THE CONVERGENCE OF INFINITE ELEMENT METHOD FOR THE NON-SIMILAR CASE*

YING LUNG-AN(应隆安)
(Department of Mathematics, Peking University)

We have considered the infinite element method for a class of elliptic systems with constant coefficients in [1]. This class can be characterized as: they have the invariance under similarity transformations of independent variables. For example, the Laplace equation and the system of plane elastic equations have this property. We have suggested a technique to solve these problems by applying this property and a self similar discretization, and proved the convergence. Not only the average convergence of the solutions has been discussed, but also the term-by-term convergence for the expansions of the solutions. The second convergence manifests the advantage of the infinite element method, that is, the local singularity of the solutions can be calculated with high precision.

We have generalized this method to the non-similar case in [2], and obtained many results parallel to that of the similar case, which include the calculation of the combined stiffness matrices and the discussion of the singularity of the solutions. For conciseness, the Helmholtz equation

$$-\Delta u+\lambda u=0 \tag{1}$$

and the linear triangular elements will be considered in this paper, but this method is good for more general equations.

We will prove the average convergence which shows the order of the convergence of the infinite element method is higher than that of the finite element method if the solutions possess singularities. At the same time, we will concentrate upon the proof of the convergence for singular components, which does not exist for the finite element method.

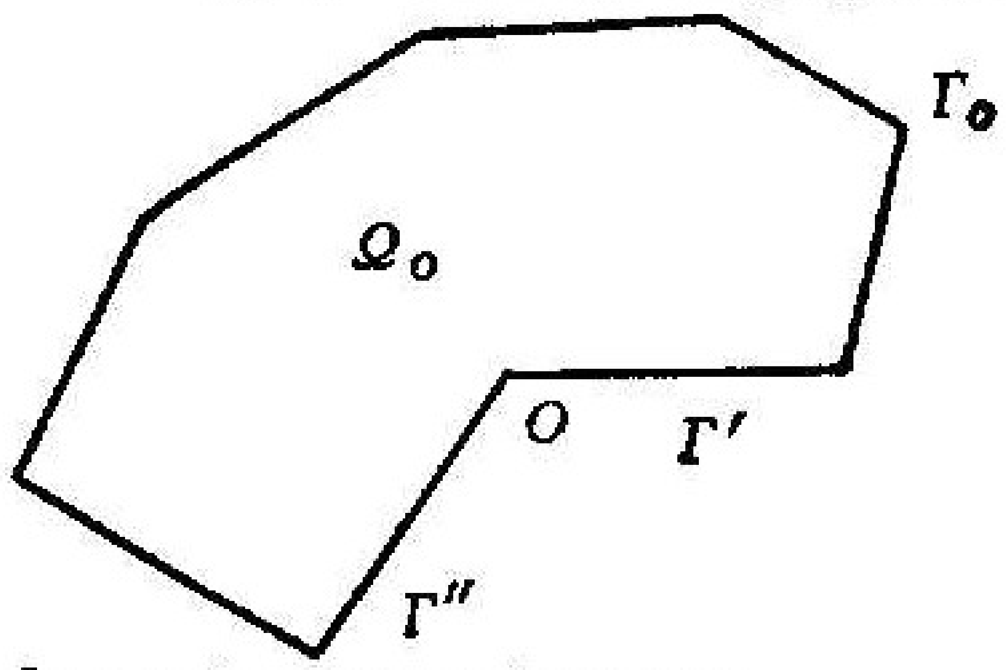
§1. Some General Statement

For conciseness we assume that the considered region D is a bounded polygon region on the x, y plane, one of the vertices of which is the origin, the inner angle of which is $\theta_0>\pi$. If we consider the boundary value problem of equation (1) on this region, the solution will, generally, possess singularity at point O. Now we devote ourselves to the calculation of this singularity. It is no harm to assume that one of the neighboring sides of point O is on the positive x-axis and the interior angle is $0<\theta<\theta_0$.

Let the neighboring sides of point O be Γ' and Γ'', we construct a neighborhood Ω_0 of point O in D, which is a polygon region and satisfies the star-shape condition with

* Received September 29, 1982.

respect to point O, that is the line segment which connects any point on $\bar{\Omega}_0$ and point O lies on $\bar{\Omega}_0$ entirely. Γ' and Γ'' are also two sides of Ω_0, the remaining part of the boundary of Ω_0 is expressed by $\Gamma_0: r=R(\theta)$, where r, θ are polar coordinates.



Some boundary conditions are assumed on the boundary of D, for definiteness we assume that the homogeneous Neumann condition $\frac{\partial u}{\partial \nu}=0$ is assumed on Γ'andΓ'', where ν denotes the normal directoin, the discussion is similar for other kind of homogeneous boundary conditions.

We make the infinite element discretization as usual: $D\backslash\bar{\Omega}_0$ is discretized into a finite number of triangular elements by the conventional way, while Ω_0 is discretized into an infinite number of triangular elements as the following: a constant $\xi, 0<\xi<1$, is taken, we construct similar curves of Γ_0 with 0 as the center and ξ, $\xi^2, \cdots, \xi^k, \cdots$ as the constants of proportionality, thus, Ω_0 is discretized into an infinite number of "layers", then we construct the line segments from point O to nodal points on Γ_0, each layer is discretized into finite number of quadrilaterals, finally, each quadrilateral is discretized into triangles, the mode of discretization for each layer is the same. The interpolation functions are linear on each element and is continuous on D.

One auxilary unbounded region is needed in the following proof, which is denoted by $\Omega=\{(r, \theta): 0<r<\infty,\ 0<\theta<\theta_0\}$, hence $\Omega_0\subset\Omega$. We also discretize region $\Omega\backslash\bar{\Omega}_0$ into infinite number of layers by the constants of proportionality $\xi^{-1}, \cdots, \xi^{-k}, \cdots$, then they are discretized into triangular elements by the same mode. The curves $r=\xi^k R(\theta)\ (k=0, \pm1, \cdots)$are denoted by Γ_k.

