FINITE DIFFERENCE METHOD OF THE BOUNDARY PROBLEMS FOR THE SYSTEMS OF GENERALIZED SCHRÖDINGER TYPE*

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§ 1

The nonlinear Schrödinger equation

$$u_t - iu_{xx} + \beta |u|^p u = 0 \tag{1}$$

and the nonlinear Schrödinger system

$$\begin{aligned} u_t - iu_{xx} + u(\alpha|u|^2 + \beta|v|^2) = 0, \\ v_t - iv_{xx} + v(\alpha|u|^2 + \beta|v|^2) = 0 \end{aligned} \tag{2}$$

of complex valued functions u and v often appear in the study of problems of physics. These equations and systems may be regarded as the special cases of the system

$$u_t = Au_{xx} + f(u) \tag{3}$$

of real valued functions, where $u(x, t) = (u_1(x, t), \cdots, u_m(x, t))$ is a m-dimensional vector valued unknown function, A is a $m \times m$ non-negatively definite and non-singular constant matrix and $f(u) = (f_1(u), \cdots, f_m(u))$ is a m-dimensional vector valued function of vector variable u. The system (3) may be called the system of generalized Schrödinger type. In [1, 2] the periodic boundary problem and the initial value problem for the system of generalized Schrödinger type of higher order are studied by the method of straight line and the method of Galërkin respectively. In [3] the first boundary value problem for the system (3) is discussed by use of the fixed-point technique and the method of integral estimations. There are many works contribute to the finite difference method for solving the problems of Schrödinger equations.

The purpose of this paper is to solve the boundary problems in rectangular domain $Q_T = \{0 \leqslant x \leqslant l, 0 \leqslant t \leqslant T\}$ for the system (3) of generalized Schrödinger type by means of finite difference method. Assume that the boundary problems (*) take one of the following boundary conditions: the first boundary condition

$$u(0, t) = u(l, t) = 0; \tag{4}$$

the second boundary conditon

$$u_x(0, t) = u_x(l, t) = 0 \tag{5}$$

and the mixed boundary conditions

$$u(0, t) = u_x(l, t) = 0 \tag{6}$$

or

$$u_x(0, t) = u(l, t) = 0. \tag{7}$$

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The initial value condition is

$$u(x, 0)=\varphi(x), \tag{8}$$

where $\varphi(x)$ is a m-dimensional vector valued initial function, satisfying the appropriate boundary condition $(*)$. We denote any given one of boundary conditions (4), (5), (6) or (7) by the symbol $(*)$.

Let us divide the rectangular domain Q_T into small grids by the parallel lines $x=x_j$ $(j=0, 1, \cdots, J)$ and $t=t_n$ $(n=0, 1, \cdots, N)$, where $x_j=jh$, $t_n=nk$, $Jh=l$, $Nk=T(j=0, 1, \cdots, J; n=0, 1, \cdots, N)$. Denote the vector valued discrete function on the grid point (x_j, t_n) by $v_j^n(j=0, 1, \cdots, J; n=0, 1, \cdots, N)$. Let us construct the finite difference system

$$\frac{v_j^{n+1}-v_j^n}{k}=A\frac{1}{h^2}\Delta_+\Delta_-v_j^{n+1}+f_j^{n+1}, \tag{3$_h$}$$

where $\Delta_+v_j=v_{j+1}-v_j$, $\Delta_-v_j=v_j-v_{j-1}$ and $f_j^{n+1}=f(v_j^{n+1})$. The finite difference boundary conditions are as follows:

$$v_0^n=v_J^n=0; \tag{4$_h$}$$

$$v_1^n-v_0^n=v_J^n-v_{J-1}^n=0; \tag{5$_h$}$$

$$v_0^n=v_J^n-v_{J-1}^n=0; \tag{6$_h$}$$

$$v_1^n-v_0^n=v_J^n=0, \tag{7$_h$}$$

where $n=1, 2, \cdots, N$. The initial condition is as

$$v_j^0=\bar{\varphi}_j, \quad j=0, 1, \cdots, J, \tag{8$_h$}$$

where $\bar{\varphi}_j=\varphi(x_j)$ $(j=0, 1, \cdots, J)$ and $\bar{\varphi}_1=\varphi(0)$ (or $\bar{\varphi}_{J-1}=\varphi(l)$) in the case of boundary condition $v_1^n-v_0^n=0$ (or $v_J^n-v_{J-1}^n=0$). Hence the discrete function $\bar{\varphi}_j(j=0, 1, \cdots, J)$ also satisfies the boundary condition $(*)$.

Now we make the following assumptions for the system (3) of generalized Schrödinger type and the initial vector valued function $\varphi(x)$.

(I) A is a $m\times m$ non-negatively definite and non-singular constant matrix.

(II) The m-dimensional vector valued function $f(u)$ of the vector variable u satisfies the condition of monotonicity

$$(u-v, f(u)-f(v))\leqslant b|u-v|^2, \tag{9}$$

where b is a constant.

(III) The components of the m-dimensional vector valued initial function $\varphi(x)$ are twice continuously differentiable in $[0, l]$. Denote $\varphi(x)\in C^{(2)}([0, l])$. And $\varphi(x)$ satisfies the appropriate boundary condition $(*)$.

The scalar product of two vectors u and v is denoted by (u, v) and $|u|^2=(u, u)$. For the discrete functions $\{u_j\}$ and $\{v_j\}$, we have the symbols $(u, v)_h=\sum_{j=0}^{J}u_jv_jh$ and $\|u\|_h^2=(u, u)_h$.

§ 2

The finite difference system $(3)_h$ and the finite difference boundary conditions $(*)_h$ can be considered as the nonlinear system of unknown vectors $v_j^{n+1}(j=0, 1, \cdots, J)$, where $v_j^n(j=0, 1, \cdots, J)$ are the known vectors. Now we are going to prove the existence of the solutions $v_j^{n+1}(j=0, 1, \cdots, J)$ for the nonlinear system $(3)_h$ and $(*)_h$.

The following lemma is easily verified by direct calculation.

