IMPLICIT DIFFERENCE SCHEMES FOR THE GENERALIZED NON-LINEAR SCHRÖDINGER SYSTEM*

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Abstract

In this paper we prove under certain weak conditions that two classes of implicit difference schemes for the generalized non-linear Schrödinger system are convergent and that an iteration method for the corresponding non-linear difference equations is convergent. Therefore, quite a complete theoretical foundation of implicit schemes for the generalized non-linear Schrödinger system is established in this paper.

Convergence of Difference Schemes

We discuss the following initial-boundary-value problem for the generalized non-linear Schrödinger system:

$$\begin{cases} iU_t+\frac{\partial}{\partial x}\left(A(x)\frac{\partial U}{\partial x}\right)+\beta(x)q(|U|^2)U+F(x,t)U=G(x,t), \\ \qquad 0\leqslant x\leqslant 1,\ 0\leqslant t\leqslant T, \\ U|_{x=0}=U|_{x=1}=0, \quad 0\leqslant t\leqslant T, \\ U|_{t=0}=E(x), \qquad 0\leqslant x\leqslant 1. \end{cases} \tag{1}$$

Here U, E are complex vectors; $A(x)$, $F(x,t)$ real symmetrical matrices; $\beta(x)$, $q(|U|^2)$ real scalar functions ($|U|$ denotes the Euclidean vector norm of U); and $G(x,t)$ is a real vector. As for $q(|U|^2)$, we consider the following functions: $|U|^2$ (and $|U|^{2p}$, p being a positive integer), $\varkappa(1-e^{-|U|^2})$, $|U|^2/(1+|U|^2)$, $\ln(1+|U|^2)$, etc.

This problem can be solved by using the following scheme

$$\begin{cases} i\frac{V_j^{n+1}-V_j^n}{\Delta t}+\frac{1}{2}\left[(A_{j+\frac{1}{2}}V_{jx}^{n+1})_{\bar{x}}+(A_{j+\frac{1}{2}}V_{jx}^n)_{\bar{x}}\right]+\frac{1}{2}\beta_j(\alpha_1 q(|V_j^{n+1}|^2) \\ \quad +(1-\alpha_1)q(|V_j^n|^2))\cdot(V_j^{n+1}+V_j^n)+\frac{1}{2}F_j^{n+\frac{1}{2}}(V_j^{n+1}+V_j^n)=G_j^{n+\frac{1}{2}}, \\ \qquad j=1,2,\cdots,J-1, \\ V_0^{n+1}=V_J^{n+1}=0, \quad J=1/\Delta x. \end{cases} \tag{2}$$

Here V_j^n denotes the approximate value of U at $x=j\Delta x$, $t=n\Delta t$; $F_j^{n+\frac{1}{2}}=F\left(j\Delta x, \left(n+\frac{1}{2}\right)\Delta t\right)$; $A_{j+\frac{1}{2}}=A\left(\left(j+\frac{1}{2}\right)\Delta x\right)$, $\beta_j=\beta(j\Delta x)$, $G_j^{n+\frac{1}{2}}=G\left(j\Delta x, \left(n+\frac{1}{2}\right)\Delta t\right)$; $V_{jx}^{n+\delta}=\frac{1}{\Delta x}(V_{j+1}^{n+\delta}-V_j^{n+\delta})$, $(A_{j+\frac{1}{2}}V_{jx}^{n+\delta})_{\bar{x}}=\frac{1}{\Delta x}(A_{j+\frac{1}{2}}V_{jx}^{n+\delta}-A_{j-\frac{1}{2}}V_{j-1,x}^{n+\delta})$, $\delta=0$ or 1; and α_1 is a

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positive constant. Clearly, the truncation error is $O(\Delta t^2+\Delta x^2)$ for $\alpha_1=\frac{1}{2}$, and $O(\Delta t+\Delta x^2)$ otherwise. Chang[1] has discussed convergence of this scheme for $\alpha_1=1$. However, one meets considerable difficulties when trying to prove the convergence of this scheme for $\alpha_1\neq 1$ by using the method in [1]. If $\alpha_1\neq 0$, one has to solve a system of non-linear equations at each step. An iteration method for solving the system is usually needed, and the iteration is required to be convergent. For the scheme with $\alpha_1=0$, only a system of linear equations needs to be solved, so this scheme is also special. Therefore, we pay our attention to the three schemes with $\alpha_1=\frac{1}{2}$, 1, 0, which are called Scheme A, Scheme B and Scheme C respectively in the following.

