ESTIMATION FOR SOLUTIONS OF ILL-POSED CAUCHY PROBLEMS OF DIFFERENTIAL EQUATIONS WITH PSEUDO-DIFFERENTIAL OPERATORS*

Part I. Case of First Order Operators

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Abstract

In this paper we discuss the estimation for solutions of the ill-posed Cauchy problems of the following differential equation

$$\frac{du(t)}{dt}=A(t)u(t)+N(t)u(t),\ \forall t\in(0,1),$$

where $A(t)$ is a p. d. o. (pseudo-differential operator (s)) of order 1 or 2, $N(t)$ is a uniformly bounded $H\to H$ linear operator. It is proved that if the symbol of the principal part of $A(t)$ satisfies certain algebraic conditions, two estimates for the solution $u(t)$ hold. One is similar to the estimate for analytic functions in the Three-circle Theorem of Hadamard. Another is the estimate of the growth rate of $\|u(t)\|$ when $A(1)u(1)\in H$.

Introduction

In this paper we will discuss the estimation for solutions of the Cauchy problems of the differential equation

$$\frac{du(t)}{dt}=A(t)u(t)+N(t)u(t),\ \forall t\in(0,1), \tag{1}$$

with the prescribed $u(0)=u_0$, where $u(t)=u(t,x)$ is a n-dimensional vector function, $x\in R^m$, $A(t)$ is a p. d. o. dependent on the parameter t, $N(t)$ is a uniformly (respectively to t) bounded linear operator $H\to H$. This Cauchy problem in general is not well-posed.

The simplest examples are the Cauchy problem of the Cauchy-Riemann equations and that of the backward heat equation. The estimate of solutions of the Cauchy problem of the Laplace equation was obtained by M. M. Lavrentiev[1]. The same estimate for the Cauchy-Riemann equations, the backward heat equation and that of (1), in which A is a differential operator with constant coefficients satisfying certain conditions were obtained in [2, 3, 4]. This estimate can be represented in the form

$$\|u(t)\|\leqslant c\|u(0)\|^{1-t}\|u(1)\|^{t}, \tag{2}$$

where c is a constant independent of $u(t)$.

The estimate (2) is significant in the investigation of approximate methods for solving the ill-posed Cauchy problems. In [4] the author discussed the difference

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schemes for solving (1), in which the differential operator A has constant coefficients. Considering the estimate (2), we gave an appropriate definition of stability, and proved the theorem of equivalence between convergence and stability for the consistent and commutative (with A) difference schemes. In [10] the finite element method for solving the Cauchy problems of the Laplace equation was discussed. So in order to investigate the approximate methods for (1) in the general case, it is important to establish the estimate of the solutions, similar to (2).

When $m=1$ and A is a first order differential operator whose coefficient matrix is variable and of simple structure, similar estimate was obtained by the author in terms of the Three-line Theorem[5]. It is very like the Three-circle Theorem of Hadamard. This estimate can be written in the form

$$\|u(t)\| \leq C \max_{i=1,2} \{\|u(0)\|^{1-\delta_i(t)} \|u(t)\|^{\delta_i(t)}\}, \tag{3}$$

where $\delta_i(t)$ are increasing functions of t, satisfying $\delta_i(0)=0$ and $\delta_i(1)=1$. If $(u(1), u(1))' = \frac{d}{dt}(u(t), u(t))|_{t=1}$ exists, then the growth rate of $\|u(t)\|$ can be estimated by

$$\|u(t)\| \leq c^* \|u(0)\| e^{K(u(1))t}, \tag{4}$$

where $K(u(1)) = c^{**}(u(1), u(1))'/(u(1), u(1))$. c, c^* and c^{**} are all constant, independent of $u(t)$.

In [5], the author discussed the second order ordinary differential inequalities and obtained two estimates for the solutions. One of them is an extension of the inequality for convex functions. These two inequalities play an important role in the estimation of the solutions of ill-posed Cauchy problems. We shall reformulate the lemma about these two inequalities (in section 2) and use them to prove the main theorems of this paper. Using the same inequalities, S. Agmon and L. Nirenberg[6] obtain the estimation for solutions of abstract differential equations in a Hilbert space. In [7] estimation of this type was also discussed. Many results about estimation for solutions of ill-posed problems in P. D. E. are outlined in [8].

The aim of this paper is to estimate the solutions of (1), in which $A(t)$ is a first order or second order p. d. o.. We shall prove that estimates similar to (3) and (4) are also valid for the solutions of (1), when the symbol of the principal part of $A(t)$ satisfies certain easily verified algebraic conditions. As mentioned above, the estimate (3) is meaningful to approximate methods, this paper can be considered as a preparation for the discussion of approximate methods of ill-posed problem (1).

This paper is divided into two parts. Part 1 is devoted to the case of first order p. d. o.. First, we review briefly some theorems about p. d. o.. Then we formulate some lemmas, which are used in the proof of the main theorems of this paper. One of them is on the second order ordinary differential inequalities. The others describe truncators and quasi-inverses. Finally we derive the estimates for solutions of (1). In part II we discuss the case, in which $A(t)$ is of second order.

§ 1. Pseudo-Differential Operators

In this section we first recall the definition of p. d. o. on vector-valued functions $u(x)$ and some theorems about them, according to [9].

Let $u(x)$ be a n-dimensional vector function, defined on R^m. Its Fourier transform is denoted by

$$\tilde{u}(\xi)=(2\pi)^{-\frac{m}{2}}\int e^{-ix\cdot\xi}u(x)\,dx.$$

The norm $\|\cdot\|_s$ is defined as

$$\|u(x)\|_s^2=\int(1+|\xi|^2)^s|\tilde{u}(\xi)|^2\,d\xi.$$

H_s, as usual, is the Hilbert space with the norm $\|\cdot\|_s$.

Let $$\varphi=\{u(x):u(x)\in H_s,\ \forall s\},$$

and $\mathscr{L}$ be the set of all linear operators L: $\varphi\to\varphi$.

Definition 1. *If $L\in\mathscr{L}$, and for each s there exists a constant c_s such that*

$$\|Lu\|_s\leqslant c_s\|u\|_{s+r},\ \forall u\in\varphi,$$

then L is called an operator of order r.

Since φ is dense everywhere in H_s, for every r-th order operator L there exists a unique extension, which is a linear operator $H_{s+r}\to H_s$. For convenience we denote this extension by the same notation L. Hence in this paper a r-th order operator L is considered also as a linear operator $H_{s+r}\to H_s$.

Now we consider operators $L(t)$ dependent on parameter t.

Definition 2. *Let $L(t)$: $[0, 1)\to\mathscr{L}$, if for each s there exists a constant c_s independent of t, such that*

$$\|L(t)u\|_s\leqslant C_s\|u\|_{s+r},\ \forall u\in\varphi \text{ and } t\in[0, 1), \tag{1.1}$$

then $L(t)$ is called a uniformly bounded (on $[0, 1)$) operator of order r.