Lemma 1. *For any* $\{u_j\}$ *and* $\{v_j\}$ $(j=0, 1, \cdots, J)$, *there are the relations*

$$\sum_{j=0}^{J-1} u_j \Delta_+ v_j = -\sum_{j=1}^{J} v_j \Delta_- u_j - u_0 v_0 + u_J v_J, \tag{10}$$

$$\sum_{j=1}^{J-1} u_j \Delta_+ \Delta_- v_j = -\sum_{j=0}^{J-1} (\Delta_+ u_j)(\Delta_+ v_j) - u_0(v_1 - v_0) + u_J(v_J - v_{J-1}). \tag{11}$$

Lemma 2. *Suppose that the matrix* A *and the vector valued function* $f(u)$ *satisfy the conditions* (I) *and* (II) *respectively. When* $1-2bk>0$, *the finite difference system* $(3)_h$ *and* $(*)_h$ *has a unique solution* $v_j^n (j=0, 1, \cdots, J; n=1, 2, \cdots, N)$.

Proof. For any m-dimensional vector z_j $(j=0, 1, \cdots, J)$, let us construct m-dimensional vectors $V_j (j=0, 1, \cdots, J)$ as follows

$$V_j = v_j^n + \lambda A \frac{k}{h^2} \Delta_+ \Delta_- z_j + \lambda k f(z_j), \quad j=1, 2, \cdots, J-1 \tag{12}$$

and V_0 and V_J are determined by the boundary consitions $(*)_h$, where $0 \leqslant \lambda \leqslant 1$. This defines a mapping $V=T_\lambda z$ of $m(J+1)$-dimensional Euclidean space into itself, where $V=\{V_j\}$, $z=\{z_j\}$.

In order to establish the existence of the solution for the nonlinear system $(3)_h$ and $(*)_h$, it is sufficient to prove the uniform boundedness for all the possible fixed point of the mapping with respect to the parameter $0 \leqslant \lambda \leqslant 1$.

Making the scalar product of the vector V_j with the vector equation

$$V_j = v_j^n + \lambda A \frac{k}{h^2} \Delta_+ \Delta_- V_j + \lambda k f(V_j) \tag{13}$$

and summing up the resulting relations for $j=1, 2, \cdots, J-1$, we get

$$\sum_{j=1}^{J-1} |V_j|^2 - \sum_{j=1}^{J-1} |(v_j^n, V_j) = \lambda \frac{k}{h^2} \sum_{j=1}^{J-1} (V_j, A\Delta_+ \Delta_- V_j) + \lambda k \sum_{j=1}^{J-1} (V_j, f(V_j)). \tag{14}$$

From the formular (11) of Lemma 1, it can be verified that

$$\sum_{j=1}^{J-1} (V_j, A\Delta_+ \Delta_- V_j) = -\sum_{j=0}^{J-1} (\Delta_+ V_j, A\Delta_+ V_j)$$
$$-(V_0, A(V_1 - V_0)) + (V_J, A(V_J - V_{J-1})).$$

Since V_j satisfies the boundary condition $(*)_h$, then we have

$$\sum_{j=1}^{J-1} (V_j, A\Delta_+ \Delta_- V_j) = -\sum_{j=1}^{J-1} (\Delta_+ V_j, A\Delta_+ V_j) \leqslant 0.$$

Using the assumption (II), the later part of the right-hand side of (14)

$$\sum_{j=1}^{J+1} (V_j, f(V_j)) \leqslant (b+\delta) \sum_{j=1}^{J-1} |V_j|^2 + \frac{(J-1)}{4\delta} |f(0)|^2,$$

where $\delta>0$. Hence it can be derived from (14) that

$$(1-2\lambda(b+\delta)k) \sum_{j=1}^{J-1} |V_j|^2 \leqslant \sum_{j=1}^{J-1} |v_j^n|^2 + \lambda k \frac{J-1}{4\delta} |f(0)|^2.$$

Suppose that k satisfies the inequality $1-2bk>0$. We can take $\delta>0$ so small that $1-2(b+\delta)k>0$. Hence $\sum_{j=1}^{J-1} |V_j|^2$ is uniformly bounded with respect to the parameter $0 \leqslant \lambda \leqslant 1$. Thus the solution of the nonlinear system $(3)_h$ and $(*)_h$ exists.

Assume that $\{V_j\}$ and $\{\bar{V}_j\}$ are two solutions of the nonlinear system $(3)_h$ and $(*)_h$. Then we have

$$V_j - v_j^n = A\frac{k}{h^2}\Delta_+\Delta_- V_j + kf(V_j) \quad (j=1, 2, \cdots, J-1)$$

and

$$\overline{V}_j - v_j^n = A\frac{k}{h^2}\Delta_+\Delta_- \overline{V}_j + kf(\overline{V}_j) \quad (j=1, 2, \cdots, J-1).$$

Substract one from the other of the above two equations, there is

$$V_j - \overline{V}_j = A\frac{k}{h^2}\Delta_+\Delta_-(V_j - \overline{V}_j) + k(f(V_j) - f(\overline{V}_j)) \quad (j=1, 2, \cdots, J-1).$$

Taking the scalar product of the vector $V_j - \overline{V}_j$ with the above vector equation and summing up the resulting relations for $j=1, 2, \cdots, J-1$, we obtain

$$\sum_{j=1}^{J-1} |V_j - \overline{V}_j|^2 = \frac{k}{h^2}\sum_{j=1}^{J-1}(V_j - \overline{V}_j,\ A\Delta_+\Delta_-(V_j - \overline{V}_j)) + k\sum_{j=1}^{J-1}(V_j - \overline{V}_j,\ f(V_j) - f(\overline{V}_j)).$$

Since $\{V_j\}$ and $\{\overline{V}_j\}$ both satisfy the boundary condition $(*)_h$ and $f(u)$ satisfies the condition of monotonicity (9), the above equality can be replaced by the inequality

$$(1-bk)\sum_{j=1}^{J-1} |V_j - \overline{V}_j|^2 \leqslant 0.$$

Therefore under the condition $1-2bk>0$, $\sum_{j=1}^{J-1} |V_j - \overline{V}_j|^2 = 0$. The lemma is proved.