The region Ω_0 with its discretization is called a combined element, since equation (1) is given, if u_h is an approximate solution by the infinite element method[2], then the "strain energy"associated with u_h on Ω_0,

$$\int_{\Omega_0}|\nabla u_h|^2 dx\, dy+\lambda\int_{\Omega_0}|u_h|^2 dx\, dy$$

would be determined by the values of u_h at the nodes of Γ_0, where ∇ is the gradient operator: $\nabla:\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$. Suppose there are m nodes on Γ_0, the values of u_h on which form a m-dimensional vector according to a definite order, say the anti-clockwise order, which is denoted by y_0. A technique in [2] is given to calculate the combined stiffness matrix $K_z(\lambda)$, then for any boundary value y_0, the strain energy of the approximate solution u_h associated with y_0 on Ω_0 is expressed by $\frac{1}{2}\bar{y}_0^T K_z(\lambda) y_0$, where T denotes transpose. Ω_0 can be treated as one element by means of the matrix $K_z(\lambda)$, we can solve this algebraic system as ordinary finite element method by assembling this combined element with conventional elements on $D\backslash\bar{\Omega}_0$. We denote the values of u_h at the nodes of Γ_k by vectors $y_k\ (k=1, 2, \cdots)$ just as y_0. When we consider the approximate solutions on Ω, we also use the symbols y_k for negative indices k.

We assume that the above discretization is normal, that is, any two closed triangular elements possess either a common vertex or a common side, or no common point at all. We assume also all the inner angles of elements have an upper bound which is less than π. We would not repeat these assumptions in the following. As usual, h denotes the length of the greatest side of triangles for a definite discretization.

§ 2. The Average Convergence

For definiteness, we consider the following boundary condition at this section: Suppose condition $\frac{\partial u}{\partial \nu}=0$ is assumed on the part of boundary Γ', Γ'' of region D, while the Dirichlet condition $u=f$ is assumed on the rest of the boundary $\Gamma=\partial D\backslash\overline{(\Gamma'\cup\Gamma'')}$.

We take the usual notations $H^s(D)$ for Sobolev spaces, where s is a real number, we denote $\mathring{H}^1(D)$ the subspace of $H^1(D)$, the element of which assumes value 0 on Γ. Here it should be noticed that this notation is different from the usual one. The norm in $H^s(D)$ is denoted by $\|\cdot\|_{s,D}$, as $s=0$, $H^0(D)=L^2(D)$, the inner product is denoted by $(\cdot)$.

The subspace of $H^1(D)$ corresponding to the infinite element discretization in the above section is denoted by $S_h(D)$, while the subspace of $\mathring{H}^1(D)$ is denoted by $\mathring{S}_h(D)$. We also define the weighted Sobolev spaces $H^{M,1}(D)$, where $M>0$ is an integer, which is the closure of the space of infinite differentiable functions $C_\infty(\overline{D})$ according to the norm

$$\|u\|_{M,1,D}=\left\{\sum_{|\alpha|=M}\int_D r^{2(M-1)}|\partial^\alpha u|^2 dx\,dy+\|u\|_{1,D}^2\right\}^{1/2},$$

where r is the distance of each point of D to point O, $\alpha=(\alpha_1,\ \alpha_2)$, $|\alpha|=\alpha_1+\alpha_2$, α_1 and α_2 are nonnegative integers, $\partial^\alpha u=\partial^{\alpha_1+\alpha_2}u/\partial x^{\alpha_1}\partial y^{\alpha_2}$.

We introduce space $H^s(\Gamma)$ on Γ as usual, the norm of which is denoted by $\|\cdot\|_{s,\Gamma}$. The subspace of $H^{1/2}(\Gamma)$ corresponding to the infinite element discretization is denoted by $S_h(\Gamma)$. It is easy to see there is a constant C, which depends on the region D only and is independent of the discretization, such that for any $f\in S_h(\Gamma)$, there is a $u\in S_h(D)$, satisfying

$$\|u\|_{1,D}\leqslant C\|f\|_{1/2,\Gamma}, \tag{2}$$

and

$$u|_\Gamma=f.$$

This is the so called property of "conformance" in [3]. Since we are considering the polygon region now, it is easy to verify that this property always holds.

We consider a sesquilinear functional

$$B(u,\ v)=\int_D\left(\frac{\partial u}{\partial x}\overline{\frac{\partial v}{\partial x}}+\frac{\partial u}{\partial y}\overline{\frac{\partial v}{\partial y}}\right)dx\,dy \tag{3}$$

on D, which corresponds to the Laplace operator. The variational formulation of our problem is:

$$\begin{cases}\text{for given } f\in H^{1/2}(\Gamma), \text{ to find } u\in H^1(D), \text{ such that } u|_\Gamma=f,\\ B(u,\ v)+(\lambda u,\ v)=0, \quad \forall v\in\mathring{H}^1(D).\end{cases} \tag{4}$$

Suppose $f_h\in S_h(\Gamma)$ is an approximate value of f, then the infinite element approximation of problem (4) is:

$$\begin{cases} \text{to find } u_h \in S_h(D), \text{ such that } u_h|_\Gamma = f_h, \\ B(u_h, v) + (\lambda u_h, v) = 0, \quad \forall v \in \mathring{S}_h(D). \end{cases} \tag{5}$$

Let λ_0 be the first eigenvalue of problem (4), then it is easy to see

Lemma 1. *There exists a unique solution u for problem (4) as $|\lambda| < |\lambda_0|$ and*

$$\|u\|_{1,D} \leqslant C\|f\|_{1/2,\Gamma},$$

where constant C depends on the region D and λ only.

Lemma 2. *There exists a unique solution u_h for problem (5) as $|\lambda| < |\lambda_0|$ and*

$$\|u_h\|_{1,D} \leqslant C\|f_h\|_{1/2,\Gamma},$$

where constant C depends on the region D and λ only and is independent of the discretization.