We first discuss the stability of Scheme A. Clearly, there are the following relations:

$$\begin{cases} \frac{i}{\Delta t}\sum_{j=1}^{J-1}[(V_j^{n+1}-V_j^n,\ V_j^{n+1}+V_j^n)+(\bar V_j^{n+1}-\bar V_j^n,\ \bar V_j^{n+1}+\bar V_j^n)] \\ \quad =\frac{2i}{\Delta t}\sum_{j=1}^{J-1}(|V_j^{n+1}|^2-|V_j^n|^2); \\ \frac{1}{2}\sum_{j=1}^{J-1}[((A_{j+\frac12}V_{jx}^{n+1})_{\bar x}+(A_{j+\frac12}V_{jx}^n)_{\bar x},\ V_j^{n+1}+V_j^n) \\ \quad -((A_{j+\frac12}\bar V_{jx}^{n+1})_{\bar x}+(A_{j+\frac12}\bar V_{jx}^n)_{\bar x},\ \bar V_j^{n+1}+\bar V_j^n)] \\ \quad =-\frac{1}{2}\sum_{j=0}^{J-1}[(A_{j+\frac12}V_{jx}^{n+1}+A_{j+\frac12}V_{jx}^n,\ V_{jx}^{n+1}+V_{jx}^n) \\ \quad -(A_{j+\frac12}\bar V_{jx}^{n+1}+A_{j+\frac12}\bar V_{jx}^n,\ \bar V_{jx}^{n+1}+\bar V_{jx}^n)]=0; \\ \sum_{j=1}^{J-1}\Big[\frac{1}{4}\beta_j(q(|V_j^{n+1}|^2)+q(|V_j^n|^2))[(V_j^{n+1}+V_j^n,\ V_j^{n+1}+V_j^n) \\ \quad -(\bar V_j^{n+1}+\bar V_j^n,\ \bar V_j^{n+1}+\bar V_j^n)]+\Big(\frac{1}{2}F_j^{n+\frac12}(V_j^{n+1}+V_j^n),\ V_j^{n+1}+V_j^n\Big) \\ \quad -\Big(\frac{1}{2}F_j^{n+\frac12}(\bar V_j^{n+1}+\bar V_j^n),\ \bar V_j^{n+1}+\bar V_j^n\Big)\Big]=0; \\ \sum_{j=1}^{J-1}[(G_j^{n+\frac12},\ V_j^{n+1}+V_j^n)-(G_j^{n+\frac12},\ \bar V_j^{n+1}+\bar V_j^n)] \\ \quad =2i\sum_{j=1}^{J-1}(G_j^{n+\frac12},\ \mathrm{Im}(V_j^{n+1}+V_j^n)). \end{cases} \tag{3}$$

Therefore, subtracting the inner product of $\bar V_j^{n+1}+\bar V_j^n$ and the conjugate equation of (2) from that of $V_j^{n+1}+V_j^n$ and (2), and summing up these differences from $j=1$ to $J-1$, we obtain

$$\frac{1}{\Delta t}\sum_{j=1}^{J-1}(|V_j^{n+1}|^2-|V_j^n|^2)=\sum_{j=1}^{J-1}(G_j^{n+\frac12},\ \mathrm{Im}(V_j^{n+1}+V_j^n)).$$

Let $\|V^n\|^2=\Delta x\sum_{j=1}^{J-1}|V_j^n|^2$, $\|G^n\|^2=\Delta x\sum_{j=1}^{J-1}|G_j^n|^2$. Then it follows from the above relation and the Schwarz inequality that

$$\|V^{n+1}\|^2-\|V^n\|^2\leqslant\Delta t\Big(\frac{1}{2}\|V^{n+1}\|^2+\frac{1}{2}\|V^n\|^2+\|G^{n+\frac12}\|^2\Big),$$

which can be rewritten as

$$\|V^{n+1}\|^2 \leqslant \frac{1+\frac{1}{2}\Delta t}{1-\frac{1}{2}\Delta t}\|V^n\|^2 + \frac{\Delta t}{1-\frac{1}{2}\Delta t}\|G^{n+\frac{1}{2}}\|^2. \tag{4}$$