§ 3

Now we turn to get a series of a priori estimates for the solutions of the finite difference system

$$\frac{v_j^{n+1} - v_j^n}{k} = A\frac{1}{h^2}\Delta_+\Delta_- v_j^{n+1} + f(v_j^{n+1}) \tag{$(3)_h$}$$

with the appropriate boundary condition $(*)_h$ and the initial condition $(8)_h$.

Making the scalar product of the vector $v_j^{n+1}kh$ with the finite difference system $(3)_h$ and summing up the resulting relations for $j=1, 2, \cdots, J-1$, we have

$$\sum_{j=1}^{J-1}(v_j^{n+1},\ v_j^{n+1} - v_j^n)h = \frac{k}{h}\sum_{j=1}^{J-1}(v_j^{n+1},\ A\Delta_+\Delta_- v_j^{n+1}) + kh\sum_{j=1}^{J-1}(v_j^{n+1},\ f(v_j^{n+1}))$$

or

$$(v^{n+1},\ v^{n+1})_h = (v^{n+1},\ v^n)_h + \frac{k}{h}(v^{n+1},\ A\Delta_+\Delta_- v^{n+1})_h + k(v^{n+1},\ f^{n+1})_h. \tag{15}$$

On account of the assumption (I) and the boundary condition $(*)_h$, using the formula (11) of Lemma 1, we get

$$(v^{n+1},\ A\Delta_+\Delta_- v^{n+1})_h = -(\Delta_+ v^{n+1},\ A\Delta_+ v^{n+1})_h \leqslant 0.$$

From the assumption (II), it follows

$$(v^{n+1},\ f^{n+1})_h \leqslant (b+\delta)\|v^{n+1}\|_h^2 + \frac{1}{4\delta}l|f(0)|^2,$$

where $\delta>0$. Then (15) can be simplified as

$$(v^{n+1},\ v^{n+1})_h \leqslant (v^{n+1},\ v^n)_h + (b+\delta)k\|v^{n+1}\|_h^2 + \frac{kl}{4\delta}|f(0)|^2$$

or

$$\|v^{n+1}\|_h^2 \leqslant \frac{\|v^n\|_h^2 + \frac{kl}{4\delta}|f(0)|^2}{1-2(b+\delta)k}.$$

From this iterative relation, we can easily obtain the estimate

$$\|v^n\|_h^2 \leqslant (1-2(b+\delta)k)^{-n}\left\{\|v^0\|_h^2 + \frac{l|f(0)|}{4\delta(b+\delta)}\right\}. \tag{16}$$

Lemma 3. *Suppose that the assumptions* (I) *and* (II) *are fulfilled and* $\varphi(x) \in C([0, l])$. *When* $nk \leqslant T$ *and* $1-4bk>0$, $\|v^n\|_h$ *is uniformly bounded with respect to* h *and* k, *i. e., there is the estimate*

$$\|v^n\|_h \leqslant K_1 e^{(b+\delta)T}\{\max_{x\in[0,l]}|\varphi(x)| + |f(0)|\}, \tag{16'}$$

where K_1 *is a constant independent of the stepsize* h *and* k.

Now we are going to estimate $\left\|\frac{\Delta_+ v^n}{h}\right\|_h$. Taking the scalar product of the vector $\Delta_+\Delta_- v_j^{n+1}\frac{k}{h}$ and the finite difference system $(3)_h$ and then summing up the resulting relations for $j=1, 2, \cdots, J-1$, there is the equality

$$\frac{1}{h}\sum_{j=1}^{J-1}(\Delta_+\Delta_- v_j^{n+1}, v_j^{n+1}-v_j^n) = \frac{k}{h^3}\sum_{j=1}^{J-1}(\Delta_+\Delta_- v_j^{n+1}, A\Delta_+\Delta_- v_j^{n+1})$$
$$+\frac{k}{h}\sum_{j=1}^{J-1}(\Delta_+\Delta_- v_j^{n+1}, f_j^{n+1}), \quad n=0, 1, \cdots, N-1. \tag{17}$$

For the left hand side of the above equality,

$$\frac{1}{h}\sum_{j=1}^{J-1}(\Delta_+\Delta_- v_j^{n+1}, v_j^{n+1}-v_j^n) = -\frac{1}{h}\sum_{j=0}^{J-1}(\Delta_+ v_j^{n+1}, \Delta_+(v_j^{n+1}-v_j^n))$$
$$-\frac{1}{h}(\Delta_+ v_0^{n+1}, v_0^{n+1}-v_0^n) + \frac{1}{h}(\Delta_- v_J^{n+1}, v_J^{n+1}-v_J^n),$$
$$n=0, 1, \cdots, N-1. \tag{18}$$

Since $\{v_j^{n+1}\}$ and $\{v_j^n\}$ both satisfy the finite difference boundary conditions $(*)_h$, the last two terms of right-hand side of (18) vanish. So we have for (18)

$$\frac{1}{h}\sum_{j=1}^{J-1}(\Delta_+\Delta_- v_j^{n+1}, v_j^{n+1}-v_j^n) = -(\omega^{n+1}, \omega^{n+1}-\omega^n)_h, \tag{19}$$

where $\omega_j = \frac{\Delta_+ v_j}{h}$ $(j=0, 1, \cdots, J-1)$. The first term of the right-hand side of (17) is non-negative. The second term of the right-hand side of (17) can be written in the form

$$\frac{k}{h}\sum_{j=1}^{J-1}(\Delta_+\Delta_- v_j^{n+1}, f_j^{n+1}) = -\frac{k}{h}\sum_{j=0}^{J-1}(\Delta_+ v_j^{n+1}, \Delta_+ f_j^{n+1})$$
$$-\frac{k}{h}(\Delta_+ v_0^{n+1}, f_0^{n+1}) + \frac{k}{h}(\Delta_- v_J^{n+1}, f_J^{n+1}). \tag{20}$$