We always denote the constants by C and do not distinguish them throughout this paper. The above two lemmas are the corollaries of the well known Lax–Milgram theorem[4], if we notice that

$$|B(u, u) + (\lambda u, u)| \geqslant \alpha\|u\|_{1,D}^2, \ \forall u \in \mathring{H}^1(D), \tag{6}$$

for $|\lambda| < |\lambda_0|$, where $\alpha > 0$ is a constant depending on λ. We estimate the error $u - u_h$ as follows:

Theorem 1. *If u, u_h are the solutions of problems (4) and (5) respectively, and $u \in H^{2,1}(D)$, then*

$$\|u - u_h\|_{1,D} \leqslant C\{hR^{-1}\|u\|_{2,1,D} + \|f - f_h\|_{1/2,\Gamma}\},$$

where R is the distance between Γ_1 and point O, C depends on D and λ only.

Proof. By the Lemma 8 of [1],

$$\|u - \Pi u\|_{1,D} \leqslant ChR^{-1}\|u\|_{2,1,D}, \tag{7}$$

where Πu is the interpolation function of u. Denote by f_I the value of Πu on Γ, then $f_I \in S_h(\Gamma)$, and

$$\|f - f_I\|_{1/2,\Gamma} \leqslant ChR^{-1}\|u\|_{2,1,D}. \tag{8}$$

Consider problem (4) and (5) with boundary value f_I, let the solutions of them be u' and u'_h respectively, if we set $w = u' - \Pi u$, $w_h = u'_h - \Pi u$, then they satisfy

$$B(w, v) + (\lambda w, v) = -B(\Pi u, v) - (\lambda \Pi u, v), \ \forall v \in \mathring{H}^1(D),$$

$$B(w_h, v) + (\lambda w_h, v) = -B(\Pi u, v) - (\lambda \Pi u, v), \ \forall v \in \mathring{S}_h(D),$$

respectively and $w \in \mathring{H}^1(D)$, $w_h \in \mathring{S}_h(D)$, owing to [5] and (6).

$$\|w - w_h\|_{1,D} \leqslant C \inf_{v \in \mathring{S}_h(D)} \|w - v\|_{1,D},$$

we take $v = 0$ particularly and obtain

$$\|w - w_h\|_{1,D} \leqslant C\|w\|_{1,D} = C\|u' - \Pi u\|_{1,D} \leqslant C\{\|u' - u\|_{1,D} + \|u - \Pi u\|_{1,D}\}.$$

Therefore

$$\|u' - u'_h\|_{1,D} = \|w - w_h\|_{1,D} \leqslant C\{\|u' - u\|_{1,D} + \|u - \Pi u\|_{1,D}\}.$$

By Lemma 1, 2

$$\|u - u'\|_{1,D} \leqslant C\|f - f_I\|_{1/2,\Gamma},$$

$$\|u_h - u'_h\|_{1,D} \leqslant C\|f_h - f_I\|_{1/2,\Gamma},$$

hence

$$\begin{aligned}\|u-u_h\|_{1,D}&\leqslant\|u-u'\|_{1,D}+\|u'-u'_h\|_{1,D}+\|u'_h-u_h\|_{1,D}\\&\leqslant C\{\|u-u'\|_{1,D}+\|u-\Pi u\|_{1,D}+\|f_h-f_I\|_{1/2,\Gamma}\}\\&\leqslant C\{\|f-f_I\|_{1/2,\Gamma}+\|u-\Pi u\|_{1,D}+\|f_h-f_I\|_{1/2,\Gamma}\}.\end{aligned}$$

By (7), (8)

$$\|u-u_h\|_{1,D}\leqslant C\{hR^{-1}\|u\|_{2,1,D}+\|f-f_h\|_{1/2,\Gamma}\}.\quad \text{Q. E. D.}$$

§ 3. The Convergence of an Auxiliary Problem

In order to discuss the convergence of the singular components for the infinite element approximate solution, we consider an auxiliary problem here, which is the Poisson equation

$$\Delta u=f \tag{9}$$

on region $\Omega=\{(r,\ \theta):0<r<\infty,\ 0<\theta<\theta_0\}$ with Neumann condition

$$\left.\frac{\partial u}{\partial \nu}\right|_{\theta=0,\theta_0}=0. \tag{10}$$

Here we assume that supp $f\subset\Omega'\subset\subset\overline{\Omega}$, that is, f is a function with compact support in $\overline{\Omega}$. We also assume $f\in L^2$. If

$$\int_\Omega f\,dx\,dy=0,$$

then the solutions of problem (9), (10) exist, and is unique up to a constant difference. It is easy to see that the solution $u\in H^2$ on any bounded closed region which excludes point O, and u is the solution of the Laplace equation as r is large enough, hence the main part of u is $\alpha+\beta r^{-\pi/\theta_0}\cos\frac{\pi}{\theta_0}\theta$ for $r\to+\infty$, where α and β are constants and α is arbitrary, so the first order derivative has the order $O(r^{-\frac{\pi}{\theta_0}-1})$ as $r\to+\infty$, and the second order derivative has the order $O(r^{-\frac{\pi}{\theta_0}-2})$. The solution near point O is $u=\alpha'+\beta' r^{\pi/\theta_0}\cos\frac{\pi}{\theta_0}\theta+u'$, where $u'\in H^2$, by [6].

We introduce the following function space to discuss the infinite element approximation of the above problem: space $H^1_s(\Omega)$ is the closure of the space of infinite differentiable and compactly supported functions $\mathring{C}_\infty(\overline{\Omega})$ according to the norm

$$\|u\|_{(s),1,\Omega}=\left\{\int_\Omega\frac{1}{(1+r)^s}|u|^2dx\,dy+\int_\Omega\left(\left|\frac{\partial u}{\partial x}\right|^2+\left|\frac{\partial u}{\partial y}\right|^2\right)dx\,dy\right\}^{1/2},$$

where $2<s<4$. Similarly, define space $H^{2,1}_s(\Omega)$ by norm

$$\|u\|_{(s),2,1,\Omega}=\left\{\|u\|^2_{(s),1,\Omega}+\max_{|\alpha|=2}\int_\Omega r^2|\partial^\alpha u|^2dx\,dy\right\}^{1/2}.$$

By the above consideration the solution of Poisson equation $u\in H^{2,1}_s(\Omega)$.