We denote the set of all uniformly bounded operators of order r and the set of all uniformly bounded (on $[0, 1)$) linear operators $H_{s+r}\to H_s$ respectively by $\mathscr{L}^r$ and $\mathscr{L}(H_{s+r}\to H_s)$. Obviously $\mathscr{L}^r\subset\mathscr{L}(H_{s+r}\to H_s)$.

Let $a_i(t, x, \xi)$ be an infinitely differentiable (with respect to x and ξ) $(n\times n)$ matrix, defined on $([0, 1)\times R^m\times(R^m\setminus(|\xi|=0)))$, and possessing the following properties:

(1) $a_i(t, x, \xi)$ is positive homogeneous of degree zero in ξ, i. e.

$$a_i(t, x, \gamma\xi)=a_i(t, x, \xi),\ \forall\gamma>0.$$

(2) There exists the limit $\lim\limits_{|x|\to\infty}a_i(t, x, \xi)=a_i(t, \infty, \xi)$.

(3) For any non-negative integer r and index $\alpha=(\alpha_1, \cdots, \alpha_m)$, $\beta=(\beta_1, \cdots, \beta_m)$,

$$(1+|x|^r)D^\alpha\partial^\beta(a_i(t, x, \xi)-a_i(t, \infty, \xi))\to0 \text{ as } |x|\to\infty, \tag{1.2}$$

uniformly on $\{t\in[0, 1),\ |\xi|=1\}$, where

$$D^\alpha=D_1^{\alpha_1}\cdots D_m^{\alpha_m},\quad \partial^\beta=\partial_1^{\beta_1}\cdots\partial_m^{\beta_m},\quad D_j=-\sqrt{-1}\,\frac{\partial}{\partial x_j},\quad \partial_j=\frac{\partial}{\partial\xi_j}.$$

Now we consider the power series of $|\xi|$ with such $a_i(t, x, \xi)$ as coefficients

$$a(t, x, \xi)=\sum_{i=0}^{N}a_i(t, x, \xi)|\xi|^{r-i}, \tag{1.3}$$

where N is any non-negative integer number.

We denote the set of all such $a(t, x, \xi)$ by $\{S\}$, and $\{S^0\}=\{a(t, x, \xi);\ N=0$ in expression (1.3) of $a\}$, $\{S_1\}=\{a(t, x, \xi): \frac{\partial}{\partial t} a(t, x, \xi)\in\{S\}\}$,

$\{DS\}=\{a(t, x, \xi)\in\{S\}:$ all $a_i(t, x, \xi)$ are diagonal$\}$, $\{DS_1\}=\{S_1\}\cap\{DS\}$.

Let $\zeta(|\xi|)$ be a c^∞ non-negative function satisfying $\zeta(|\xi|)=0$ for $|\xi|\leqslant\frac{1}{2}$ and $\zeta(|\xi|)=1$ for $|\xi|\geqslant 1$.

We define p. d. o. according to [9].

Definition 3. *For every $a(t, x, \xi)\in\{S\}$, the p. d. o. $A(t)=\zeta(|D|)a(t, x, D)$ is defined as follows*

$$\widetilde{A(t)u}(\xi) = (2\pi)^{-\frac{m}{2}}\int e^{-ix\cdot\xi}\zeta(|\xi|)a(t, x, \xi)u(x)dx, \tag{1.4}$$

or

$$\widetilde{A(t)u}(\xi)=\zeta(|\xi|)\{a(t, \infty, \xi)\tilde{u}(\xi)+(2\pi)^{-\frac{m}{2}}\int \tilde{a}'(t, \xi-\eta, \xi)\tilde{u}(\eta)d\eta, \tag{1.4'}$$

where $\tilde{a}'(t, \eta, \xi)$ is the Fourier transform of $a'(t, x, \xi)=a(t, x, \xi)-a(t, \infty, \xi)$ with respect to x. $a(t, x, \xi)$ is called symbol of $A(t)$.

In [9] p. d. o. are defined for more generel $a(t, x, \xi)$. But in our paper we consider only the case, in which $a(t, x, \xi)$ has only finite number of terms in the expression (1.3).

Definition 4. *For every $a(t, x, \xi)\in\{S\}$, the operator $A^{(*)}(t)$ is defined by*

$$\widetilde{A^{(*)}(t)}u(\xi)=\zeta(|\xi|)a^*(t, \infty, \xi)\tilde{u}(\xi)+(2\pi)^{-\frac{m}{2}}\int \tilde{a}'^*(t, \eta-\xi, \eta)\zeta(|\eta|)\tilde{u}(\eta)d\eta, \tag{1.5}$$

where $a^*(t, \infty, \xi)$ and $a'^*(t, x, \xi)$ are the conjugate transposes of $a(t, \infty, \xi)$ and $a'(t, x, \xi)$ respectively.

The operator $A^{(*)}(t)$ is the formal adjoint of $A(t)$, i. e.

$$(A^{(*)}u, v)=(u, Av), \forall u, v\in\varphi. \tag{1.6}$$

Using (1.4'), (1.5) we can easily verify that

$$(\widetilde{A^{(*)}u}(\xi), \tilde{v}(\xi))=(\tilde{u}(\xi), \widetilde{Av}(\xi)), \forall u, v\in\varphi,$$

from which (1.6) is obtained directly.

Obviously, if A is an operator of order r, $A^{(*)}$ is also an operator of order r.

When $a(t, x, \xi)\in\{S_1\}$, due to the uniform convergence property (1.2), there exists the limit

$$\lim_{\Delta t\to 0}\frac{\widetilde{A(t+\Delta t)u}(\xi)-\widetilde{A(t)u}(\xi)}{\Delta t}=\lim_{\Delta t\to 0}(2\pi)^{-\frac{m}{2}}\int e^{-ix\cdot\xi}\zeta(|\xi|)$$

$$\cdot\sum_{i=0}^{N}\frac{a_i(t+\Delta t, x, \xi)-a_i(t, x, \xi)}{\Delta t}|\xi|^{r-i}u(x)dx$$

$$=(2\pi)^{-\frac{m}{2}}\int e^{-ix\cdot\xi}\zeta(|\xi|)\sum_{i=0}^{N}\frac{\partial}{\partial t}a_i(t, x, \xi)|\xi|^{r-i}u(x)dx$$

$$=\left(J(|D|)\frac{\partial}{\partial t}a(t, x, D)\right)\widetilde{u(\xi)}.$$

This means that the operator $A'(t)=\lim\limits_{\Delta t\to 0}\frac{A(t+\Delta t)-A(t)}{\Delta t}$ exists, and is a p. d. o. with the symbol $\frac{\partial}{\partial t}a(t, x, \xi)$.