Using the assumption (II),

$$\frac{1}{h}\sum_{j=0}^{J-1}(\Delta_+ v_j^{n+1}, \Delta_+ f_j^{n+1}) \leqslant \frac{b}{h}\sum_{j=0}^{J-1}|\Delta_+ v_j^{n+1}|^2 = b\|\omega^{n+1}\|_h^2.$$

For the case of the second boundary condition (5) and the corresponding finite difference boundary condition $(5)_h$, we have $\Delta_+ v_0^{n+1} = \Delta_- v_J^{n+1} = 0$. Hence (20) becomes

$$-\frac{1}{h}\sum_{j=0}^{J-1}(\Delta_+\Delta_- v_j^{n+1}, f_j^{n+1}) \leqslant b\|\omega^{n+1}\|_h^2. \tag{21}$$

For the case of the first boundary condition (4), the mixed boundary condition (6) or (7) and the corresponding finite difference boundary conditions, if the system (3) is homogeneous, i. e., $f(0)$ is a zero vector, then (21) is also valid. Finally, we have

$$(\omega^{n+1}, \omega^{n+1}-\omega^n)_h \leqslant bk\|\omega^{n+1}\|_h^2$$

and

$$\|\omega^n\|_h \leqslant (1-2kb)^{-\frac{n}{2}} \left\| \frac{\Delta_+ v^0}{h} \right\|_h. \tag{22}$$

Lemma 4. *Under the conditions* (I), (II) *and* $\varphi(x) \in C^{(1)}([0, l])$, *for the finite difference system* $(3)_h$, $(5)_h$ *and* $(8)_h$, *corresponding to the second boundary problem* (5) *for the system* (3) *of generalized Schrödinger type, the approximate solution* $\{v_j^n\}$ $(j=0, 1, \cdots, J;\ n=0, 1, \cdots N)$ *has the estimate*

$$\left\| \frac{\Delta_+ v^n}{h} \right\|_h \leqslant K_2 \quad (n=0, 1, \cdots, N), \tag{22$'$}$$

where K_2 *is independent of the stepsizes* h *and* k.

Lemma 5. *Suppose that the conditions of Lemma 4 are satisfied and suppose that the system* (3) *is homogeneous, i. e.,* $f(0)=0$. *The solutions of the finite difference systems* $(3)_h$, $(4)_h$, $(8)_h$; $(3)_h$, $(6)_h$, $(8)_h$ *and* $(3)_h$, $(7)_h$, $(8)_h$, *corresponding to the first boundary problem* (4) *and the mixed boundary problems* (6) *and* (7) *for the generalized Schrödinger system* (3), *have the uniform estimate* (22)$'$ *with respect to the stepsizes* h *and* k.

Then we turn to estimate $\|z^{n+1}\|_h$, where $z_j^{n+1} = \dfrac{v_j^{n+1}-v_j^n}{k}$ $(j=0, 1, \cdots, J;\ n=0, 1, \cdots, N)$. At the grid points (x_j, t_{n+1}) and (x_j, t_n), we have

$$\frac{v_j^{n+1}-v_j^n}{k} = A\frac{1}{h^2}\Delta_+\Delta_- v_j^{n+1} + f(v_j^{n+1}),$$

$$\frac{v_j^n - v_j^{n-1}}{k} = A\frac{1}{h^2}\Delta_+\Delta_- v_j^n + f(v_j^n).$$

Then the system for z_j^{n+1} takes the form

$$\frac{z_j^{n+1}-z_j^n}{k} = A\frac{1}{h^2}\Delta_+\Delta_- z_j^{n+1} + \frac{1}{k}(f(v_j^{n+1}) - f(v_j^n))$$

$$(j=1, 2, \cdots, J-1;\ n=1, 2, \cdots, N-1). \tag{23}$$

Let $v_j^{-1} (j=1, 2, \cdots, J-1)$ be defined by the following equality

$$z_j^0 = \frac{\bar{\varphi}_j - v_j^{-1}}{k} = A\frac{1}{h^2}\Delta_+\Delta_-\bar{\varphi}_j + f(\bar{\varphi}_j) \qquad (j=1, 2, \cdots, J-1). \tag{24}$$

Thus the system (23) is valid for the case $n=0$. Obviously, $\{z_j^n\}$ $(n=1, 2, \cdots, N)$ satisfies the boundary condition $(*)_h$. Taking the scalar product of the vector $z_j^{n+1}kh$ and the vector equation (23), we have

$$\sum_{j=1}^{J-1}(z_j^{n+1}, z_j^{n+1}-z_j^n)h = \frac{k}{h}\sum_{j=1}^{J-1}(z_j^{n+1}, A\Delta_+\Delta_- z_j^{n+1}) + hk\sum_{j=1}^{J-1}\left(z_j^{n+1}, \frac{f_j^{n+1}-f_j^n}{k}\right),$$

where

$$h\sum_{j=1}^{J-1}\left(z_j^{n+1}, \frac{f_j^{n+1}-f_j^n}{k}\right) \leqslant b\|z^{n+1}\|_h^2.$$

Hence there is

$$(z^{n+1}, z^{n+1}-z^n)_h \leqslant b\|z^{n+1}\|_h^2$$

or

$$\|z^n\|_h \leqslant (1-2bk)^{-\frac{n}{2}}\|z^0\|_h. \tag{25}$$

Similarly, it leads to the following lemma.

Lemma 6. *Suppose that the conditions* (I), (II) *and* (III) *are fulfilled. For the solutions* v_j^n $(j=0, 1, \cdots, J; n=0, 1, \cdots, N)$ *of the finite difference system* $(3)_h$, $(*)_h$ *and* $(8)_h$, *holds the estimate*

$$\left\|\frac{v^{n+1}-v^n}{k}\right\|_h \leqslant K_3, \tag{25$'$}$$

where K_3 *is independent of* h *and* k.

Lemma 7. *Under the conditions of Lemma* 6, *the estimate*

$$\left\|\frac{\Delta_+\Delta_-v^n}{h^2}\right\|_h \leqslant K_4 \tag{26}$$

holds for $n=0, 1, \cdots, N$, *where* K_4 *is independent of* h *and* k.