Denote by $\mathbb{C}$ the space of complex number, since $\mathbb{C}\subset H^{2,1}_s(\Omega)\subset H^1_s(\Omega)$, we can define the quotient space $H=H^1_s(\Omega)/\mathbb{C}$. Denote by $\tilde{u}$ the element of H, then there is an equivalent norm

$$\|\tilde{u}\|_H=\left\{\int_\Omega\left(\left|\frac{\partial u}{\partial x}\right|^2+\left|\frac{\partial u}{\partial y}\right|^2\right)dx\,dy\right\}^{1/2},\quad u\in\tilde{u}. \tag{11}$$

We refer readers to [7] for the proof of (11), although bounded region was considered there, the above result would be easy to obtain if we repeat the proof.

Similarly, let $B(\tilde{u}, \tilde{v})$ be the sesquilinear functional on H,

$$B(\tilde{u}, \tilde{v})=\int_{\Omega}\left(\frac{\partial u}{\partial x}\overline{\frac{\partial v}{\partial x}}+\frac{\partial u}{\partial y}\overline{\frac{\partial v}{\partial y}}\right)dx\,dy, \quad u\in\tilde{u},\ v\in\tilde{v}.$$

A semilinear functional on space H may be formed associated with the function f as

$$\langle f, \tilde{v}\rangle=\int_{\Omega} f\bar{v}\,dx\,dy=(f, v), \quad v\in\tilde{v}.$$

Thus, the solution $\tilde{u}$ of the above boundary value problem of the Poisson equation is also the solution of the variational problem

$$B(\tilde{u}, \tilde{v})+\langle f, \tilde{v}\rangle=0, \quad \forall\tilde{v}\in H. \tag{12}$$

The solution of the variational problem (12) exists and is unique, and we have $u\in H_s^{2,1}(\Omega)$ for any $u\in\tilde{u}$. It is easy to verify by the closed graph theorem that

Lemma 3. *$f\mapsto u$ is a bounded linear operator from $L^2(\Omega')$ to $H_s^{2,1}(\Omega)/\mathbb{C}$.*

We have made an infinite element discretization for region Ω in § 2, the corresponding subspace of $H_s^1(\Omega)$ is denoted by $S_h(\Omega)$, the quotient space is $H_h=S_h(\Omega)/\mathbb{C}$. Let $u\in H_s^{2,1}(\Omega)$, denote by Πu the interpolation function, then we can prove

Lemma 4. *$u-\Pi u\in H^1(\Omega)$ for any $u\in H_s^{2,1}(\Omega)$ and*

$$\|u-\Pi u\|_{1,\Omega}\leqslant ChR^{-1}\|u\|_{(s),2,1,\Omega},$$

where h is the length of the largest side of the elements between Γ_0 and Γ_1, R is the distance between Γ_1 and point O.

Proof. Denote any triangular element by Δ_i, then we obtain as in [1]

$$\|u-\Pi u\|_{1,\Delta_i}^2\leqslant Ch_i^2\max_{|\alpha|=2}\|\partial^\alpha u\|_{0,\Delta_i}^2,$$

where h_i is the length of the largest side of Δ_i. But

$$\max_{|\alpha|=2}\|\partial^\alpha u\|_{0,\Delta_i}^2=\max_{|\alpha|=2}\int_{\Delta_i}|\partial^\alpha u|^2\,dx\,dy\leqslant r_i^{-2}\max_{|\alpha|=2}\int_{\Delta_i}r^2|\partial^\alpha u|^2\,dx\,dy,$$

where r_i is the distance between Δ_i and point O. By the geometric similar relation

$$h_i r_i^{-1}\leqslant hR^{-1},$$

hence

$$\|u-\Pi u\|_{1,\Delta_i}^2\leqslant Ch^2R^{-2}\max_{|\alpha|=2}\int_{\Delta_i}r^2|\partial^\alpha u|^2\,dx\,dy,$$

summing them up with respect to i we obtain the desired result. Q. E. D.

Now we consider the approximate problem of (12): to find $\tilde{u}_h\in H_h$, such that

$$B(\tilde{u}_h, \tilde{v})+\langle f, \tilde{v}\rangle=0, \quad \forall\tilde{v}\in H_h. \tag{13}$$

Lemma 5. *The solution of problem (13) exists and is unique, and*

$$\|\tilde{u}-\tilde{u}_h\|_H\leqslant ChR^{-1}\|u\|_{(s),2,1,\Omega}, \quad u\in\tilde{u},$$

where $\tilde{u}$, $\tilde{u}_h$ are the solutions of problem (12), (13).

The existence and uniqueness in this lemma is the corollary of the Lax-Milgram theorem, the error estimation can be obtained by Lemma 4 and [5]. (The proof of Theorem 1 is referred.)

§ 4. The Convergence of the Singular Components

We consider the local property of the solutions near point O, the considered region is Ω_0, the space is $H^s(\Omega_0)$. Like the definition in § 2, $\mathring{H}^1(\Omega_0)$ is the subspace of $H^1(\Omega_0)$, the elements of which assume the value 0 on Γ_0. We also have subspace $S_h(\Omega_0)$ and $\mathring{S}_h(\Omega_0)$ corresponding to the infinite element discretization.

First of all, let us recall the results in [1] for the Laplace equation.

If u and u_h are the solutions of the following:

$$\begin{cases} B(u, v)=0, & \forall v\in \mathring{H}^1(\Omega_0), \\ u\in H^1(\Omega_0), & u|_{\Gamma_0}=f, \end{cases} \tag{14}$$

$$\begin{cases} B(u_h, v)=0, & \forall v\in \mathring{S}_h(\Omega_0), \\ u_h\in S_h(\Omega_0), & u_h|_{\Gamma_0}=f_h, \end{cases} \tag{15}$$

then u can be expanded into a series near point O as

$$u=\alpha+\beta r^{\pi/\theta_0}\cos\frac{\pi}{\theta_0}\theta+\gamma r^{2\pi/\theta_0}\cos\frac{2\pi}{\theta_0}\theta+\cdots, \tag{16}$$

y_k, the values of u_h at the nodes of Γ_k, can be expressed as

$$y_k=\alpha_h g^{(1)}+\beta_h\lambda_2^k g^{(2)}+\cdots, \tag{17}$$

where α, β, γ, α_h, β_h are constants. $\lambda_1(=1)$, λ_2, $\cdots$, λ_m are the eigenvalues of X_0[2] arranged by the decreasing norm, $g^{(1)}(=(1, 1, \cdots, 1)^T)$, $g^{(2)}$, $\cdots$, $g^{(m)}$ are the corresponding eigenvectors.