Suppose $\Delta t \leqslant \frac{2}{3}$. In this case

$$\frac{1+\frac{1}{2}\Delta t}{1-\frac{1}{2}\Delta t} \leqslant 1+\frac{3}{2}\Delta t, \qquad \frac{\Delta t}{1-\frac{1}{2}\Delta t} \leqslant \frac{3}{2}\Delta t.$$

If $\max\|G^{n+\frac{1}{2}}\|^2 \leqslant g$, it follows therefore from (4) that

$$\begin{aligned} \|V^{n+1}\|^2 &\leqslant \left(1+\frac{3}{2}\Delta t\right)^{n+1}\|V^0\|^2 + \frac{\left(1+\frac{3}{2}\Delta t\right)^{n+1}-1}{1+\frac{3}{2}\Delta t-1}\cdot\frac{\Delta t}{1-\frac{1}{2}\Delta t}g \\ &\leqslant e^{\frac{3}{2}(n+1)\Delta t}(\|V^0\|^2+g), \end{aligned} \tag{5}$$

which means that the scheme is stable.

We come to the discussion of convergence. The truncation error of this scheme is $O(\Delta t^2)+O(\Delta x^2)$, so the exact solution satisfies the relation

$$\begin{aligned} &i\frac{U_j^{n+1}-U_j^n}{\Delta t} + \frac{1}{2}[(A_{j+\frac{1}{2}}U_{jx}^{n+1})_{\bar{x}} + (A_{j+\frac{1}{2}}U_{jx}^n)_{\bar{x}}] + \frac{1}{4}\beta_j[q(|U_j^{n+1}|^2)+q(|U_j^n|^2)] \\ &\cdot(U_j^{n+1}+U_j^n) + \frac{1}{2}F_j^{n+\frac{1}{2}}\cdot(U_j^{n+1}+U_j^n) = G_j^{n+\frac{1}{2}} + O_j^n(\Delta t^2) + O_j^n(\Delta x^2). \end{aligned}$$

Thus the error $\varepsilon_j^n \equiv U_j^n - V_j^n$ satisfies the equation

$$\begin{aligned} &i\frac{\varepsilon_j^{n+1}-\varepsilon_j^n}{\Delta t} + \frac{1}{2}[(A_{j+\frac{1}{2}}\varepsilon_{jx}^{n+1})_{\bar{x}} + (A_{j+\frac{1}{2}}\varepsilon_{jx}^n)_{\bar{x}}] + \frac{\beta_j}{4}[q(|U_j^{n+1}|^2) \\ &+q(|U_j^n|^2) - q(|V_j^{n+1}|^2) - q(|V_j^n|^2)](U_j^{n+1}+U_j^n) \\ &+\frac{\beta_j}{4}(q(|V_j^{n+1}|^2)+q(|V_j^n|^2))(\varepsilon_j^{n+1}+\varepsilon_j^n) \\ &+\frac{1}{2}F_j^{n+\frac{1}{2}}(\varepsilon_j^{n+1}+\varepsilon_j^n) = O_j^n(\Delta t^2)+O_j^n(\Delta x^2). \end{aligned} \tag{6}$$

For ε_j certain formulas similar to (3) still hold. Subtracting the inner product of $\bar{\varepsilon}_j^{n+1} + \bar{\varepsilon}_j^n$ and the conjugate equation of (6) from that of $\varepsilon_j^{n+1}+\varepsilon_j^n$ and (6), and summing up these differences from $j=1$ to $J-1$, we obtain

$$\begin{aligned} &\frac{1}{\Delta t}\sum_{j=1}^{J-1}(|\varepsilon_j^{n+1}|^2 - |\varepsilon_j^n|^2) + \sum_{j=1}^{J-1}\Big[\frac{\beta_j}{4}(q(|U_j^{n+1}|^2)+q(|U_j^n|^2) \\ &\quad -q(|V_j^{n+1}|^2)-q(|V_j^n|^2))\cdot\mathrm{Im}(U_j^{n+1}+U_j^n,\ \varepsilon_j^{n+1}+\varepsilon_j^n)\Big] \\ &= \sum_{j=1}^{J-1}\mathrm{Im}(O_j^n(\Delta t^2)+O_j^n(\Delta x^2),\ (\varepsilon_j^{n+1}+\varepsilon_j^n)). \end{aligned} \tag{7}$$