Theorem 1.1. *The p. d. o. $A(t)$ with the symbol $a(t, x, \xi)\in\{S\}$ having expression (1.3) is a uniformly bounded (on $[0, 1)$) operator of order r, i. e. $A(t)\in\mathscr{L}^r$, where r is the power index of first term in expression (1.3) of $a(t, x, \xi)$.*

We denote the set of all such p. d. o. uniformly bounded (on $[0, 1)$), having symbols $a(t, x, \xi)\in\{S\}$ by $\mathscr{L}^r_s$. And analogously denote corresponding sets of p. d. o. by $\mathscr{L}^r_{s_1}$, $\mathscr{L}^r_{Ds}$, $\mathscr{L}^r_{Ds_1}$, $\mathscr{L}^r_{s*}$, $\cdots$. Obviously $\mathscr{L}^r_{Ds_1}\subset\mathscr{L}^r_{s_1}\subset\mathscr{L}^r_s\subset\mathscr{L}^r$, $\mathscr{L}^r_{Ds}\subset\mathscr{L}^r_s$ and $A'(t)\in\mathscr{L}^r_s$, if $A(t)\in\mathscr{L}^r_{s_1}$.

Theorem 1.2. *Let $A(t)\in\mathscr{L}^r_s$ have the symbol $a(t, x, \xi)$ and $A^{(*)}_\rho(t)$ be a p. d. o. with the symbol*

$$a^*_\rho(t, x, \xi)=\sum_{|\alpha|=0}^{\rho}\frac{(-1)^\alpha}{\alpha!}D^\alpha\partial^\alpha a^*(t, x, \xi),$$

then $A^{(*)}(t)-A^{(*)}_\rho(t)\in\mathscr{L}^{r-\rho-1}$.

It is clear that $A^{(*)}_\rho(t)\in\mathscr{L}^r_s$. Since $a(t, x, \xi)$ is a power series of $|\xi|$, substituting the corresponding power series for $a^*(t, x, \xi)$ into the sum for $a^{(*)}_\rho(t, x, \xi)$ and then differentiating, we obtain for $a^{(*)}_\rho(t, x, \xi)$ the finite power series of $|\xi|$ having the same form of (1.3) whose first term is of $|\xi|^r$.

Theorem 1.3. *Let $A(t)\in\mathscr{L}^{r(a)}_s$, $B(t)\in\mathscr{L}^{r(b)}_s$ have symbols $a(t, x, \xi)$ and $b(t, x, \xi)$ respectively, $P_k(t)$ and $Q_k(t)$ be the p. d. o. with the symbols*

$$p_k(t, x, \xi)=\sum_{|\alpha|=k}\frac{(-1)^k}{\alpha!}D^\alpha a(t, x, \xi)\partial^\alpha b(t, x, \xi),$$

$$q_k(t, x, \xi)=\sum_{|\alpha|=k}\frac{(-1)^k}{\alpha!}D^\alpha b(t, x, \xi)\partial^\alpha a(t, x, \xi)$$

respectively, then the operators $A(t)B(t)-\sum_{k=0}^{\rho-1}P_k(t)$, $B(t)A(t)-\sum_{k=0}^{\rho-1}Q_k(t)$ and $(A(t)B(t)-B(t)A(t))-\sum_{k=0}^{\rho-1}(P_k(t)-Q_k(t))$ all belong to $\mathscr{L}^{r(a)+r(b)-\rho}_s$.

It is clear that $P_k(t)\in\mathscr{L}^{r(a)+r(b)-k}_s$ and $Q_k(t)\in\mathscr{L}^{r(a)+r(b)-k}_s$.

The proofs of Theorems 1.1, 1.2, 1.3 are omitted.

In [9], the p. d. o. for scalar functions $a(x, \xi)$ are discussed. Here we discuss the p. d. o. for matrix functions $a(t, x, \xi)$ dependent on the parameter t. The proofs of Theorem 1, Theorem 2(ii) and Lemma 5.1 of [9] can be transcribed over to the corresponding Theorems 1.1, 1.2, 1.3 without any difficulties.

§ 2. Lemmas

In this section we present some lemmas. The first two lemmas are often used in estimating the solutions of ill-posed problem (1). The last two lemmas are related to truncators and quasi-inverses.

Lemma 2.1. *Let $g(t)$ be a twice differentiable function on $[0, 1]$, $f(t)$ and $h(t)$ be measurable functions in this interval, satisfying*

$$\operatorname*{ess\,sup}_{(0,1)}|f(t)|<\text{const } M,\quad \operatorname*{ess\,sup}_{(0,1)}|h(t)|<M.$$

If $g(t)$ satisfies $\qquad g''(t)\geqslant f(t)g'(t)+h(t)\qquad$ *a. e. in* $(0, 1)$,

then $\qquad g(t)\leqslant\max_{i=1,2}\{(1-\delta_i(t))g(0)+\delta_i(t)g(1)+M_2\},\ \forall t\in(0, 1),\qquad$ (2.1)

and
$$g(t)\leq g(0)+g'(1)\int_0^t e^{-\int_\eta^1 f(\xi)d\xi}\,d\eta+M_3 t,\ \forall t\in(0,1),\tag{2.2}$$
where $M_2=\frac{M}{2}\exp 2M$, $M_3=M\exp 2M$, $\delta_i(t)=1/(1+\gamma_i(t))$, $\gamma_1(t)=(e^M-e^{Mt})/(1-e^{-Mt})$, $\gamma_2(t)=(e^{-Mt}-e^{-M})/(e^{Mt}-1)$.

When $M\to 0$, we have that $M_2\to 0$ and $\delta_i(t)\to t$. Inequality (2.1) is reduced to the inequality for convex functions.

The proof can be found in [5]. A similar proof can be found also in [6].

Let $\mathscr{B}(t)$ be a linear operator dependent on t, defined on $D(\mathscr{B})$ with values in a Hilbert space $\mathscr{H}$, $D(\mathscr{B})$ is dense in $\mathscr{H}$.

We denote by $\{Y\}$ the set of all $y(t)$ possessing following properties:

(1) $y(t)\in C([0,1];\mathscr{H})\cap C^1([0,1);\mathscr{H})$ and $\mathscr{B}(t)y(t)\in C([0,1);\mathscr{H})$.

(2) $$2\operatorname{Re}\left(y(t),\ \frac{dy(t)}{dt}-\mathscr{B}(t)y(t)\right)_{\mathscr{H}}=\widetilde{M}_4(t)\,\|y(t)\|^2_{\mathscr{H}},\tag{2.3}$$

where $\widetilde{M}_4(t)$ is a bounded measurable function, $\operatorname*{ess\,sup}_{(0,1)}|\widetilde{M}_4(t)|\leq\text{const } M_4$.

For convenience in lemma 2.2 (only) $\|\cdot\|_{\mathscr{H}}$ and $(\cdot,\cdot)_{\mathscr{H}}$ are abbreviated to $\|\cdot\|$ and $(\cdot,\cdot)$.