Each term in (17) corresponds to a function in $S_h(\Omega_0)$, which is the solution of problem (15) under some boundary condition. For instance, the first term corresponds to the constant α_h, which is the solution of (15) with $f_h=\alpha_h$. The function corresponding to the second term is denoted by u_h^*, the value of u_h^* on Γ_0 is denoted by f_h^*, then u_h^* is the solution of problem (15) as $f_h=f_h^*$.

We define operators Q_1 and Q_2 such that

$$Q_1u=\alpha,\quad Q_2u=\beta r^{\pi/\theta_0}\cos\frac{\pi}{\theta_0}\theta,$$

regarding them as the functions on Γ_0, we have the operators P_1, P_2 as

$$P_1f=\alpha,\quad P_2f=\beta r^{\pi/\theta_0}\cos\frac{\pi}{\theta_0}\theta.$$

Just the same, let $Q_{1,h}u_h=\alpha_h$, $Q_{2,h}u_h=u_h^*$, and $P_{1,h}f_h=\alpha_h$, $P_{2,h}f_h=f_h^*$ on Γ_0. Here Q_1, $Q_{1,h}$, $\cdots$ are operators in space $H^1(\Omega_0)$, P_1, $P_{1,h}$, $\cdots$ are operators in space $H^{1/2}(\Gamma_0)$. And it is easy to see that the range of definition for operators P_1, P_2 is the entire space $H^{1/2}(\Gamma_0)$.

The first order derivatives of Q_2u and $Q_{2,h}u_h$ is unbounded near point O, therefore we call them the singular components of u and u_h respectively. We have proved in [1] that the operators $P_{1,h}$, $P_{2,h}$ can be extended to operators on space $H^{1/2}(\Gamma_0)$ such that

$$\|P_j-P_{j,h}\|\leqslant Ch,\quad j=1, 2, \cdots,$$

applying this result we can prove

$$\|Q_ju-Q_{j,h}u_h\|_{1,\Omega_0}\leqslant C\{\|u-u_h\|_{1,\Omega_0}+h\|u\|_{1,\Omega_0}\},\quad j=1, 2, \cdots.$$

Now we consider $j=1$, 2 only.

We consider the following two solutions u and u_h for the Helmholtz equation:

$$\begin{cases} B(u, v)+(\lambda u, v)=0, & \forall v\in \mathring{H}^1(\Omega_0), \\ u\in H^1(\Omega_0), & u|_{\Gamma_0}=f, \end{cases} \tag{18}$$

$$\begin{cases} B(u_h, v)+(\lambda u_h, v)=0, & \forall v\in \mathring{S}_h(\Omega_0), \\ u_h\in S_h(\Omega_0), & u_h|_{\Gamma_0}=f_h. \end{cases} \tag{19}$$

There are similar expansions:

$$u=\alpha+\beta r^{\pi/\theta_0}\cos\frac{\pi}{\theta_0}\theta+\cdots, \tag{20}$$

$$y_k=\alpha_h g^{(1)}+\beta_h\lambda_2^k g^{(2)}+\cdots, \tag{21}$$

where λ_2, $g^{(1)}$, $g^{(2)}$ are the same as that in (17)[2], the difference is that each term in the expansion is no longer the solution of the problem, thus only the residue in (20), (21) is an infinitesimal with higher order as $r\to 0$. Nevertheless, we also define operators Q_1, Q_2, $Q_{1,h}$, $Q_{2,h}$ as previously and we will prove the same convergence for the singular components in this section.

(21) may be rewritten as

$$y_k=\alpha_h g^{(1)}+\beta_h\xi^{k\alpha_2}g^{(2)}+\cdots, \tag{22}$$

where $\alpha_2=\log\lambda_2/\log\xi$. r has the same order with ξ^k as $r\to 0$, comparing (20) and (22) we know the index α_2 corresponds to π/θ_0. We have proved in [1] that

Theorem 2. $|\alpha_2-\pi/\theta_0|\leqslant Ch.$

We are now in a position to consider the convergence of the singular components. For this purpose we consider some auxiliary solutions at first.

Because $\xi\to 1$ as $h\to 0$, we may assume $\xi\geqslant 0.5$. Take a constant l such that $\eta=\xi^l\in\left[\frac{1}{4}, \frac{1}{2}\right]$, construct the similar curves of Γ_0 with O as the center, and η^k ($k=0, \pm 1, \pm 2, \cdots$) as the constants of proportionality, which are denoted by Γ'_k. $\Omega^{(k)}$ denote the region between Γ'_{k-1} and Γ'_k, Ω_k denote the region $\{(r, \theta): 0<r<\eta^k R(\theta), 0<\theta<\theta_0\}$, these notations denote the regions themselves and their areas as well. Let

$$C_{k-1}=\frac{1}{\Omega^{(k-1)}}\int_{\Omega_k}\lambda u_h\,dx\,dy,$$

$$f_k=\begin{cases}\lambda u_h+C_k, & \Omega^{(k)},\\ -C_{k-1}, & \Omega^{(k-1)},\\ 0, & \Omega\setminus\overline{(\Omega^{(k)}\cup\Omega^{(k-1)})},\end{cases}$$

for $k=1, 2, \cdots$, then

$$\int_\Omega f_k\,dx\,dy=0.$$

Besides, since

$$\int_{\Omega^{(k-1)}}|C_{k-1}|^2dx\,dy=\frac{1}{\Omega^{(k-1)}}\left|\int_{\Omega_k}\lambda u_h\,dx\,dy\right|^2$$

$$\leqslant\frac{1}{\Omega^{(k-1)}}\int_{\Omega_k}dx\,dy\int_{\Omega_k}|\lambda u_h|^2dx\,dy\leqslant C\int_{\Omega_k}|u_h|^2dx\,dy,$$