When $q(|U|^2)=|U|^2$, $\varkappa(1-e^{-|U|^2})$, $|U|^2/(1+|U|^2)$ or $\ln(1+|U|^2)$, $|q'|$ is bounded for any $|U|^2$. Suppose $|q'|$ is bounded by Q. In this case we have the relation

$$
\begin{aligned}
|q(|U_j^n|^2)-q(|V_j^n|^2)| &= |q'(\xi_j^n)(|U_j^n|^2-|V_j^n|^2)| \\
&\leqslant Q(|U_j^n|+|V_j^n|)\,||U_j^n|-|V_j^n||\leqslant Q(2|U_j^n|+||U_j^n|-|V_j^n||)\,||U_j^n|-|V_j^n|| \\
&\leqslant Q(2|U_j^n|+|\varepsilon_j^n|)\,|\varepsilon_j^n|,
\end{aligned}
$$

where $\xi_j^n=|U_j^n|^2+\theta(|V_j^n|^2-|U_j^n|^2),\ 0<\theta<1.$

Therefore, if $\max\{|U_j^{n+1}|,\ |U_j^n|\}\leqslant W$ and $|\beta_j|\leqslant\widetilde{B}$, we get

$$
\begin{aligned}
&\Big|\sum_{j=1}^{J-1}\frac{\beta_j}{4}(q(|U_j^{n+1}|^2)+q(|U_j^n|^2)-q(|V_j^{n+1}|^2)-q(|V_j^n|^2))\,\mathrm{Im}(U_j^{n+1}+U_j^n,\ \varepsilon_j^{n+1}+\varepsilon_j^n)\Big| \\
&\leqslant\sum_{j=1}^{J-1}\frac{\widetilde{B}Q}{4}[(2W+|\varepsilon_j^{n+1}|)|\varepsilon_j^{n+1}|+(2W+|\varepsilon_j^n|)|\varepsilon_j^n|]\cdot 2W(|\varepsilon_j^{n+1}|+|\varepsilon_j^n|) \\
&\leqslant\frac{\widetilde{B}QW}{2}\max\{4W,\ 1\}\sum_{j=1}^{J-1}(|\varepsilon_j^{n+1}|^3+|\varepsilon_j^{n+1}|^2|\varepsilon_j^n|+|\varepsilon_j^{n+1}|^2 \\
&\quad+|\varepsilon_j^{n+1}||\varepsilon_j^n|+|\varepsilon_j^n|^2+|\varepsilon_j^n|^2|\varepsilon_j^{n+1}|+|\varepsilon_j^n|^3).
\end{aligned}
\tag{8}
$$

Because $|\varepsilon_j^{n+1}|^3\leqslant\frac{1}{2}(|\varepsilon_j^{n+1}|^2+|\varepsilon_j^{n+1}|^4)$, $|\varepsilon_j^{n+1}|^2|\varepsilon_j^n|\leqslant\frac{1}{2}(|\varepsilon_j^{n+1}|^4+|\varepsilon_j^n|^2)$ etc., it follows from (7) and (8) that

$$
\frac{1}{\Delta t}\sum_{j=1}^{J-1}(|\varepsilon_j^{n+1}|^2-|\varepsilon_j^n|^2)\leqslant k\sum_{j=1}^{J-1}(|\varepsilon_j^{n+1}|^4+|\varepsilon_j^{n+1}|^2+|\varepsilon_j^n|^4+|\varepsilon_j^n|^2)+\frac{c}{\Delta x}(\Delta t^4+\Delta x^4),
$$

where k and c are two positive constants. Obviously, we can further obtain from the above

$$
\|\varepsilon^{n+1}\|^2-\|\varepsilon^n\|^2\leqslant k\Delta t\Big(\|\varepsilon^n\|^2+\frac{1}{\Delta x}\|\varepsilon^n\|^4+\|\varepsilon^{n+1}\|^2+\frac{1}{\Delta x}\|\varepsilon^{n+1}\|^4\Big)+c\Delta t(\Delta t^4+\Delta x^4).
\tag{9}
$$