Lemma 2.2. *If* $y(t)\in\{Y\}$, *the derivative* $\frac{d}{dt}\operatorname{Re}(y(t),\mathscr{B}(t)y(t))$ *exists and satisfies*
$$\frac{d}{dt}\operatorname{Re}(y(t),\mathscr{B}(t)y(t))\geq 2\|\mathscr{B}(t)y(t)\|^2-M_5\|y(t)\|\,\|\mathscr{B}(t)y(t)\|-M_6\|y(t)\|^2,\tag{2.4}$$
then
$$\|y(t)\|\leq M_7\max_{i=1,2}\{\|y(0)\|^{1-\delta_i(t)}\|y(1)\|^{\delta_i(t)}\},\tag{2.5}$$
where $\delta_i(t)$ *are the functions defined in lemma* 2.1.

Moreover, if there exists $\mathscr{B}(1)y(1)\in\mathscr{H}$, *then*
$$\|y(t)\|\leq M_8\|y(0)\|\exp\{M_9[\|\mathscr{B}(1)y(1)\|/\|y(1)\|]t\}.\tag{2.6}$$
The constants M_7, M_8, M_9 *and* M *in the expression of* $\delta_i(t)$ *depend only on the constants* M_4, M_5 *and* M_6.

Proof. Set
$$\psi(t)=\ln\|y(t)\|^2$$
and
$$g'(t)=\frac{2\operatorname{Re}(y(t),\mathscr{B}(t)y(t))}{\|y(t)\|^2}=\frac{2\|\mathscr{B}(t)y(t)\|}{\|y(t)\|}\cos\theta\tag{2.7}$$
with initial value $g(0)=\psi(0)$, here $\cos\theta=\frac{\operatorname{Re}(y(t),\mathscr{B}(t)y(t))}{\|y(t)\|\,\|\mathscr{B}(t)y(t)\|}$. Then from (2.7) and (2.4), we obtain
$$\begin{aligned}
g''(t)&\doteq(2/\|y(t)\|^4)\left\{\frac{d}{dt}\operatorname{Re}(y(t),\mathscr{B}(t)y(t))\|y(t)\|^2\right.\\
&\qquad\left.-\operatorname{Re}(y(t),\mathscr{B}(t)y(t))\frac{d}{dt}(y(t),y(t))\right\}\\
&\geq(1/\|y(t)\|^4)\{4\|\mathscr{B}(t)y(t)\|^2\|y(t)\|^2-4\|\mathscr{B}(t)y(t)\|^2\|y(t)\|^2\cos^2\theta\\
&\qquad-2M_5\|\mathscr{B}(t)y(t)\|\,\|y(t)\|^3-2M_6\|y(t)\|^4-\widetilde{M}_4(t)g'(t)\,\|y(t)\|^4\}\\
&=4\|\mathscr{B}(t)y(t)\|^2\sin^2\theta/\|y(t)\|^2-2M_5[\|\mathscr{B}(t)y(t)\|\sin^2\theta/\|y(t)\|\\
&\qquad+\frac{1}{2}g'(t)\cos\theta]-\widetilde{M}_4(t)g'(t)-2M_6\\
&\geq-(M_5\cos\theta+\widetilde{M}_4(t))g'(t)-(2M_6+(M_5/2)^2\sin^2\theta).
\end{aligned}$$

Hence from lemma 2.1 we have inequalities (2.1) and (2.2) with

$$M=\max\{M_4+M_5,\ 2M_6+(M_5/2)^2\}.$$

Using (2.7) and (2.3) we obtain $|\psi(t)-g(t)|\leqslant M_4 t$. Consequently from (2.1) and (2.2) we have

$$\psi(t)\leqslant\max_{i=1,2}\{(1-\delta_i(t))\psi(0)+\delta_i(t)\psi(1)\}+M_2+2M_4$$

and

$$\psi(t)\leqslant\psi(0)+[(\psi'(1)+M_4)e^M+M_3]t,$$

from which the desired estimates (2.5), (2.6) result with the constants

$$M_7^2=\exp[M_2(M_4,\ M_5,\ M_6)+2M_4],$$
$$M_8^2=\exp[M_3(M_4,\ M_5,\ M_6)+2M_4e^M],$$
$$M_9=\frac{1}{2}\exp M(M_4,\ M_5,\ M_6).$$

In [5] the author used this method to prove the estimates (2.5), (2.6) for the case in which $\mathscr{B}(t)$ is a first order differential operator with variable coefficient in one-dimensional x.

Now we describe two lemmas about truncators and quasi-inverses.

Definition 5. *The operators $T(K)$ and $R(K)$ are defined as follows*

$$\widetilde{T(K)u}(\xi)=t_K(|\xi|)\tilde{u}(\xi),\ \widetilde{R(K)u}(\xi)=(1-t_K(|\xi|))\tilde{u}(\xi), \tag{2.8}$$

where $K>0$ is a parameter, $t_K(|\xi|)$ is a non-negative $C^\infty(R)$ function satisfying $t_K(|\xi|)=1$ for $|\xi|\leqslant K$ and $t_K(|\xi|)=0$ for $|\xi|\geqslant K+1$. $T(K)$ is called truncator.

Obviously $T(K)+R(K)=I$ — the identity operator.

Lemma 2.3. *Let $L(t)\in\mathscr{L}^r$, then*

(1) *for any real ρ and s, hold*

$$\|L(t)T(K)u\|_s\leqslant C'_{s,r,\rho}\|T(K)u\|_{s-\rho},\ \forall u\in\varphi, \tag{2.9}$$

and

$$\|T(K)L(t)u\|_s\leqslant C_{s,r,\rho}\|u\|_{s-\rho},\ \forall u\in\varphi, \tag{2.10}$$

where $C'_{s,r,\rho}=C_sC_{(r+\rho)}$, $C_{s,r,\rho}=C_{s-\rho-r}C_{(r+\rho)}$, $C_{(r+\rho)}=\max\{1,\ [1+(K+1)^2]^{\frac{r+\rho}{2}}\}$, C_s (and $C_{s-\rho-r}$) is the constant in (1.1).

