THE ERROR BOUND OF THE FINITE ELEMENT METHOD FOR A TWO-DIMENSIONAL SINGULAR BOUNDARY VALUE PROBLEM*

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1. Introduction

The finite element method for one-dimensional singular boundary value problems have been studied by several authors (for instance, see [4], [10], [8], [11]). The finite element method for a two-dimensional singular boundary value problem is proposed in [12]. Recently [9], [16], [1], [15] and [3] have given the relevant theoretical studies. In [9], the error of order $O(h^k)$ has been proved for the Lagrange elements of degree k provided that the solution of the boundary value problem is in $C^{k+1}(\bar{\Omega})$. [16] has proved the convergence of the linear finite element method provided only that the solution of the boundary value problem belongs to a weighted Sobolev space. For problem (1.1) in the present paper, [1] has proved that the error is of order $O(h)$ for a variant linear element including a logarithmic term. For the ordinary linear element, [15] and [3] have also obtained the error of order $O(h)$. In this paper we extend the result of [15] and [3] to the elements of high degree.

We consider the following model problem:

$$\begin{cases} \Omega: & Lu \equiv -\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\beta_1\frac{\partial u}{\partial r}\right)+\frac{\partial}{\partial z}\left(\beta_2\frac{\partial u}{\partial z}\right)\right]=f, \\ \Gamma_1: & u=0, \end{cases} \tag{1.1}$$

where Ω is a bounded open domain with $r>0$ in (r, z)-plane, $\Gamma_1=\partial\Omega\backslash\Gamma_0$, $\Gamma_0=\partial\Omega\cap\{(r, z): r=0\}$.

In order to formulate the weak from of problem (1.1) we introduce some weighted Sobolev spaces. The similar spaces have been studied in [2], [5], [13] and [14].

2. Weighted Sobolev Spaces V_1^m

Define
$$V^0(\Omega)=\{v: v \text{ is measurable in } \Omega,\ \|v\|_{V^0(\Omega)}<\infty\},$$
$$V_1^m(\Omega)=\{v\in V^0(\Omega): \|v\|_{V_1^m(\Omega)}<\infty\},\ m=1, 2, \cdots,$$

where
$$\|v\|_{V^0(\Omega)}=\left(\int_\Omega v^2 r\,dr\,dz\right)^{1/2},$$
$$\|v\|_{V_1^1(\Omega)}=\left(\sum_{|\alpha|\le 1}\|\partial^\alpha v\|^2_{V^0(\Omega)}\right)^{1/2},$$

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$$\|v\|_{V_1^m(\Omega)}=\Big(\sum_{|\alpha|<m}\|\partial^\alpha v\|^2_{V^0(\Omega)}+\sum_{j=1}^{m-1}\Big\|r^{j-m}\frac{\partial^j v}{\partial r^j}\Big\|^2_{V^0(\Omega)}\Big)^{1/2},\quad m=2,\ 3,\ \cdots.$$

Sometimes we use V^0, V_1^m instead of $V^0(\Omega)$, $V_1^m(\Omega)$.

Using the arguments similar to those in [13], [14] and [5] we can prove the following propositions.

Proposition 2.1. The spaces V^0, V_1^m are Banach spaces.

Proposition 2.2. If Ω has a locally Lipschitz boundary then $C^\infty(\overline{\Omega})$ is dense in $V_1^m(\Omega)$.

Now we may as usual define the trace on the boundary of Ω for the elements of $V_1^m(\Omega)$. Then we may introduce the following spaces corresponding to problem (1.1):

$$V_{1,0}^1(\Omega)=\{v\in V_1^1(\Omega):\ v=0 \text{ on } \Gamma_1\}.$$

From now on we assume that Ω has a locally Lipschitz boundary, that $f\in V^0(\Omega)$, and that β_1, β_2 are bounded, measurable in Ω and there exists a positive constant β_0 such that $\beta_1\geqslant\beta_0$, $\beta_2\geqslant\beta_0$.