Moving all terms on $\|\varepsilon^{n+1}\|$ to the left hand side and all terms on $\|\varepsilon^n\|$ to the right hand side, and denoting $\Big(1-k\Delta t-\frac{k\Delta t}{\Delta x}\|\varepsilon^n\|^2\Big)\|\varepsilon^n\|^2$ by y^n, we have

$$
\begin{aligned}
y^{n+1}&=\Big(1-k\Delta t-\frac{k\Delta t}{\Delta x}\|\varepsilon^{n+1}\|^2\Big)\|\varepsilon^{n+1}\|^2 \\
&\leqslant\Big(1+k\Delta t+\frac{k\Delta t}{\Delta x}\|\varepsilon^n\|^2\Big)\|\varepsilon^n\|^2+c\Delta t(\Delta t^4+\Delta x^4) \\
&=\frac{1+k\Delta t+(k\Delta t/\Delta x)\|\varepsilon^n\|^2}{1-k\Delta t-(k\Delta t/\Delta x)\|\varepsilon^n\|^2}\,y^n+c\Delta t(\Delta t^4+\Delta x^4).
\end{aligned}
\tag{10}
$$

When $\|\varepsilon^n\|^2=O(\Delta x)$ and Δt is sufficiently small, (10) can be rewritten as

$$
y^{n+1}\leqslant(1+\alpha\Delta t)y^n+c\Delta t(\Delta t^4+\Delta x^4),
\tag{11}
$$

where α is a positive constant.

Before going to further proof, we make two notes. The first is that when $\|U\|$ is bounded, there exists a constant c_4 independent of Δt and Δx, such that

$$
\|\varepsilon^{n+1}\|^2\leqslant c_4.
\tag{12}
$$

Because $\|U\|^2$ is bounded and

$$
\|\varepsilon^{n+1}\|^2\leqslant(\|V^{n+1}\|+\|U^{n+1}\|)^2\leqslant 2(\|V^{n+1}\|^2+\|U^{n+1}\|^2),
$$

(12) follows immediately from (5).

The second is that if $a>0$, $b>0$, $c>0$ and $\frac{4ac}{b^2}<1$, then

$$x > \frac{b}{a} - \frac{2c}{b} \tag{13}$$

or

$$x < \frac{2c}{b}, \tag{14}$$

when $-ax^2+bx-c\leqslant 0$.

It is easy to prove this conclusion. Because of $b^2-4ac>0$, the equation $-ax^2+bx-c=0$ has two real roots. Suppose the two roots are x_1 and x_2, and $x_1<x_2$. Noticing $-a<0$, we see that if and only if $x\leqslant x_1$ or $x\geqslant x_2$, the inequality $-ax^2+bx-c\leqslant 0$ holds. Moreover,

$$x_1=\frac{b-\sqrt{b^2-4ac}}{2a}=\frac{b-b\sqrt{1-4ac/b^2}}{2a}<\frac{b-b(1-4ac/b^2)}{2a}=\frac{2c}{b},$$

$$x_2=\frac{b+\sqrt{b^2-4ac}}{2a}=\frac{b+b\sqrt{1-4ac/b^2}}{2a}>\frac{b+b(1-4ac/b^2)}{2a}=\frac{b}{a}-\frac{2c}{b}.$$

Therefore (13) or (14) must hold.

From what we have proved, we obtain the following conclusion. If $\Delta t=O(\Delta x^{\frac{1}{2}+\delta})$, $\delta>0$ and $\Delta t, \Delta x$ are so small that $\frac{1}{2}<1-k\Delta t-\frac{k\Delta t}{\Delta x^{1/2}}$, and if $y^{n+1}<c_5\ (\Delta t^4+\Delta x^4)$ and $\|\varepsilon^n\|^2\leqslant 4c_5(\Delta t^4+\Delta x^4)$, then $\|\varepsilon^{n+1}\|^2\leqslant 4c_5(\Delta t^4+\Delta x^4)$ must hold.

In fact, if $y^{n+1}=\left(1-k\Delta t-\frac{k\Delta t}{\Delta x}\|\varepsilon^{n+1}\|^2\right)\|\varepsilon^{n+1}\|^2\leqslant c_5(\Delta t^4+\Delta x^4)$, we have from (13) and (14)

$$\|\varepsilon^{n+1}\|^2<\frac{2c_5(\Delta t^4+\Delta x^4)}{1-k\Delta t}<4c_5(\Delta t^4+\Delta x^4) \tag{15}$$

or

$$\|\varepsilon^{n+1}\|^2>\frac{1-k\Delta t}{\frac{k\Delta t}{\Delta x}}-\frac{2c_5(\Delta t^4+\Delta x^4)}{1-k\Delta t}>\frac{\Delta x}{2k\Delta t}-4c_5(\Delta t^4+\Delta x^4). \tag{16}$$