(2) *for any real $\rho\geqslant s+r$, holds*

$$\|L(t)R(K)u\|_s\leqslant C_s(1+K^2)^{\frac{s+r-\rho}{2}}\|R(K)u\|_\rho,\ \forall u\in\varphi. \tag{2.11}$$

Proof. Since $|\widetilde{T(K)u}(\xi)|=0$ for $|\xi|\geqslant K+1$, it follows that

$$\|L(t)T(K)u\|_s\leqslant C_s\|T(K)u\|_{s+r}=C_s\|[(1+|\xi|^2)^{\frac{r+\rho}{2}}][(1+|\xi|^2)^{\frac{s-\rho}{2}}\widetilde{T(K)u}(\xi)\|$$
$$\leqslant C_sC_{(r+\rho)}\|(1+|\xi|^2)^{\frac{s-\rho}{2}}\widetilde{T(K)u}(\xi)\|=C'_{s,r,\rho}\|T(K)u\|_{s-\rho}.$$

Now set $L(t)u=v$, then from the above extimate we obtain

$$\|T(K)L(t)u\|_s=\|T(K)v\|_s\leqslant C_{(r+\rho)}\|T(K)v\|_{s-\rho-r}\leqslant C_{(r+\rho)}\|v\|_{s-\rho-r}$$
$$\leqslant C_{(r+\rho)}C_{s-\rho-r}\|u\|_{s-\rho}=C_{s,r,\rho}\|u\|_{s-\rho}.$$

For (2) we have that $|\widetilde{R(K)u}(\xi)|=0$ for $|\xi|\leqslant K$. Hence for $\rho\geqslant s+r$ it follows that

$$\|L(t)R(K)u\|_s \leqslant C_s\|R(K)u\|_{s+r} = C_s\|(1+|\xi|^2)^{\frac{s+r-\rho}{2}}[(1+|\xi|^2)^{\frac{\rho}{2}}\widetilde{R(K)u}(\xi)]\|$$
$$\leqslant C_s(1+K^2)^{\frac{s+r-\rho}{2}}\|(1+|\xi|^2)^{\frac{\rho}{2}}\widetilde{R(K)u}(\xi)\| = C_s(1+K^2)^{\frac{s+r-\rho}{2}}\|R(K)u\|_\rho.$$

Lemma 2.4. *Let $A(t)\in\mathscr{L}^0_{s_0}$ have the symbol $a(t, x, \xi)$ satisfying $|\det a(t, x, \xi)| \geqslant$ const $\gamma>0$ $\forall$ all (t, x, ξ), then for sufficiently large K there exist the operators $A^{(-1)}(t)\in\mathscr{L}^0_{s_0}$, $D_{(K)}(t)\in\mathscr{L}(H\to H)$ and $L_{(-1)}(t)\in\mathscr{L}^{-1}$ such that*

$$D_{(K)}(t)A^{(-1)}(t)A(t)u = u + D_{(K)}(t)L_{(-1)}(t)T(K)u,\ \forall u\in H. \tag{2.12}$$

Before the proof we note first that if the support of $\tilde{u}(\xi)\subset\{|\xi|\geqslant K+1\}$, then the relation (2.12) is reduced to $u=[D_{(K)}(t)A^{(-1)}(t)]A(t)u$. So the operator $A^{(-1)}_{(K)}(t) = D_{(K)}(t)A^{(-1)}(t)$ is called the quasi-inverse of $A(t)$ in H.

Proof. Under the condition of lemma, it is obvious that there exists $b(t, x, \xi) = a^{-1}(t, x, \xi)\in\{S^0\}$. So the p. d. o. $A^{(-1)}(t)$ with the symbol $b(t, x, \xi)$ exists and belongs to $\mathscr{L}^0_{s_0}$.

Applying Theorem 1.3 (in this case $Q_0(t)$ is the identity operator I) we have

$$L_{(-1)}(t) = A^{(-1)}(t)A(t) - I = (A^{(-1)}(t)A(t) - Q_0(t))\in\mathscr{L}^{-1}.$$

From lemma 2.3 it results that

$$\|L_{(-1)}(t)R(K)u\| \leqslant C_0(L_{(-1)})(1+K^2)^{-\frac{1}{2}}\|R(K)u\| \leqslant C_0(1+K^2)^{-\frac{1}{2}}\|u\|,\ \forall u\in\varphi.$$

Since $C_0(1+K^2)^{-\frac{1}{2}}\leqslant 1-s<1$ for sufficiently large K, there exists in $\mathscr{L}(H\to H)$ the operator

$$D_{(K)}(t) = (I+L_{(-1)}(t)R(K))^{-1} = \sum_{j=0}^{\infty}(-1)^j(L_{(-1)}(t)R(K))^j,$$

whose norm satisfies $\|D_{(K)}(t)\|_{\mathscr{L}(H\to H)}\leqslant 1/s$.

Now we have

$$\begin{aligned} D_{(K)}(t)A^{(-1)}(t)A(t)u &= D_{(K)}(t)[I+L_{(-1)}(t)]u \\ &= D_{(K)}(t)\{I+L_{(-1)}(t)[R(K)+T(K)]\}u \\ &= [I+D_{(K)}(t)L_{(-1)}(t)T(K)]u,\ \forall u\in H. \end{aligned}$$

The lemma is proved.

§ 3. Estimation of Solutions for the Case of 1-st Order p. d. o.

This section deals with the estimation for solutions of the differential equation

$$\frac{du(t)}{dt} = A(t)u(t) + N(t)u(t),\ \forall t\in(0, 1) \tag{3.1}$$

with first order p. d. o. $A(t)\in\mathscr{L}^1_{s_1}$, where $N(t)\in\mathscr{L}(H\to H)$. $u(t)=u(t, x)$ is a n-dimensional vector function.

We denote by $\{V_1\}$ the set of all solutions of (3.1), which belong to $C([0, 1]; H)\cap C^1([0, 1); H)\cap C([0, 1); H_1)$.

Now we make the following hypothesis.

Hypothesis I.

(1) The symbol $a_0(t, x, \xi)|\xi|$ of $A(t)$ belongs to $\{S^0_1\}$.

(2) The $(n\times n)$ matrix $a_0(t, x, \xi)$ which is positive homogeneous of degree zero in ξ, is uniformly diagonalizable. This means that there exists a $(n\times n)$ matrix

$p(t, x, \xi)\in\{S_1^0\}$, which is positive homogeneous of degree zero in ξ, uniformly nonsingular and consisting of the left eigenvectors of $a_0(t, x, \xi)$, i. e.

$$|\det p(t, x, \xi)|\geqslant \text{const } \gamma>0, \ \forall \text{ all } (t, x, \xi),$$

and

$$p(t, x, \xi)a_0(t, x, \xi)=J(t, x, \xi)p(t, x, \xi),$$

where $J(t, x, \xi)\in\{DS_1^0\}$ is a diagonal matrix consisting of the eigenvalues $\lambda_j(t, x, \xi)$ of $a_0(t, x, \xi)$.

(3) Indices $(j=1, 2, \cdots, n)$ can be divided into two groups $\{I_1\}$ and $\{I_2\}$ such that

$$\operatorname{Re}\lambda_j(t, x, \xi)\equiv 0, \ \forall j\in\{I_1\}$$

and

$$|\operatorname{Re}\lambda_j(t, x, \xi)|\geqslant \text{const } \gamma_1>0, \ \forall j\in\{I_2\} \text{ and all} (t, x, \xi).$$

(4) In the interval $(0,1)$, $N'(t)=\lim\limits_{\Delta t\to 0}\dfrac{N(t+\Delta t)-N(t)}{\Delta t}$ exists and belongs to $\mathscr{L}(H\to H)$.

