ratio
FEM		8	19	0.38450718	1.69778724
		16	69	0.22647548	
Coupling	$R=0.5$	8	18	0.14283466	3.82161713
		16	51	0.03737545	
	$R=0.8$	8	18	0.08247280	4.21704760
		16	51	0.01955700	
	$R=0.99$	16	51	0.01459408	1.85738649
		24	75	0.00785732	

Table 2. The jump through crack

Method	Number of nodes	x	1.00	0.75	0.50
FEM	19	$U(x, +0)-U(x, -0)$	3.24017620	2.66844702	2.09671783
		Relative error	0.18995595	0.22968574	0.25869810
	69	$U(x, +0)-U(x, -0)$	3.64538860	3.11208820	2.46181011
		Relative error	0.08865285	0.10161757	0.12961845
Coupling $R=0.5$	18	$U(x, +0)-U(x, -0)$	3.74924565	3.20859575	2.66794586
		Relative error	0.06268859	0.07375824	0.05673844
	51	$U(x, +0)-U(x, -0)$	3.93803215	3.41174126	2.78799343
		Relative error	0.01549196	0.01511518	0.01429520
Exact value		$u(x, +0)-u(x, -0)$	4.00000000	3.46410180	2.82842636

Table 3. The jump through crack near singularity

Method	Number of nodes	x	0.1	0.01	0.001	0.0001
FEM	19	$U(x, +0)-U(x, -0)$	0.41934365	0.04193436	0.00419344	0.00041934
		Relative error	0.66847951	0.89516430	0.96684820	0.98951650
	69	$U(x, +0)-U(x, -0)$	0.63291341	0.06329137	0.00632913	0.00063291
		Relative error	0.49963825	0.84177158	0.94996386	0.98417725
Coupling $R=0.5$	18	$U(x, +0)-U(x, -0)$	1.20084095	0.37975490	0.12008911	0.03797540
		Relative error	0.05065231	0.05061275	0.05061270	0.05061500
	51	$U(x, +0)-U(x, -0)$	1.24850368	0.39481878	0.12485290	0.03948190
		Relative error	0.01297163	0.01295309	0.01295165	0.01295250
Coupling $R=0.8$	18	$U(x, +0)-U(x, -0)$	1.22803402	0.38834321	0.12280506	0.03883439
		Relative error	0.02915431	0.02914197	0.02914124	0.02914025
	51	$U(x, +0)-U(x, -0)$	1.25573349	0.39709890	0.12557393	0.03970994
		Relative error	0.00725597	0.00725275	0.00725142	0.00725150
Coupling $R=0.99$	51	$U(x, +0)-U(x, -0)$	1.25793552	0.39779496	0.12579346	0.03977967
		Relative error	0.00551511	0.00551260	0.00551588	0.00550825
	75	$U(x, +0)-U(x, -0)$	1.26164341	0.39896685	0.12616473	0.03989674
		Relative error	0.00258377	0.00258288	0.00258073	0.00258150
Exact value		$u(x, +0)-u(x, -0)$	1.26491165	0.40000000	0.12649117	0.04000000

4. Conclusions

a. The coupling of the canonical boundary element method with the finite element method keeps the advantage of the finite element method, which is applicable to relatively arbitrary domain, and removes the limitation of canonical boundary reduction, thereby its range of application is extended greatly.

b. This coupling is applicable to problems over infinite domains and domain with crack or concave angle, which is due to the merit of the canonical boundary reduction, and avoids the shortcoming of classical finite element method, by which the accuracy will be damaged seriously when we deal will singularity. In Tables 1—3 we have seen that the result obtained by finite element method is far inferior to those results obtained by coupling. Especially near the singularity, the former can not embody the behaviour of the solution at all, but the latter still preserves ideal accuracy.

c. This coupling can be brought into the calculating system of the finite element method; in fact, the subdomain in which the canonical boundary reduction is carried out is exactly a "large element"[4]. Compared with the finite element method, the coupling hardly increase the complexity of programing. When their numbers of nodes are equal, their computation times are nearly equal, let alone the number of nodes associated with coupling is far less.

d. In order to decrease nodes and raise accuracy, the subdomain in which the canonical boundary reduction is carried out should be as large as possible. For example, in Tables 1—3 the result obtained by taking $R=0.8$ is better than $R=0.5$; when $R=0.99$, the result is the best.

This coupling is applicable to solving harmonic and biharmonic problems over infinite domains too.

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