we have

$$\|f_k\|_{0,\Omega}\leqslant C\|u_h\|_{0,\Omega_0} \tag{23}$$

uniformly for k. For $\tilde{v}\in H$, $v\in\tilde{v}$, set

$$\langle f_k, \tilde{v}\rangle=\int_\Omega f_k\bar{v}\,dx\,dy,$$

then by Lemma 3, the solution of problem

$$B(\tilde{u}_k, \tilde{v})+\langle f_k, \tilde{v}\rangle=0, \quad \forall \tilde{v}\in H$$

exists and is unique. If $u_k\in\tilde{u}_k$, let $w_k=(I-Q_1-Q_2)u_k$, $w=\sum_{k=1}^{\infty}w_k$, where I is the identity operator. Then, if and only if this series converges in $H^1(\Omega_0)$, w is the solution of problem

$$B(w, v)+(\lambda u_h, v)=0, \quad \forall v\in \dot{H}^1(\Omega_0). \tag{24}$$

Lemma 6. $\|w\|_{1,\Omega_0}\leqslant C\|u_h\|_{0,\Omega_0}$,

where C is independent of the discretization, and $w=o(r^{\pi/\theta_0})$ near point O.

Proof. We define a similar transformation of the independent variables, $\eta^k x\mapsto x$, $\eta^k y\mapsto y$, let $f\mapsto f_k^*$, $\tilde{u}_k\mapsto\tilde{u}_k^*$ under this transformation, then $\tilde{u}_k^*$ satisfies

$$B(\tilde{u}_k^*, \tilde{v})+\langle\eta^{2k}f_k^*, \tilde{v}\rangle=0, \quad \forall\tilde{v}\in H. \tag{25}$$

We know from (23)

$$\|f_k^*\|_{0,\Omega}\leqslant C\eta^{-k}\|u_h\|_{0,\Omega_0},$$

by Lemma 3 (taking Ω_{-2} as Ω') we obtain

$$\|\tilde{u}_k^*\|_{H_s^{2,1}(\Omega)/\mathbb{C}}\leqslant C\|\eta^{2k}f_k^*\|_{0,\Omega}\leqslant C\eta^k\|u_h\|_{0,\Omega_0},$$

that is, there exists $u_k^*\in\tilde{u}_k^*$, such that

$$\|u_k^*\|_{(s),2,1,\Omega}\leqslant C\eta^k\|u_h\|_{0,\Omega_0}. \tag{26}$$

We notice that u_k^* is the solution of the Laplace equation on Ω_0, let $w_k^*=(I-Q_1-Q_2)u_k^*$, then since Q_1, Q_2 are bounded operators:

$$\|Q_1u_k^*\|_{1,\Omega_0}+\|Q_2u_k^*\|_{1,\Omega_0}\leqslant C\eta^k\|u_h\|_{0,\Omega_0},$$

set

$$Q_1u_k^*=\alpha_k^*, \quad Q_2u_k^*=\beta_k^*r^{\pi/\theta_0}\cos\frac{\pi}{\theta_0}\theta,$$

then

$$|\alpha_k^*|+|\beta_k^*|\leqslant C\eta^k\|u_h\|_{0,\Omega_0}. \tag{27}$$

Under the inverse transformation $x\mapsto\eta^k x$, $y\mapsto\eta^k y$, we have $w_k^*\mapsto w_k$, $u_k^*\mapsto u_k$, by the definition of space $H_s^{2,1}(\Omega)$

$$\|u_k\|_{(s),2,1,\Omega}\leqslant C\eta^{-k\left(\frac{s}{2}-1\right)}\|u_k^*\|_{(s),2,1,\Omega}. \tag{28}$$

By (26)

$$\|u_k\|_{(s),2,1,\Omega}\leqslant C\eta^{k\left(2-\frac{s}{2}\right)}\|u_h\|_{0,\Omega_0}. \tag{29}$$

Because $\alpha_k^*\mapsto\alpha_k^*$, $\beta_k^*r^{\pi/\theta_0}\cos\frac{\pi}{\theta_0}\theta\mapsto\beta_k^*\eta^{-k\pi/\theta_0}r^{\pi/\theta_0}\cos\frac{\pi}{\theta_0}\theta$ under this inverse transformation, by (27) and (29)

$$\|w_k\|_{1,\Omega_0}\leqslant C\left(\eta^{k\left(2-\frac{s}{2}\right)}+\eta^k+\eta^{k\left(1-\frac{\pi}{\theta_0}\right)}\right)\|u_h\|_{0,\Omega_0},$$

noticing that $s<4$ and $\theta_0>\pi$, we sum them up with respect to k and obtain

$$\|w\|_{1,\Omega_0}\leqslant C\|u_h\|_{0,\Omega_0}.$$

Now we discuss the property of w as $r\to 0$. Consider the following auxiliary problem: if $f\in L^\infty(\Omega')$, $\int_{\Omega'}f\,dx\,dy=0$, v is a bounded solution of

$$\Delta v=f,$$

$$\left.\frac{\partial v}{\partial\nu}\right|_{\theta=0,\theta_0}=0$$

on Ω, then thanks to the discussion in § 3, v exists and is unique up to a constant difference. Let $\Omega'=\Omega^{(-1)}\cup\Gamma'_{-1}\cup\Omega^{(-2)}$, v satisfies the Laplace equation on Ω_0, we have the expansion (16). Besides, $v\in W^1_\infty$ on any compact subset which excludes point O, so v is a continuous function. The solution $v-Q_1v$, which equals to zero at point O, is determined by the problem uniquely, therefore $f\mapsto v-Q_1v$ determines a closed operator from $L^\infty(\Omega')$ to $C(\overline{\Omega})$, we have

$$|v-Q_1v|\leqslant C\sup|f|$$

on Ω by the closed graph theorem.