The estimate (26) follows immediately from the system $(3)_h$, since A is a non-singular constant matrix, thus the inverse matrix A^{-1} exists.

§ 4

In this section we continue to give some maximum norm estimates for the solutions of the finite difference system.

Lemma 8. *Under the assumptions* (I), (II) *and* (III), *the solutions* v_j^n $(j=0, 1, \cdots, J;\ n=0, 1, \cdots, N)$ *of the finite difference system* $(3)_h$, $(*)_h$ *and* $(8)_h$ *have the following estimation relation*

$$\max_{j=0,1,\cdots,J} |v_j^n| \leqslant K_5, \quad n=0, 1, \cdots, N; \tag{27}$$

$$\max_{j=0,1,\cdots,J-1} |\Delta_+v_j^n| \leqslant K_6h, \quad n=0,1, \cdots, N; \tag{28}$$

$$\max_{j=1,2,\cdots,J-1} |\Delta_+\Delta_-v_j^n| \leqslant K_7h^{\frac{3}{2}}, \quad n=0,1,\cdots,N; \tag{29}$$

$$\max_{n=0,1,\cdots,N-1} |v_j^{n+1}-v_j^n| \leqslant K_8k^{\frac{3}{4}}, \quad j=0,1, \cdots, J; \tag{30}$$

$$\max_{n=0,1,\cdots,N-1} |\Delta_+v_j^{n+1}-\Delta_+v_j^n| \leqslant K_9k^{\frac{1}{4}}h, \quad j=0,1, \cdots, J-1, \tag{31}$$

where K'*s are independent of* h *and* k.

Proof. For any discrete function $u_j (j=0, 1, \cdots, J)$, we have

$$u_m^2-u_s^2=\sum_{j=s}^{m-1} (u_{j+1}+u_j)\frac{\Delta_+u_j}{h}h \leqslant 2\|u\|_h\left\|\frac{\Delta_+u}{h}\right\|_h,$$

where $\|u\|_h^2=\sum_{j=0}^{J}u_j^2h$, $\left\|\frac{\Delta_+u}{h}\right\|_h^2=\sum_{j=0}^{J-1}\left(\frac{\Delta_+u_j}{h}\right)^2h$. If $|u_j|\geqslant a\geqslant 0$ for all $j=0, 1, \cdots, J$, then

$$\|u\|_h^2=\sum_{j=0}^{J}u_j^2h \geqslant (J+1)ha^2 \geqslant la^2,$$

where $Jh=l$. So there is a certain u_s, such that

$$|u_s| \leqslant \frac{\|u\|_h}{\sqrt{l}}. \tag{32}$$

Taking this u_s, we obtain

$$u_m^2 \leqslant \frac{\|u\|_h^2}{l}+2\|u\|_h\left\|\frac{\Delta_+u}{h}\right\|_h.$$

Hence there is the relation

$$\max_{j=0,1,\cdots,J} |u_j| \leqslant C_1 \|u\|_h^{\frac{1}{2}} \left(\left\| \frac{\Delta_+ u}{h} \right\|_h + \|u\|_h \right)^{\frac{1}{2}}, \tag{33}$$

where C_1 is a constant. (27) is an immediate consequence of (33).

Replacing $\{u_j\}$ by $\left\{\frac{\Delta_+ u_j}{h}\right\}$ $(j=0, 1, \cdots, J-1)$ in the general relation (33), we get

$$\max_{j=0,1,\cdots,J-1} \left| \frac{\Delta_+ u_j}{h} \right| \leqslant C_1 \left\| \frac{\Delta_+ u}{h} \right\|_h^{\frac{1}{2}} \left\{ \left\| \frac{\Delta_+ \Delta_- u}{h^2} \right\|_h + \left\| \frac{\Delta_+ u}{h} \right\|_h \right\}^{\frac{1}{2}}, \tag{34}$$

where C_1 is a constant. The right-hand side is uniformly bounded with respect to h from the Lemma 4, 5 and 7. So (28) is valid.

The estimate (29) follows directly from

$$\left| \frac{\Delta_+ u_j}{h} - \frac{\Delta_+ u_{j-1}}{h} \right| = h \left| \frac{\Delta_+ \Delta_- u_j}{h^2} \right| \leqslant h^{\frac{1}{2}} \left\| \frac{\Delta_+ \Delta_- u}{h^2} \right\|_h.$$

Suppose that the discrete function $u_j (j=0, 1, \cdots, J)$ satisfies the boundary condition $(*)_h$. For u_j the relation (11) takes the simple form

$$\sum_{j=1}^{J-1} u_j \Delta_+ \Delta_- u_j = - \sum_{j=0}^{J-1} (\Delta_+ u_j)^2.$$

Then we have

$$\left\| \frac{\Delta_+ u}{h} \right\|_h^2 \leqslant \|u\|_h \left\| \frac{\Delta_+ \Delta_- u}{h^2} \right\|_h. \tag{35}$$

Substituting this inequality into the relation (33), we obtain

$$\max_{j=0,1,\cdots,J} |u_j| \leqslant C_2 \|u\|_h^{\frac{3}{4}} \left(\left\| \frac{\Delta_+ \Delta_- u}{h^2} \right\|_h + \|u\|_h \right)^{\frac{1}{4}}, \tag{36}$$

where C_2 is a constant.

Since the discrete function $\{v_j^{n+1} - v_j^n\}$ $(j=0, 1, \cdots, J)$ satisfies the boundary condition $(*)_h$, we have from (36) the corresponding relation for $\{v_j^{n+1} - v_j^n\}$,

$$\begin{aligned} \max_{j=0,1,\cdots,J} |v_j^{n+1} - v_j^n| &\leqslant C_2 \|v^{n+1} - v^n\|_h^{\frac{3}{4}} \left\{ \left\| \frac{\Delta_+ \Delta_- (v^{n+1} - v^n)}{h^2} \right\|_h + \|v^{n+1} - v^n\|_h \right\}^{\frac{1}{4}} \\ &\leqslant C_2 k^{\frac{3}{4}} \left\| \frac{v^{n+1} - v^n}{k} \right\|_h^{\frac{3}{4}} \left\{ \left\| \frac{\Delta_+ \Delta_- v^{n+1}}{h^2} \right\|_h \right. \\ &\quad \left. + \left\| \frac{\Delta_+ \Delta_- v^n}{h^2} \right\|_h + \|v^{n+1}\|_h + \|v^n\|_h \right\}^{\frac{1}{4}}. \end{aligned}$$

This gives the estimate (30).