Lemma 2.3. (Ref. [6]) *There exists a constant $C>0$ such that*

$$\int_\Omega\Big[\Big(\frac{\partial v}{\partial r}\Big)^2+\Big(\frac{\partial v}{\partial z}\Big)^2\Big]r\,dr\,dz\geqslant C\|v\|^2_{V_1^1(\Omega)},\quad \forall v\in V_{1,0}^1(\Omega).$$

The proof of the following lemma is similar to that of theorem 2.2 in [5].

Lemma 2.4. *If $v\in V_1^m$, $m\geqslant 2$, then*

$$\frac{\partial^j v}{\partial r^j}=0 \ \text{on}\ \Gamma_0,\qquad j=1,\ 2,\ \cdots,\ m-1.$$

It is easy to prove that $V_1^2(\Omega)\subset C^0(\overline{\Omega})$. (Ref. [15]).

3. The Weak Form of the Problem and the Discrete Problem

Define the bilinear form $B_1(u,\ v)$ and the linear functional $F(v)$ as follows:

$$B_1(u,\ v)=\int_\Omega\Big(\beta_1\frac{\partial u}{\partial r}\ \frac{\partial v}{\partial r}+\beta_2\frac{\partial u}{\partial z}\ \frac{\partial v}{\partial z}\Big)r\,dr\,dz,\qquad \forall u,\ v\in V_1^1(\Omega),$$

$$F(v)=\int_\Omega fvr\,dr\,dz,\quad \forall v\in V_1^1(\Omega).$$

The weak form of problem (1.1) is

Problem (3.1). Find $u\in V_{1,0}^1(\Omega)$ such that

$$B_1(u,\ v)=F(v),\ \forall v\in V_{1,0}^1(\Omega).$$

By lemma 2.3 we know that $B_1(u,\ v)$ is coercive on $V_{1,0}^1(\Omega)\times V_{1,0}^1(\Omega)$. So we may easily prove the following theorem using the Lax-Milgram theorem.

Theorem 3.1. *Problem (3.1) has a unique solution.*

From now on we assume that Ω is a polygon.

Let $T_h=\{C_1,\ \cdots,\ C_n\}$ be a normal triangulation of Ω (Ref. [6]). Denote by h_i and θ_i respectively the size of the maximal edge and the minimal inner angle of C_i. Let $h=\max h_i$, $\theta=\min\theta_i$. Define the finite element spaces $V_1^{m,h}$ of degree m as follows:

$$V_1^{1,h}=\{v_h\in C^0(\overline{\Omega}):\ v_h \text{ is a linear function on } C_i, i=1,\ \cdots,\ n;\ v_h=0 \text{ on } \Gamma_1\},$$

$$V_1^{m,h}=\{v_h\in C^{m-1}(\overline{\Omega}):\ v_h \text{ is a polynomial of degree } m \text{ on } C_i,$$
$$i=1,\ \cdots,\ n;\ v_h=0 \text{ on } \Gamma_1\},\ m=2,\ 3,\ \cdots.$$

The discrete problem corresponding problem (3.1) and the finite element spaces of degree m is

Problem (3.1'). Find $u_h \in V_1^{m,h}$ such that

$$B_1(u_h, v_h) = F(v_h), \quad \forall v_h \in V_1^{m,h}.$$

Similarly to theorem 3.1 we have

Theorem 3.2. *Problem (3.1') has a unique solution.*

4. The Error Bound of the Finite Element Solutions

Assume for the triangulations T_h that

$$\theta \geq \theta_0 > 0, \quad \theta_0 \text{ independent of } h. \tag{4.1}$$

Given any triangle $C \in T_h$. Denote still by h its maximal edge. Given $m_0 = (m+1)\cdot(m+2)/2$ nodes P_k $(k=1, \cdots, m_0)$ as usual. Denote by φ_k the basis functions for Lagrange interpolation corresponding the nodes P_k, $k=1, \cdots, m_0$.