Therefore, we only need to prove that the second inequality cannot hold. To do so, we introduce another inequality. Because $|\varepsilon_j^{n+1}|^3\leqslant\frac{1}{2}\left(\frac{1}{\Delta x^{1/2}}|\varepsilon_j^{n+1}|^2+\Delta x^{1/2}|\varepsilon_j^{n+1}|^4\right)$ etc, we can obtain from (7) and (8) the following inequality similar to (9)

$$\|\varepsilon^{n+1}\|^2-\|\varepsilon^n\|^2\leqslant k\Delta t\Big(\|\varepsilon^n\|^2+\frac{1}{\Delta x^{1/2}}\|\varepsilon^n\|^2+\frac{1}{\Delta x^{1/2}}\|\varepsilon^n\|^4+\|\varepsilon^{n+1}\|^2$$
$$+\frac{1}{\Delta x^{1/2}}\|\varepsilon^{n+1}\|^2+\frac{1}{\Delta x^{1/2}}\|\varepsilon^{n+1}\|^4\Big)+c\Delta t(\Delta t^4+\Delta x^4),$$

i. e.,

$$\left(1-k\Delta t-\frac{k\Delta t}{\Delta x^{1/2}}-\frac{k\Delta t}{\Delta x^{1/2}}\|\varepsilon^{n+1}\|^2\right)\|\varepsilon^{n+1}\|^2\leqslant c_6(\Delta t^4+\Delta x^4), \tag{17}$$

where c_6 is a constant a little greater than $4c_5$. Therefore $\|\varepsilon^{n+1}\|^2$ must satisfy

$$\|\varepsilon^{n+1}\|^2\leqslant\frac{2c_6(\Delta t^4+\Delta x^4)}{1-k\Delta t-\frac{k\Delta t}{\Delta x^{1/2}}}<4c_6(\Delta t^4+\Delta x^4) \tag{18}$$

or

$$\|\varepsilon^{n+1}\|^2 > \frac{1-k\Delta t-\dfrac{k\Delta t}{\Delta x^{1/2}}}{\dfrac{k\Delta t}{\Delta x^{1/2}}} - \frac{2c_6(\Delta t^4+\Delta x^4)}{1-k\Delta t-\dfrac{k\Delta t}{\Delta x^{1/2}}} > \frac{\Delta x^{1/2}}{2k\Delta t} - 4c_6(\Delta t^4+\Delta x^4). \tag{19}$$

We have supposed $\Delta t = O(\Delta x^{\frac{1}{2}+\delta})$, $\delta>0$, so $\dfrac{\Delta x^{1/2}}{2k\Delta t} - 4c_6(\Delta t^4+\Delta x^4) > c_4$ if Δt and Δx is sufficiently small. This means that (19) can be rewritten as

$$\|\varepsilon^{n+1}\|^2 > c_4 \tag{20}$$

when Δt and Δx is sufficiently small.

Therefore $\|\varepsilon^{n+1}\|^2$ must satisfy (15) or (16), (18) or (20), and (12). This means that the only possibility is that $\|\varepsilon^{n+1}\|^2$ satisfies (15).

We can now complete our proof by induction. Owing to $\|\varepsilon^0\|^2=0$, it follows from (11) and the above results that

$$y^1 \leqslant ((1+\alpha\Delta t)^1-1)\frac{c}{\alpha}(\Delta t^4+\Delta x^4) < \frac{c}{\alpha}e^{\alpha T}(\Delta t^4+\Delta x^4),$$

$$\|\varepsilon^1\|^2 \leqslant 4\frac{c}{\alpha}e^{\alpha T}(\Delta t^4+\Delta x^4), \quad T>\Delta t.$$

Suppose for $l=1, 2, \cdots, n$

$$\begin{cases} y^l \leqslant ((1+\alpha\Delta t)^l-1)\dfrac{c}{\alpha}(\Delta t^4+\Delta x^4) < \dfrac{c}{\alpha}e^{\alpha T}(\Delta t^4+\Delta x^4), & (21)\\ \|\varepsilon^l\|^2 \leqslant 4\dfrac{c}{\alpha}e^{\alpha T}(\Delta t^4+\Delta x^4). & (22)\end{cases}$$