We make some remarks about this hypothesis. Condition (1) does not restrict the generality. If $A(t)$ has symbol $a(t, x, \xi)\in\{S_1\}$, we can take the principal part of $A(t)$ with the symbol $a_0(t, x, \xi)|\xi|\in\{S_1^0\}$ as the new $A(t)$, and the rest of $A(t)$ can be included in the new $N(t)$. Condition (2) means that there exist n simple characteristic combinations $v(t)=P(t)u(t)$, where $P(t)$ is a p. d. o. of order zero with the symbol $p(t, x, \xi)$. In the appendix of Part II we shall indicate that if the eigenvalues of $a_0(t, x, \xi)$ are distinct and $J(t, x, \xi)\in\{DS_1^0\}$, then such $p(t, x, \xi)\in\{S_1^0\}$ exists. Condition (3) means that for the equation (3.1) there exists no characteristic combination having mixed type character. Some of these characteristic combinations, corresponding to $\{I_1\}$, have hyperbolic character, the others, corresponding to $\{I_2\}$, have nondegenerate elliptic character.

Theorem 3.1. *Suppose that hypothesis I holds, then*

$$\|u(t)\|\leqslant M_1^{(1)}\max_{i=1,2}\{\|u(0)\|^{1-\delta_i(t)}\|u(1)\|^{\delta_i(t)}\}, \ \forall u(t)\in\{V_1\}. \tag{3.2}$$

Moreover, if $u(1)\in H_1$, *then*

$$\|u(t)\|\leqslant M_2^{(1)}\|u(0)\|\exp\{M_3^{(1)}[\|u(1)\|_1/\|u(1)\|]t\}. \tag{3.3}$$

Here $\delta_i(t)$ *are the functions defined in lemma* 2.1, *the constants* $M_1^{(1)}$, $M_2^{(1)}$, $M_3^{(1)}$ *and* M *in the expression of* $\delta_i(t)$ *are independent of* $u(t)$.

Proof. We first diagonalize equation (3.1) and then put it in the case discussed in lemma 2.2.

Since the condition I (2) is satisfied, there exists $p^{-1}(t, x, \xi)\in\{S_1^0\}$. Let $P(t)$, $P^{(-1)}(t)$, $E(t)$, $E_R(t)$ and $E_I(t)$ be the p. d. o. with the symbols $p(t, x, \xi)$, $p^{-1}(t, x, \xi)$, $J(t, x, \xi)|\xi|$, $\operatorname{Re}J(t, x, \xi)|\xi|$ and $\sqrt{-1}\operatorname{Im}J(t, x, \xi)|\xi|$ respectively. Then $P(t)$ and $P^{(-1)}(t)$ belong to $\mathscr{L}^0_{s1}$, $E(t)$, $E_R(t)$ and $E_I(t)$ belong to $\mathscr{L}^1_{s1}$. Their derivatives $P'(t)$ and $E'_R(t)$, $E'_I(t)$ with the symbols $\dfrac{\partial}{\partial t}p(t, x, \xi)$ and $\operatorname{Re}\dfrac{\partial}{\partial t}J(t, x, \xi)|\xi|$, $\sqrt{-1}\operatorname{Im}\dfrac{\partial}{\partial t}J(t, x, \xi)|\xi|$ belong to $\mathscr{L}^0_{s0}$ and $\mathscr{L}^1_{s0}$ respectively. According to lemma 2.4, there exist for sufficiently large K the operators $D_{(K)}(t)\in\mathscr{L}(H\to H)$ and $L_{(-1)}(t)\in\mathscr{L}^{-1}$ such that

$$D_{(K)}(t)P^{(-1)}(t)P(t)u=u+D_{(K)}(t)L_{(-1)}(t)T(K)u, \ \forall u\in H, \tag{3.4}$$

where $D_{(K)}(t)P^{(-1)}(t)$ is the quasi-inverse of $P(t)$ in H.

Now let $u(t)\in\{V_1\}$ and set

$$P(t)u(t)=v(t),\ T(K)u(t)=w(t). \tag{3.5}$$

Then

$$\begin{aligned}\frac{dv(t)}{dt}&=P(t)A(t)u(t)+[P(t)N(t)+P'(t)]u(t)\\&=E(t)P(t)u(t)+\{[P(t)A(t)-E(t)P(t)]\\&\quad+[P(t)N(t)+P'(t)]\}[D_{(K)}(t)P^{(-1)}(t)P(t)u(t)\\&\quad-D_{(K)}(t)L_{(-1)}(t)T(K)u(t)],\\\frac{dw(t)}{dt}&=T(K)\frac{du(t)}{dt}=T(K)[A(t)+N(t)]\\&\quad\cdot[D_{(K)}(t)P^{(-1)}(t)P(t)u(t)-D_{(K)}(t)L_{(-1)}(t)T(K)u(t)].\end{aligned}$$

Therefore

$$\frac{dv(t)}{dt}=(E_R(t)+E_I(t))v(t)+N_{1,1}(t)v(t)+N_{1,2}(t)w(t), \tag{3.6}$$

$$\frac{dw(t)}{dt}=N_{2,1}(t)v(t)+N_{2,2}(t)w(t), \tag{3.7}$$

where

$$\begin{aligned}N_{1,1}(t)&=\{[P(t)A(t)-E(t)P(t)]+[P(t)N(t)\\&\quad+P'(t)]\}D_{(K)}(t)P^{(-1)}(t),\\N_{1,2}(t)&=-\{\cdots\cdots\}D_{(K)}(t)L_{(-1)}(t),\\N_{2,1}(t)&=T(K)[A(t)+N(t)]D_{(K)}(t)P^{(-1)}(t),\\N_{2,2}(t)&=-T(K)[A(t)+N(t)]D_{(K)}(t)L_{(-1)}(t).\end{aligned}$$

It follows from lemma 2.3 that $N_{2,i}(t)\in\mathscr{L}(H\to H)$. Taking $\rho=1$ and applying theorem 1.3 we obtain that $E(t)P(t)-R(t)\in\mathscr{L}^0$ and $P(t)A(t)-R(t)\in\mathscr{L}^0$, where $R(t)$ is a p. d. o. with the symbol $J(t,x,\xi)|\xi|p(t,x,\xi)=p(t,x,\xi)a_0(t,x,\xi)|\xi|$. Hence $P(t)A(t)-E(t)P(t)\in\mathscr{L}^0\subset\mathscr{L}(H\to H)$. Since $P(t)$, $P^{(-1)}(t)$, $P'(t)$ and $N(t)$ all belong to $\mathscr{L}^0$, $L_{(-1)}(t)\in\mathscr{L}^{-1}$, $D_{(K)}(t)\in\mathscr{L}(H\to H)$, it is easily verified that $N_{1,i}(t)\in\mathscr{L}(H\to H)$.

The system (3.6), (3.7) is diagonalized. Now we put it in the case of lemma 2.2.