Now we turn to the last estimate (31) of the lemma. Substituting (35) into the right-hand part of the inequality (34), there is

$$\max_{j=0,1,\cdots,J-1} \left| \frac{\Delta_+ u_j}{h} \right| \leqslant C_3 \|u\|_h^{\frac{1}{4}} \left(\left\| \frac{\Delta_+ \Delta_- u}{h^2} \right\|_h + \|u\|_h \right)^{\frac{3}{4}}.$$

Because the discrete function $\{v_j^{n+1} - v_j^n\}$ $(j=0, 1, \cdots, J)$ satisfies the boundary condition $(*)_h$, the similar inequality for $\{v_j^{n+1} - v_j^n\}$ is

$$\max_{j=0,1,\cdots,J-1}\left|\frac{\Delta_+v_j^{n+1}-\Delta_+v_j^n}{h}\right|\leqslant C_3\|v^{n+1}-v^n\|_h^{\frac{1}{4}}\left\{\left\|\frac{\Delta_+\Delta_-(v^{n+1}-v^n)}{h^2}\right\|_h+\|v^{n+1}-v^n\|_h\right\}^{\frac{3}{4}}$$

$$\leqslant C_3k^{\frac{1}{4}}\left\|\frac{v^{n+1}-v^n}{k}\right\|_h^{\frac{1}{4}}\left\{\left\|\frac{\Delta_+\Delta_-v^{n+1}}{h^2}\right\|_h+\left\|\frac{\Delta_+\Delta_-v^n}{h^2}\right\|_h+\|v^{n+1}\|_h+\|v^n\|_h\right\}^{\frac{3}{4}}.$$

This implies the estimate (31).

Hence the lemma is proved.

§ 5

Let us put $v_{hk}(x, t)=v_j^{n+1}$ for $(x, t)\in Q_j^n=\{jh<x\leqslant(j+1)h;\ nk<t\leqslant(n+1)k\}$ $(j=0, 1, \cdots, J-1;\ n=0, 1, \cdots, N-1)$. Then $v_{hk}(x, t)$ is a m-dimensional vector valued piecewise constant function in the rectangular domain $Q_T=\{0\leqslant x\leqslant l,\ 0\leqslant t\leqslant T\}$. By the similar way, we can extend the m-dimensional vector valued discrete functions $\frac{\Delta_+v_j^{n+1}}{h}$, $\frac{v_j^{n+1}-v_j^n}{k}$, $\frac{\Delta_+\Delta_-v_j^{n+1}}{h^2}$ at grid point (x_j, t_{n+1}) to the corresponding m-dimensional vector valued piecewise constant functions $\bar{v}_{hk}(x, t)$, $\tilde{v}_{hk}(x, t)$, $\bar{\bar{v}}_{hk}(x, t)$ in the rectangular domain Q_T respectively. It follows directly from the Lemmas 3—7 for the estimations of the discrete function $v_j^n(j=0, 1, \cdots, J;\ n=0, 1, \cdots, N)$, that thus constructed m-dimensional vector valued piecewise constant functions $v_{hk}(x, t)$, $\bar{v}_{hk}(x, t)$, $\tilde{v}_{hk}(x, t)$ and $\bar{\bar{v}}_{hk}(x, t)$ have the following estimate

$$\sup_{0\leqslant t\leqslant T}\|v_{hk}(\cdot, t)\|_{L_2(0,l)}+\sup_{0\leqslant t\leqslant T}\|\bar{v}_{hk}(\cdot, t)\|_{L_2(0,l)}+\sup_{0\leqslant t\leqslant T}\|\tilde{v}_{hk}(\cdot, t)\|_{L_2(0,l)}+\sup_{0\leqslant t\leqslant T}\|\bar{\bar{v}}_{hk}(\cdot, t)\|_{L_2(0,l)}\leqslant K_{10}, \tag{37}$$

where K_{10} is independent of h and k.

We can select a sequence $\{h_i, k_i\}$, such that as $i\to\infty$, $\sqrt{h_i^2+k_i^2}\to 0$ and $v_{hk}(x, t)$, $\bar{v}_{hk}(x, t)$, $\tilde{v}_{hk}(x, t)$ and $\bar{\bar{v}}_{hk}(x, t)$ converge weakly to $u(x, t)$, $\bar{u}(x, t)$, $\tilde{u}(x, t)$ and $\bar{\bar{u}}(x, t)$ in $L_p((0, T);\ L_2(0, l))$ respectively, where $1\leqslant p<\infty$. Since the norm of the weak limiting function of the sequence not exceeds the lower limit of norms of the functions, the norms of functions $u(x, t)$, $\bar{u}(x, t)$, $\tilde{u}(x, t)$ and $\bar{\bar{u}}(x, t)$ in the functional space $L_p((0,T);\ L_2(0, l))$are uniformly bounded with respect to $1\leqslant p<\infty$. Hence they are bounded in the functional space $L_\infty((0, T);\ L_2(0, l))$, i. e.,

$$\sup_{0\leqslant t\leqslant T}\|u(\cdot, t)\|_{L_2(0,l)}+\sup_{0\leqslant t\leqslant T}\|\bar{u}(\cdot, t)\|_{L_2(0,l)}+\sup_{0\leqslant t\leqslant T}\|\tilde{u}(\cdot, t)\|_{L_2(0,l)}+\sup_{0\leqslant t\leqslant T}\|\bar{\bar{u}}(\cdot, t)\|_{L_2(0,l)}\leqslant K_{10}. \tag{38}$$

Now we want to prove that $\bar{u}=u_x$, $\tilde{u}=u_t$ and $\bar{\bar{u}}=u_{xx}$ and that $u(x, t)$ satisfies the boundary condition (*) and the intial condition (8).