Lemma 4.1. (Ref. [7]) *There exists a constant C_0 such that*

$$|\partial^\alpha \varphi_j| \leq C_0 h^{-|\alpha|} \quad if\, |\alpha| \leq m. \tag{4.2}$$

By simple calculation we obtain the following lemma.

Lemma 4.2. *Assume that α_1 and α_2 are non-negative integers with $\alpha_1+\alpha_2 \leq m$, that (r_j, z_j) are the coordinates of the nodes P_j. Then*

$$\sum_{j=1}^{m_0} \varphi_j(r, z)(r_j - r)^{\alpha_1}(z_j - z)^{\alpha_2} = 0.$$

The following lemma is an extension of a result in [3] (see also [15]).

Lemma 4.3. *Suppose that $v \in V_1^{m+1}(C)$, $m \geq 1$, and v_I is the Lagrange interpolation of degree m for v on C corresponding to the nodes P_k, $k=1, \cdots, m_0$. Then*

$$\|v - v_I\|_{V_1^i(C)} \leq M h^{m+1-i} |v|_{V_1^{m+1}(C)}, \quad i=0, 1, \tag{4.3}$$

where M independent of C, v, $V_1^0(C) = V^0(C)$, and

$$|v|_{V_1^{m+1}(C)} = \Big(\sum_{|\alpha|=m+1} \|\partial^\alpha v\|^2_{V^0(C)}\Big)^{1/2}.$$

Proof. It is sufficient to prove the conclusion for $v \in C^\infty(C)$. Given any point $P \in C$. Using the Taylor's formula with integral remainder we have

$$v(P_j) - v(P) = d_j v(P) + \cdots + \frac{1}{m!} d_j^m v(P) + \frac{1}{m!}\int_0^1 (1-t)^m d_j^{m+1} v(M_j)\, dt,$$
$$j=1, \cdots, m_0. \tag{4.4}$$

where
$$d_j = (r_j - r)\frac{\partial}{\partial r} + (z_j - z)\frac{\partial}{\partial z},$$
$$d_j^n = d_j \cdot d_j^{n-1}, \quad n=2, 3, \cdots,$$
$$M_j = P_j t + P(1-t).$$

It follows from the properties of the basis functions, lemma 4.2 and (4.4) that

$$v_I(P) - v(P) = \sum_{j=1}^{m_0} \varphi_j(P)[v(P_j) - v(P)]$$
$$= \frac{1}{m!} \sum_j \int_0^1 (1-t)^m \varphi_j(P) d_j^{m+1} v(M_j)\, dt. \tag{4.5}$$

Differentiating (4.5) we obtain

$$\frac{\partial v_I}{\partial r}-\frac{\partial v}{\partial r}=\frac{1}{m!}\sum_j\int_0^1(1-t)^m\left[\frac{\partial\varphi_j}{\partial r}d_j^{m+1}-(m+1)\varphi_j d_j^m\cdot\frac{\partial}{\partial r}\right]v(M_j)dt$$
$$+\frac{1}{m!}\sum_j\int_0^1(1-t)^m\varphi_j d_j^{m+1}\left[\frac{\partial v(M_j)}{\partial r}(1-t)\right]dt.$$

Integrating by parts the integrals in the second sum, noting lemma 4.2 and that

$$\frac{d}{dt}[d_j^m v(M_j)]=d_j^{m+1}v(M_j),$$

we have
$$\frac{\partial v_I}{\partial r}-\frac{\partial v}{\partial r}=\frac{1}{m!}\sum_j\int_0^1(1-t)^m\frac{\partial\varphi_j}{\partial r}d_j^{m+1}v(M_j)dt.$$