Set

$$y(t)=\begin{pmatrix}v(t)\\w(t)\end{pmatrix},\quad \mathscr{B}(t)=\begin{pmatrix}E_R(t)&\theta_{n\times n}\\\theta_{n\times n}&\theta_{n\times n}\end{pmatrix}.$$

In this case $y(t)=y(t,x)$ is a $2n$-dimensional vector,

$$\mathscr{H}=H\times H,\quad D(\mathscr{B})=\mathscr{H}_1=H_1\times H_1.$$

We have to verify the conditions (2.3) and (2.4). Let $e_0(t,x,\xi)$ be a $(n\times n)$ diagonal matrix with the elements on the diagonal

$$e_0^{(j,j)}(t,x,\xi)=\begin{cases}0, & \text{for } j\in\{I_1\},\\1/\mathrm{Re}\,\lambda_j(t,x,\xi), & \text{for } j\in\{I_2\}.\end{cases}$$

Obviously, $e_0(t,x,\xi)\in\{DS_1^0\}$ is positive homogeneous of degree zero in ξ. Set

$$e_L(t,x,\xi)=\frac{e_0(t,x,\xi)}{|\xi|}\sum_{|\alpha|=1}(-1)D^\alpha\partial^\alpha(\mathrm{Re}\,J(t,x,\xi)|\xi|)\in\{DS_1\},$$

$$e_C(t, x, \xi) = \frac{e_0(t, x, \xi)}{|\xi|} \sum_{|\alpha|=1} (-\sqrt{-1}) [D^\alpha(\operatorname{Re} J(t, x, \xi)|\xi|)\partial^\alpha(\operatorname{Im} J(t, x, \xi)|\xi|)$$
$$- D^\alpha(\operatorname{Im} J(t, x, \xi)|\xi|)\partial^\alpha(\operatorname{Re} J(t, x, \xi)|\xi|)] \in \{DS_1\},$$
$$e_P(t, x, \xi) = \frac{e_0(t, x, \xi)}{|\xi|} \operatorname{Re} \frac{\partial}{\partial t}(J(t, x, \xi)|\xi|) \in \{DS_1\}.$$

It is easily verified that $e_L(t, x, \xi)$ is positive homogeneous of degree (-1) in ξ, $e_C(t, x, \xi)$ and $e_P(t, x, \xi)$ are positive homogeneous of degree zero in ξ. Therefore $E_L(t) \in \mathscr{L}^{-1}_{DS_1}$, $E_C(t) \in \mathscr{L}^0_{DS_1}$ and $E_D(t) \in \mathscr{L}^0_{DS_1}$, where $E_L(t)$, $E_C(t)$ and $E_D(t)$ are the p. d. o. with the symbols $e_L(t, x, \xi)$, $e_C(t, x, \xi)$ and $e_P(t, x, \xi)$ respectively.

Let

$$N_I(t) = E_I^{(*)}(t) + E_I(t), \tag{3.9}$$
$$N_D(t) = E'_R(t) - E_D(t)E_R(t), \tag{3.10}$$
$$N_C(t) = E_R^{(*)}(t)E_I(t) + E_I^{(*)}(t)E_R(t) - E_C(t)E_R(t), \tag{3.11}$$
$$N_{(-1)}(t) = E_R^{(*)}(t) - (E_R(t) + E_L(t)E_R(t)). \tag{3.12}$$

Taking $\rho=1$ and applying theorem 1.2 for $E_R(t)$ we obtain

$$E_R^{(*)}(t) - E_{R\rho}^{(*)}(t) \in \mathscr{L}^{1-1-1} = \mathscr{L}^{-1}.$$

Taking $\rho=1$ and applying theorem 1.3 for $E_L(t)$ and $E_R'(t)$, we obtain that the difference between $E_L(t)E_R(t)$ and the p. d. o. with the symbol $\sum_{|\alpha|=1}(-1)D^\alpha\partial^\alpha(\operatorname{Re} J(t, x, \xi)|\xi|)$ belongs to $\mathscr{L}^{-1+1-1} = \mathscr{L}^{-1}$. And $E_{R\rho}^{(*)}(t) - E_R(t)$ just has the symbol $\sum_{|\alpha|=1}(-1)D^\alpha\partial^\alpha(\operatorname{Re} J(t, x, \xi)|\xi|)$. Finally we have $N_{(-1)}(t) = (E_R^{(*)}(t) - E_{R\rho}^{(*)}(t)] + \{[E_{R\rho}^{(*)}(t) - E_R(t)] - E_L(t)E_R(t)\} \in \mathscr{L}^{-1}$. Analogously, applying theorems 1.2 and 1.3 we can verify that $N_I(t)$, $N_D(t)$ and $N_C(t)$ belong to $\mathscr{L}^0 \subset \mathscr{L}(H \to H)$.

Now we estimate

$$2\operatorname{Re}\left(y(t), \frac{dy(t)}{dt} - \mathscr{B}(t)y(t)\right)_{\mathscr{H}} = 2\operatorname{Re}\{(v(t), E_I(t)v(t))$$
$$+ (v(t), N_{1,1}(t)v(t) + N_{1,2}(t)w(t)) + (w(t), N_{2,1}(t)v(t) + N_{2,2}(t)w(t))\}$$
$$= 2\operatorname{Re}\{(v(t), \frac{1}{2}N_I(t)v(t)) + (v(t), N_{1,1}(t)v(t) + N_{1,2}(t)w(t))$$
$$+ (w(t), N_{2,1}(t)v(t) + N_{2,2}(t)w(t))\} = \widetilde{M}_4(t)\|y(t)\|^2_{\mathscr{H}}.$$

Since $N_{ij}(t) \in \mathscr{L}(H \to H)$ and $N_I(t) \in \mathscr{L}(H \to H)$, we have the estimate $|\widetilde{M}_4(t)| \leqslant \text{const}\, M_4$, where the constant M_4 depends only on $A(t)$ and $N(t)$.