Let $\Phi(x, t)$ be a smooth function with finite support in domain $\{0<x<l;\ 0\leqslant t<T\}$. It is easy to write the relation

$$\sum_{n=0}^{N-1}\sum_{j=0}^{J}\left(\Phi_j^n\frac{v_j^{n+1}-v_j^n}{k}+v_j^{n+1}\frac{\Phi_j^{n+1}-\Phi_j^n}{k}\right)hk=\sum_{n=0}^{N-1}\sum_{j=0}^{J-1}\frac{\Phi_j^{n+1}v_j^{n+1}-\Phi_j^nv_j^n}{k}kh$$

$$=-\sum_{j=0}^{J}\Phi_j^0v_j^0h.$$

We define the piecewise constant functions $\Phi_{hk}(x, t)$ and $\widetilde{\Phi}_{hk}(x, t)$ corresponding to the discrete functions Φ_j^{n+1}, $\frac{\Phi_j^{n+1}-\Phi_j^n}{k}$ respectively. Then we get the integral relation

$$\iint_{Q_T} (\Phi_{hk}\tilde{v}_{hk}+\widetilde{\Phi}_{hk}v_{hk})dx\,dt= -\int_0^l \Phi_{hk}(x, k)\bar{\varphi}_h(x)dx,$$

where $\bar{\varphi}_h(x)$ is a m-dimensional vector valued piecewise constant function and $\bar{\varphi}_h(x)=\bar{\varphi}_j$ in $(jh, (j+1)h]$ $(j=0, 1, \cdots, J-1)$. Since $\Phi_{hk}(x, t)$ converges uniformly to $\Phi(x, t)$, $\Phi_{hk}(x, k)$ converges uniformly to $\Phi(x, 0)$ and $\widetilde{\Phi}_{hk}(x, t)$ converges uniformly to $\Phi_t(x, t)$ as $h^2+k^2\to 0$, then passing the limit process as $h_i^2+k_i^2\to 0$ and regarding that $v_{hk}(x, t)$ and $\tilde{v}_{hk}(x, t)$ converge weakly to $u(x, t)$ and $\bar{u}(x, t)$ respectively, we obtain the integral equality

$$\iint_{Q_T} (\Phi\tilde{u}+\Phi_t u)dx\,dt= -\int_0^l \Phi(x, 0)\varphi(x)dx, \tag{39}$$

where as $h_i\to 0$, $\bar{\varphi}_h(x)$ converges uniformly to $\varphi(x)$ in $[0, l]$. If $\Phi(x, t)$ is of finite support in open rectangular domain $\{0<x<l;\ 0<t<T\}$, the relation (39) shows that, the m-dimensional vector valued function $u(x, t)$ has the generalized derivative $u_t(x, t)=\tilde{u}(x, t)$.

By the same approach, we can prove that $u_x(x, t)=\bar{u}(x, t)$ and $u_{xx}(x, t)=\bar{\bar{u}}(x, t)$.

Now we again construct a m-dimensional vector valued function $v^*_{hk}(x, t)$ corresponding to the discrete function v_j^n as follows: in every small rectangular grid $\bar{Q}_j^n=\{jh\leqslant x\leqslant (j+1)h,\ nk\leqslant t\leqslant (n+1)k\}$, $(j=0, 1, \cdots, J-1;\ n=0, 1, \cdots, N-1)$, $v^*_{hk}(x, t)$ is obtained by the linear expansion in both directions x and t from the values of the discrete function v_j^n at four corners of Q_j^n. From the estimates (27), (28) and (30), we see that the set of m-dimensional vector valued functions $\{v^*_{hk}(x, t)\}$ is not only uniformly bounded but also equicontinuous. Then as $h_i^2+k_i^2\to 0$, the sequence $\{v^*_{h_ik_i}(x, t)\}$ converges uniformly to a m-dimensional vector valued function $u^*(x, t)$ in Q_T. From the construction of the functions $v^*_{hk}(x, t)$ and $v_{hk}(x, t)$, we have

$$|v^*_{hk}(x, t)-v_{hk}(x, t)|\leqslant K_{11}(h+k^{\frac{3}{4}}). \tag{40}$$

This means that $u^*(x, t)=u(x, t)$ and $\{v_{h_ik_i}(x, t)\}$ converges also uniformly to $u(x, t)$ in Q_T.

Similarly, we can see that $\{\bar{v}_{h_ik_i}(x, t)\}$ converges uniformly to $u_x(x, t)$ in Q_T.

The uniform convergence shows that the limiting function $u(x, t)$ satisfies the boundary condition (*) and the initial condition (8).

Then we turn to prove that $u(x, t)$ is the generalized global solution of the boundary problem (*), (8) of the system (3) of the Schrödinger type. We take

$$\sum_{j=1}^{J-1}\sum_{n=0}^{N-1}\Phi_j^{n+1}\frac{v_j^{n+1}-v_j^n}{k}=\sum_{j=1}^{J-1}\sum_{n=0}^{N-1}\Phi_j^{n+1}A\frac{\Delta_+\Delta_-v_j^{n+1}}{h^2}+\sum_{j=1}^{J-1}\sum_{n=0}^{N-1}\Phi_j^{n+1}f_j^{n+1}.$$

Let $F_{hk}(x, t)=f_j^{n+1}=f(v_j^{n+1})$ in Q_j^n. So $F_{hk}(x, t)$ is a m-dimensional vector valued piecewise constant function in the rectangular domain Q_T. Then we have

$$\iint_{Q_T}\Phi_{hk}\tilde{v}_{hk}\,dx\,dt=\iint_{Q_T}\Phi_{hk}A\bar{\bar{v}}_{hk}\,dx\,dt+\iint_{Q_T}\Phi_{hk}F_{hk}\,dx\,dt.$$

Since as $h_i^2+k_i^2\to 0$, $\Phi_{hk}(x, t)$ converges uniformly to $\Phi(x, t)$ in Q_T, $\tilde{v}_{hk}(x, t)$ and $\bar{\bar{v}}_{hk}(x, t)$ converge weakly to $u_t(x, t)$ and $u_{xx}(x, t)$ respectively and $F_{hk}(x, t)$ converges uniformly to $f(u(x, t))$ in Q_T, then passing to limit as $h_i^2+k_i^2 \to 0$, we get the integral relation

$$\iint_{Q_T} \Phi[u_t - Au_{xx} - f(u)]\,dx\,dt = 0. \tag{41}$$

This means that $u(x, t)$ satisfies the system (3) in generalized sense. Therefore $u(x, t)$ is the generalized global solution of the boundary problem (*), (8) of the system (3) of Schrödinger type. This completes the proof of the existence of the solution of the above mentioned problem.