Then it follows from lemma 4.1 that

$$\left|\frac{\partial v_I}{\partial r}-\frac{\partial v}{\partial r}\right|\leqslant M_0^*h^{-1}\sum_j\int_0^1(1-t)^m|d_j^{m+1}v(M_j)|dt$$

and it is easy to see that

$$\int_C\left|\frac{\partial v_I}{\partial r}-\frac{\partial v}{\partial r}\right|^2 r\,dr\,dz\leqslant M_1^*h^{-2}\sum_j\int_C\left(\int_0^1(1-t)^m|d_j^{m+1}v(M_j)|dt\right)^2 r\,dr\,dz$$
$$=M_1^*h^{-2}\sum_j\int_C\left(\int_0^1(1-t)^{-1/4}(1-t)^{m+1/4}|d_j^{m+1}v(M_j)|dt\right)^2 r\,dr\,dz$$
$$\leqslant M_2^*h^{-2}\sum_j\int_0^1(1-t)^{2m+1/2}dt\int_C|d_j^{m+1}v(M_j)|^2 r\,dr\,dz$$
$$=M_2^*h^{-2}\sum_j\int_0^1(1-t)^{2m+1/2}dt\int_C\left|\left[(r_j-r)\frac{\partial}{\partial r}\right.\right.$$
$$\left.\left.+(z_j-z)\frac{\partial}{\partial z}\right]^{m+1}v(M_j)\right|^2 r\,dr\,dz, \tag{4.6}$$

where M_0^*, M_1^* and M_2^* are some constants. Carry out variable transformation in the last integrals in (4.6) as follows:

$$\xi=r_jt+r(1-t),\quad \eta=z_jt+z(1-t).$$

Then $M_j=(\xi,\eta)$, and the triangle C reduces to a similar triangle $C_{j,t}$ with the similarity transformation center P_j. Obviously $C_{j,t}\subset C$. Hence the right side of (4.6) becomes

$$M_2^*h^{-2}\sum_j\int_0^1(1-t)^{2m+\frac{1}{2}}dt\int_{C_{j,t}}\left|\left[(r_j-\xi)\frac{\partial}{\partial\xi}+(z_j-\eta)\frac{\partial}{\partial\eta}\right]^{m+1}v(M_j)\right|^2\frac{(\xi-r_jt)}{(1-t)^3}d\xi\,d\eta$$
$$=M_2^*h^{-2}\sum_j\int_0^1(1-t)^{2m-\frac{5}{2}}dt\int_{C_{j,t}}(\xi-r_jt)\left|\left[(r_j-\xi)\frac{\partial}{\partial\xi}+(z_j-\eta)\frac{\partial}{\partial\eta}\right]^{m+1}v(M_j)\right|^2 d\xi\,d\eta$$
$$\leqslant M_3^*h^{2m}\sum_j\int_0^1(1-t)^{4m-\frac{1}{2}}dt\int_{C_{j,t}}\sum_{|\alpha|=m+1}|\partial^\alpha v|^2\xi\,d\xi\,d\eta$$
$$\leqslant M_4^*h^{2m}\int_C\sum_{|\alpha|=m+1}|\partial^\alpha v|^2\xi\,d\xi\,d\eta$$
$$=M_4^*h^{2m}|v|^2_{V_1^{m+1}(C)}.$$

It is proved that

$$\int_C\left|\frac{\partial v_I}{\partial r}-\frac{\partial v}{\partial r}\right|^2 r\,dr\,dz\leqslant M_4^*h^{2m}|v|^2_{V_1^{m+1}(C)}.$$

Similarly we have

$$\int_C \left| \frac{\partial v_I}{\partial z} - \frac{\partial v}{\partial z} \right|^2 r\,dr\,dz \leqslant M_5^* h^{2m} |v|^2_{V_1^{m+1}(C)},$$

and starting with (4.5) we derive

$$\int_C |v_I - v|^2 r dr dz \leqslant M_6^* h^{2m+2} |v|_{V_1^{m+1}(C)}.$$

Now (4.3) has been proved.

According to this lemma it is easy to prove the main result of this paper by using a well-known argument.

Theorem 4.4. Assume that u and u_h are respectively the solution of problem (3.1) and (3.1′), $u \in V_1^{m+1}(\Omega)$. Then

$$\|u - u_h\|_{V_1^1} = O(h^m).$$

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