In addition, from (3.10), (3.12) and (3.11) we have

$$\frac{d}{dt}\operatorname{Re}(y(t), \mathscr{B}(t)y(t))_{\mathscr{H}} = \frac{d}{dt}\operatorname{Re}(v(t), E_R(t)v(t))$$
$$= \operatorname{Re}\left\{(v(t), E'_R(t)v(t)) + \left(\frac{dv(t)}{dt}, E_R(t)v(t)\right) + \left(v(t), E_R(t)\frac{dv(t)}{dt}\right)\right\}$$
$$= \operatorname{Re}\{(v(t), (E_D(t)E_R(t) + N_D(t))v(t)) + ((E_R(t) + E_I(t) + N_{1,1}(t))v(t)$$
$$+ N_{1,2}(t)w(t), E_R(t)v(t)) + (E_R^{(*)}(t)v(t), (E_R(t)$$
$$+ E_I(t) + N_{1,1}(t))v(t) + N_{1,2}(t)w(t))\}$$
$$= \operatorname{Re}\{[(E_D^*v, E_Rv) + (v, N_Dv)] + [(E_Rv, E_Rv) + (E_Iv, E_Rv)$$
$$+ (N_{1,1}v + N_{1,2}w, E_Rv)] + ((E_R + E_LE_R + N_{(-1)})v, (E_R + E_I + N_{1,1})v + N_{1,2}w)\}$$

$$
\begin{aligned}
&=\mathrm{Re}\{2(E_R v,\ E_R v)+((E_R^{(*)}E_I+E_I^{(*)}E_R)v,\ v)\\
&\quad+2(N_{1,1}v+N_{1,2}w,\ E_R v)+(E_L E_R v,\ (E_R+E_I)v)\\
&\quad+(E_D^* v,\ E_R v)+(E_L E_R v,\ N_{1,1}v+N_{1,2}w)\\
&\quad+(N_{(-1)}v,\ (E_R+E_I+N_{1,1})v+N_{1,2}w)+(v,\ N_D v)\}\\
&=\mathrm{Re}\{2(E_R v,\ E_R v)+((E_C E_R+N_C)v,\ v)\\
&\quad+2(N_{1,1}v+N_{1,2}w,\ E_R v)+(E_R v,\ E_L^{(*)}(E_R+E_I)v)\\
&\quad+(E_D^* v,\ E_R v)+((E_R^{(*)}+E_I^{(*)})N_{(-1)}v,\ v)\\
&\quad+(N_{(-1)}v,\ N_{1,1}v+N_{1,2}w)+(E_L E_R v,\ N_{1,1}v+N_{1,2}w)+(v,\ N_D v)\}\\
&=\mathrm{Re}\{2(E_R v,\ E_R v)+(E_R v,\ (E_C^*+E_L^{(*)}(E_R+E_I)+E_D^*)v+2(N_{1,1}v+N_{1,2}w))\\
&\quad+[(N_C+(E_R^{(*)}+E_I^*)N_{(-1)}+N_D)v,\ v)\\
&\quad+((N_{(-1)}+E_L E_R)v,\ N_{1,1}v+N_{1,2}w)]\}\\
&=2\|E_R v\|^2+\widetilde{M}_5(t)\,\|E_R v\|\,(\|v\|^2+\|w\|^2)^{1/2}+\widetilde{M}_6(t)\,(\|v\|^2+\|w\|^2)\\
&=2\|\mathscr{B}(t)y(t)\|_{\mathscr{H}}^2+\widetilde{M}_5(t)\,\|\mathscr{B}(t)y(t)\|_{\mathscr{H}}\|y(t)\|_{\mathscr{H}}+\widetilde{M}_6(t)\,\|y(t)\|_{\mathscr{H}}^2.
\end{aligned}
$$

Since E_R and E_I (also $E_R^{(*)}$ and $E_I^{(*)}$) belong to $\mathscr{L}^1$, E_C, E_D (also E_C^* and E_D^*), N_C and N_D belong to $\mathscr{L}^0$, $N_{ij}(t)\in\mathscr{L}(H\to H)$, E_L (also $E_L^{(*)}$) and $N_{(-1)}$ belong to $\mathscr{L}^{-1}$, we have that $E_L^{(*)}(E_R+E_I)$, $E_C^*+E_L^{(*)}(E_R+E_I)+E_D^*$, $(E_R^{(*)}+E_I^{(*)})N_{(-1)}$ and $E_L E_R$ belong to $\mathscr{L}^0\subset\mathscr{L}(H\to H)$. Consequently $N_C+(E_R^{(*)}+E_I^{(*)})\,N_{(-1)}+N_D$ and $E_L E_R+N_{(-1)}$ belong to $\mathscr{L}^0\subset\mathscr{L}(H\to H)$. Finally we have the estimates

$$|\widetilde{M}_5(t)|\leqslant M_5,\quad |\widetilde{M}_6(t)|\leqslant M_6,$$

where M_5 and M_6 are constants dependent only on $A(t)$ and $N(t)$. Thus we proved that (2.3) and (2.4) hold. Then from lemma 2.2 it follows that

$$
\begin{aligned}
\|y(t)\|_{\mathscr{H}}&=(\|v(t)\|^2+\|w(t)\|^2)^{\frac{1}{2}}\leqslant M_7\max_{i=1,2}\{\|y(0)\|_{\mathscr{H}}^{1-\delta_i(t)}\|y(1)\|_{\mathscr{H}}^{\delta_i(t)}\}\\
&=M_7\max_{i=1,2}\{[(\|v(0)\|^2+\|w(0)\|^2)^{\frac{1}{2}}]^{1-\delta_i(t)}\\
&\quad\cdot[(\|v(1)\|^2+\|w(1)\|^2)^{\frac{1}{2}}]^{\delta_i(t)}\}
\end{aligned}
\tag{3.13}
$$

and

$$
\begin{aligned}
\|y(t)\|_{\mathscr{H}}&=(\|v(t)\|^2+\|w(t)\|^2)^{\frac{1}{2}}\leqslant M_8\|y(0)\|_{\mathscr{H}}\exp\{M_9 t\|\mathscr{B}(1)y(1)\|_{\mathscr{H}}/\|y(1)\|_{\mathscr{H}}\}\\
&=M_8(\|v(0)\|^2+\|w(0)\|^2)^{\frac{1}{2}}\exp\{M_9 t\|E_R(1)v(1)\|/(\|v(1)\|^2+\|w(1)\|^2)^{\frac{1}{2}}\}.
\end{aligned}
\tag{3.14}
$$

Moreover, from (3.4) and (3.5) we have

$$
\begin{aligned}
\|u(t)\|&\leqslant\|D_{(K)}(t)P^{(-1)}(t)v(t)\|+\|D_{(K)}(t)L_{(-1)}(t)w(t)\|\leqslant M_{10}(\|v(t)\|^2+\|w(t)\|^2)^{\frac{1}{2}}\\
&=M_{10}(\|P(t)u(t)\|^2+\|T(K)u(t)\|^2)^{\frac{1}{2}}\leqslant M_{11}\|u(t)\|.
\end{aligned}
\tag{3.15}
$$

And
$$\|E_R(1)v(1)\|=\|E_R(1)P(1)u(1)\|\leqslant M_{12}\|u(1)\|_1. \tag{3.16}$$

From (3.13), (3.14), (3.15) and (3.16) we obtain the desired estimates (3.2) and (3.3) with the constants $M_1^{(1)}=M_7M_{10}M_{11}$, $M_2^{(1)}=M_8M_{10}M_{11}$ and $M_3^{(1)}=M_9M_{10}M_{12}$.

These constants and constant M in the expression of $\delta_i(t)$ depend only on $A(t)$ (consequently its symbol $a_0(t,\ x,\ \xi)$) and $N(t)$, but not on $u(t)$.

The theorem is thus proved.

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