Suppose that there are two solutions $u(x, t)$ and $v(x, t)$ of the boundary problem (3), (*) and (8). Then for $w(x, t) = u(x, t) - v(x, t)$, we have

$$\iint_{Q_\tau} \Phi[w_t - Aw_{xx} - f(u) + f(v)]\,dx\,dt = 0,$$

where $0 < \tau \leqslant T$. Taking the test function to be a m-dimensional vector valued function $w(x, t)$, we get

$$\iint_{Q_\tau} (w,\ w_t - Aw_{xx} - f(u) + f(v))\,dx\,dt = 0.$$

So
$$\|w(\cdot,\ \tau)\|^2_{L_2(0,l)} \leqslant b \int_0^\tau \|w(\cdot,\ t)\|^2_{L_2(0,l)}\,dt,$$

where $w(x, 0) = 0$. Then $w(x, t) \equiv 0$ or $u(x, t) \equiv v(x, t)$.

Since the generalized solution $u(x, t)$ of the boundary problem (*), (8) for the system (3) of Schrödinger type is unique, the above mentioned convergence takes place as $h^2 + k^2 \to 0$.

Theorem 1. *Under the conditions* (I), (II) *and* (III), *the solution* v_j^n ($j = 0, 1 \cdots, J$; $n = 0, 1, \cdots, N$) *of the finite difference system* $(3)_h$, $(5)_h$ *and* $(8)_h$ *converges to a unique generalized global solution* $u(x, t) \in Z = L_\infty((0, T);\ W_2^{(2)}(0, t)) \cap W_\infty^{(1)}((0, T);\ L_2(0, l))$ *of the second boundary problem* (5), (8) *for the system* (3) *of Schrödinger type, when* $h^2 + k^2$ *tends to zero, i. e., there exists a unique solution* $u(x, t)$ *of the second boundary problem* (5), (8) *of the system* (3). *The m-dimensional vector valued discrete function* v_j^n *is the approximate solution of the second boundary problem* (3), (5), (8) *and* $\{v_j^n\}$ *and* $\left\{\frac{\Delta_+ v_j^n}{h}\right\}$ *converge uniformly to* $u(x, t)$ *and* $u_x(x, t)$ *respectively as* $h^2 + k^2 \to 0$.

Theorem 2. *Suppose that the conditions* (I), (II) *and* (III) *are satisfied and suppose that the system* (3) *is homogeneous, i. e.,* $f(0) = 0$. *The m-dimensional vector valued discrete solutions* v_j^n *of the finite difference system* $(3)_h$, $(4)_h$, $(8)_h$; $(3)_h$, $(6)_h$, $(8)_h$ *or* $(3)_h$, $(7)_h$, $(8)_h$ *converge to a unique generalized global solution* $u(x, t) \in Z$ *of the first boundary problem* (4), (8) *and the mixed boundary problem* (6), (8) *or* (7), (8) *for the homogeneous system* (3) *of Schrödinger type, when* $h^2 + k^2 \to 0$, *i. e., these boundary problems have a unique generalized global solution* $u(x, t) \in Z$ *and* v_j^n *and* $\frac{\Delta_+ v_j^n}{h}$ *converge uniformaly to* $u(x, t)$ *and* $u_x(x, t)$ *respectively as* $h^2 + k^2 \to 0$.

§ 6

In order to obtain the existence and uniqunes of the generalized global solution $u(x, t)$ of the boundary problem (*), (8) for the system (3) of Schrödinger type, the condition (III) can be weakened.

(III′) The m-dimensional vector valued initial function $\varphi(x) \in W_2^{(2)}(0, l)$ satisfies the appropriate boundary condition $(*)$.

Let $\{\varphi_s(x)\}$ be a sequence of m-dimensional vector valued twice continuously differentiable functions, convergent to $\varphi(x)$ in the functional space $W_2^{(2)}(0, l)$. And $\varphi_s(x)$ for every s satisfies the appropriate boundary condition $(*)$.

Denote the unique generalized global solution of the system (3) of Schrödinger type with the boundary condition $(*)$ and the intial condition

$$u(x, 0) = \varphi_s(x), \tag{8}$$

by $u_s(x, t) \in Z$.

Passing to the limit as $h^2 + k^2 \to 0$, the estimations (16)′, (22)′, (25)′ and (26) in § 3 become the following estimation for the solutions $u_s(x, t)$ of the boundary problem (3), $(*)$ and $(8)_s$:

$$\|u_s\|_Z \leqslant K_{12}\{\|\varphi\|_{W_2^{(2)}(0,l)} + \|f(\varphi)\|_{L_2(0,l)}\}, \tag{42}$$

where K_{12} is independent of s.

Considering the limiting procedure of $s \to \infty$, we get the following theorems.

Theorem 3. *Under the conditions* (I), (II) *and* (III′), *the second boundary problem* (5), (8) *of the system* (3) *of Schrödinger type has a m-dimensional vector valued unique generalized global solution* $u(x, t) \in Z$.

Theorem 4. *Suppose that the conditions* (I), (II) *and* (III′) *are fulfilled and suppose that the system* (3) *is homogeneous, i. e.,* $f(0) = 0$. *Then the first boundary problem* (4), (8) *and the mixed boundary problems* (6), (8) *or* (7), (8) *for the homogeneous system* (3) *of the Schrödinger type have a unique m-dimensional vector valued generalized global solution* $u(x, t) \in Z$.

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