Next, we consider $r^{-2\pi/\theta_0}(v-Q_1v-Q_2v)$, which is bounded and determined by the problem uniquely by the expansion (16). In the same way,

$$|r^{-2\pi/\theta_0}(v-Q_1v-Q_2v)|\leqslant C\sup|f|$$

on Ω by the closed graph theorem, that is

$$|v-Q_1v-Q_2v|\leqslant Cr^{2\pi/\theta_0}\sup|f|.$$

Then we consider $Q_2v=\beta r^{\pi/\theta_0}\cos\frac{\pi}{\theta_0}\theta$, since $v-Q_1v$ satisfies the Laplace equation in Ω_0, taking the value of $v-Q_1v$ on Γ_0 as the boundary value, we get a well-posed problem, it is easy to see

$$|\beta|\leqslant C\max_{\Gamma_0}|v-Q_1v|,$$

hence

$$|\beta|\leqslant C\sup|f|.$$

Applying the above result to problem (25), we obtain

$$|u_k^*-Q_1u_k^*|\leqslant C\sup|\eta^{2k}f_k^*|,$$

$$|w_k^*|\leqslant Cr^{2\pi/\theta_0}\sup|\eta^{2k}f_k^*|,$$

$$|\beta_k|\leqslant C\sup|\eta^{2k}f_k^*|.$$

But $|f_k^*|\leqslant C_h$, where C_h is a constant which depends on the discretization generally, so we get

$$|u_k^*-Q_1u_k^*|\leqslant C_h\eta^{2k},$$

$$|w_k^*|\leqslant C_h\eta^{2k}r^{2\pi/\theta_0},$$

$$|\beta_k^*|\leqslant C_h\eta^{2k},$$

returning to functions w_k we get

$$|w_k|\leqslant\begin{cases}C_h\left(\dfrac{r}{\eta^k}\right)^{2\pi/\theta_0}\eta^{2k}, & r<\eta^kR(\theta),\\ C_h\left(\dfrac{r}{\eta^k}\right)^{\pi/\theta_0}\eta^{2k}, & r>\eta^kR(\theta).\end{cases}$$

Summing them up with respect to k for a fixed point (r,θ), we get

$$|w|\leqslant\sum_{k<\frac{\log(r/R(\theta))}{\log\eta}}C_hr^{2\pi/\theta_0}\eta^{2k\left(1-\frac{\pi}{\theta_0}\right)}+\sum_{k>\frac{\log(r/R(\theta))}{\log\eta}}C_hr^{\pi/\theta_0}\eta^{k\left(2-\frac{\pi}{\theta_0}\right)},$$

if we write down the sum of the above two series, then

$$|w|\leqslant C_hr^{2\pi/\theta_0}$$

is obvious, which completes the proof of this lemma. Q. E. D.

Now we consider the infinite element approximation of w, construct $\tilde{u}_{kh}\in H_h$, which satisfies

$$B(\tilde{u}_{kh}, \tilde{v})+\langle f_k, \tilde{v}\rangle=0, \quad \forall \tilde{v}\in H_h.$$

Let $u_{kh}\in\tilde{u}_{kh}$, $w_{kh}=(I-Q_{1,h}-Q_{2,h})u_{kh}$, $w_h=\sum_{k=1}^{\infty} w_{kh}$, then if and only if this series converges in $S_h(\Omega_0)$, w_h is the solution of the following problem:

$$B(w_h, v)+(\lambda u_h, v)=0, \quad \forall v\in \mathring{S}_h(\Omega_0). \tag{30}$$

Lemma 7. $$\|w-w_h\|_{1,\Omega_0}\leqslant Ch\|u_h\|_{0,\Omega_0},$$

where C is independent of the discretization, and for sufficient small h, w_h is an infinitesimal with the order higher than the first two terms of (21).

Proof. Corresponding to (25), we have $\tilde{u}^*_{kh}\in H_h$ which satisfies

$$B(\tilde{u}^*_{kh}, \tilde{v})+\langle \eta^{2k}f^*_k, \tilde{v}\rangle=0, \quad \forall \tilde{v}\in H_h. \tag{31}$$

By Lemma 5 $$\|\tilde{u}^*_k-\tilde{u}^*_{kh}\|_H\leqslant Ch\|u^*_k\|_{(s),2,1,\Omega}, \quad u^*_k\in\tilde{u}^*_k,$$

where the factor R^{-1} has been absorbed into the constant C, by (26)

$$\|\tilde{u}^*_k-\tilde{u}^*_{kh}\|_H\leqslant Ch\eta^k\|u_h\|_{0,\Omega_0},$$

hence there exists $u^*_{kh}\in\tilde{u}^*_{kh}$, such that

$$\|u^*_k-u^*_{kh}\|_{(s),1,\Omega}\leqslant Ch\eta^k\|u_h\|_{0,\Omega_0}. \tag{32}$$

By the trace theorem

$$\|u^*_k-u^*_{kh}\|_{1/2,\Gamma_0}\leqslant Ch\eta^k\|u_h\|_{0,\Omega_0}. \tag{33}$$

By (26) and (32)

$$\|u^*_{kh}\|_{(s),1,\Omega}\leqslant C\eta^k\|u_h\|_{0,\Omega_0}.$$

In virtue of the proof of theorem 5 in [1]

$$\|P_j-P_{j,h}\|\leqslant Ch, \quad j=1, 2,$$

hence $$\|P_ju^*_k-P_{j,h}u^*_{kh}\|_{1/2,\Gamma_0}\leqslant Ch\eta^k\|u_h\|_{0,\Omega_0}, \quad j=1, 2.$$

By the Theorem 4 of [1]

$$\|Q_ju^*_k-Q_{j,h}u^*_{kh}\|_{1,\Omega_0}\leqslant Ch\eta^k\|u_h\|_{0,\Omega_0}, \quad j=1, 2. \tag{34}$$

Under the transformation of independent variables $x\mapsto\eta^kx$, $y\mapsto\eta^ky$, we have $u^*_k\mapsto u_k$, $u^*_{kh}\mapsto u_{kh}$, $Q_ju^*_k\mapsto Q_ju_k$, $Q_{j,h}u^*_{kh}\mapsto Q_{j,h}u_{kh}$, but we know from the special pattern of these functions

$$Q_1u^*_k\equiv Q_1u_k,$$
$$Q_{1,h}u^*_{kh}\equiv Q_{1,h}u_{kh},$$
$$\eta^{-k\pi/\theta_0}Q_2u^*_k\equiv Q_2u_k,$$
$$\eta^{-k\alpha_1}Q_{2,h}u^*_{kh}\equiv Q_{2,h}u_{kh}.$$

Now $$w-w_h=\sum_{k=1}^{\infty}(u_k-u_{kh})+\sum_{k=1}^{\infty}(Q_1u_k-Q_{1,h}u_{kh})+\sum_{k=1}^{\infty}(Q_2u_k-Q_{2,h}u_{kh}).$$

By (28) and (32)

$$\|u_k-u_{kh}\|_{(s),2,1,\Omega}\leqslant Ch\eta^{k\left(2-\frac{s}{2}\right)}\|u_h\|_{0,\Omega_0}.$$

By (26) and (34)

$$\|Q_1u_k-Q_{1,h}u_{kh}\|_{1,\Omega_0}\leqslant Ch\eta^k\|u_h\|_{0,\Omega_0},$$

$$\|Q_2u_k-Q_{2,h}u_{kh}\|_{1,\Omega_0}\leqslant Ch\eta^{k\left(1-\frac{\pi}{\theta_0}\right)}\|u_h\|_{0,\Omega_0}+|\eta^{-k\pi/\theta_0}-\eta^{-k\alpha_1}|\cdot C\eta^k\|u_h\|_{0,\Omega_0},$$

but by Theorem 2

$$|\eta^{-k\pi/\theta_0}-\eta^{-k\alpha_2}|=\left|\int_{-k\alpha_2}^{-k\pi/\theta_0}\eta^t\log\eta\,dt\right|\leqslant Ckh\eta^{-k\pi/\theta_0}.$$

Therefore $\|w-w_h\|_{1,\Omega_0}\leqslant Ch\|u_h\|_{0,\Omega_0}.$

The proof of the second conclusion of this lemma is the same as that of Theorem 4.3 at [2], we omit it here. One condition is needed: λ_2 is a single root and $|\lambda_2|>\xi^2$, but by Theorem 2 this condition always holds as h small enough. Q. E. D.

Finally we prove the convergence of singular components.

Theorem 3.

$$\|Q_ju-Q_{j,h}u_h\|_{1,\Omega_0}\leqslant C\{\|u-u_h\|_{1,\Omega_0}+h\|u\|_{1,\Omega_0}\},\quad j=1,\ 2.$$

Proof. We get $w_0\in H^1(\Omega_0)$ like w with u instead of u_h, which satisfies

$$B(w_0,\ v)+(\lambda u,\ v)=0,\quad \forall v\in\mathring{H}^1(\Omega_0).$$

Making use of the inequality of Lemma 6 we obtain

$$\|w_0\|_{1,\Omega_0}\leqslant C\|u\|_{0,\Omega_0},\tag{35}$$

and

$$\|w_0-w\|_{1,\Omega_0}\leqslant C\|u-u_h\|_{0,\Omega_0}.\tag{36}$$

By Lemma 7

$$\|w_0-w_h\|_{1,\Omega_0}\leqslant C\{\|u-u_h\|_{0,\Omega_0}+h\|u_h\|_{0,\Omega_0}\}.\tag{37}$$

$u-w_0$ and u_h-w_h are the solutions of problems (14) and (15) with respect to some suitable boundary values, by the Theorem 5 of [1]

$$\|Q_j(u-w_0)-Q_{j,h}(u_h-w_h)\|_{1,\Omega_0}\leqslant C\{\|u-w_0-u_h+w_h\|_{1,\Omega_0}+h\|u-w_0\|_{1,\Omega_0}\},$$
$$j=1,\ 2.$$

By (35) and (37)

$$\|Q_j(u-w_0)-Q_{j,h}(u_h-w_h)\|_{1,\Omega_0}\leqslant C\{\|u-u_h\|_{1,\Omega_0}+h\|u\|_{1,\Omega_0}\},\quad j=1,\ 2.\tag{38}$$

By Lemma 7

$$Q_{j,h}(u_h-w_h)=Q_{j,h}u_h,\ \ j=1,\ 2.\tag{39}$$

By Lemma 6

$$Q_jw_0=Q_j(w_0-w),\quad j=1,\ 2.\tag{40}$$

w_0-w satisfies equation and bondary value

$$\Delta(w_0-w)=\lambda(u-u_h),$$
$$\left.\frac{\partial(w_0-w)}{\partial\nu}\right|_{\theta=0,\theta_0}=0$$

on Ω_0. Thanks to the estimation on singular components in [6]

$$\|Q_j(w_0-w)\|_{1,\Omega_0}\leqslant C\{\|u-u_h\|_{0,\Omega_0}+\|w_0-w\|_{1/2,\Gamma_0}\},\quad j=1,\ 2,$$

by (36) and the trace theorem

$$\|Q_j(w_0-w)\|_{1,\Omega_0}\leqslant C\|u-u_h\|_{0,\Omega_0},\quad j=1,\ 2.\tag{41}$$

The conclusion of this theorem follows from (38)—(41). Q. E. D.

References

[1] Ying Lung-an, The infinite element method for equations with constant coefficients, Proceedings of the China-France Symposium on FEM, 1982. (to appear)

[2] Ying Lung-an, The infinite element method of nonsimilar case, Proceedings of the China-France Symposium on FEM, 1982. (to appear)

[3] Ying Lung-an, The convergence of the infinite similar element method, *Acta Mathematicae Applacatae Sinica*, 2, 2, (1979), 149.

[4] P. D. Lax, A. N. Milgram, Parabolic equations, Contributions to the theory of partial differential equations, Princeton, 1954.

[5] I. Babuška, Error-bounds for finite element method, *Numer. Math.*, 16, 322, 1971.

[6] R. B. Kellogg, Singularities in interface problems, Numerical solution of partial differential equations II, 351, Academic Press, NY, 1971.

[7] J. Nečas, Les Methods Directs on Théorie des Equations Elliptiquess, Messon, Paris, 1976.