We now prove that (21) and (22) hold also for $l=n+1$ if $(n+1)\Delta t\leqslant T$. If (21) and (22) hold for $l=1, 2, \cdots, n$, we obtain from (11)

$$\begin{aligned} y^{n+1} &\leqslant (1+\alpha\Delta t)y^n + c\Delta t(\Delta t^4+\Delta x^4)\\ &\leqslant [(1+\alpha\Delta t)^{n+1}-(1+\alpha\Delta t)+\alpha\Delta t]\frac{c}{\alpha}(\Delta t^4+\Delta x^4)\\ &= ((1+\alpha\Delta t)^{n+1}-1)\frac{c}{\alpha}(\Delta t^4+\Delta x^4) < \frac{c}{\alpha}e^{\alpha T}(\Delta t^4+\Delta x^4). \end{aligned}$$

Thus (21) holds for $l=n+1$. Moreover, we have proved that if (21) holds for $l=n+1$ and (22) holds for $l=n$, then (22) holds for $l=n+1$. Consequently, (22) and (21) must hold for all $(n+1)\Delta t<T$. (22) shows that

$$\|\varepsilon^l\| \leqslant 2\left(\frac{c}{\alpha}e^{\alpha T}\right)^{1/2}\Delta t^2\sqrt{1+\frac{\Delta x^4}{\Delta t^4}} \to 0 \text{ as } \Delta t\to 0.$$

Therefore we have the following conclusion: if $|q'|$ is bounded and $\Delta t=O(\Delta x^{\frac{1}{2}+\delta})$, $\delta>0$, then the approximate solution converges to the exact solution of the partial differential equations with a convergence rate of $O\left(\Delta t^2\sqrt{1+\dfrac{\Delta x^4}{\Delta t^4}}\right)$.

When $\alpha_1\neq\dfrac{1}{2}$, one can prove by the above method that the approximate solution converges to the exact solution if $\Delta t=O(\Delta x^{\frac{1}{2}+\delta})$, $\delta>0$. However, the rate of convergence is $O(\Delta t\sqrt{1+\Delta x^4/\Delta t^2})$ because the truncation error is $O(\Delta t+\Delta x^2)$. When $\alpha_1=0$, $q(|V_j^{n+1}|^2)$ disappears in the equations. Consequently, the convergence can be proved by a simpler method under the weaker condition of $\Delta t=O(\Delta x^{\frac{1}{2}+\delta})$, $\delta\geqslant 0$.

In order to solve system (1), the following two-step difference scheme can also be used. In the first step, the system (1) is approximated by

$$\begin{cases} i\dfrac{V_j^{n+\alpha_1}-V_j^n}{\alpha_1\Delta t}+\dfrac{1}{2}((A_{j+\frac{1}{2}}V_{j\bar{x}}^{n+\alpha_1})_x+(A_{j+\frac{1}{2}}V_{j\bar{x}}^n)_x) \\ \quad +\dfrac{1}{2}\beta_j q(|V_j^n|^2)(V_j^{n+\alpha_1}+V_j^n)+\dfrac{1}{2}F_j^{n+\alpha_1}(V_j^{n+\alpha_1}+V_j^n)=G_j^{n+\alpha_1}, \\ \qquad\qquad j=1,2,\cdots,J-1, \\ V_0^{n+\alpha_1}=V_J^{n+\alpha_1}=0; \end{cases} \tag{23}$$

in the second step, it is approximated by

$$\begin{cases} i\dfrac{V_j^{n+1}-V_j^n}{\Delta t}+\dfrac{1}{2}((A_{j+\frac{1}{2}}V_{j\bar{x}}^{n+1})_x+(A_{j+\frac{1}{2}}V_{j\bar{x}}^n)_x) \\ \quad +\dfrac{1}{2}\beta_j q(|V_j^{n+\alpha_1}|^2)(V_j^{n+1}+V_j^n) \\ \quad +\dfrac{1}{2}F_j^{n+\alpha_1}(V_j^{n+1}+V_j^n)=G_j^{n+\alpha_1}, \quad j=1,2,\cdots,J-1, \\ V_0^{n+1}=V_J^{n+1}=0, \end{cases} \tag{24}$$

where $\alpha_1>0$. Obviously, the truncation error of this scheme is $O(\Delta t^2+\Delta x^2)$ if $\alpha_1=\frac{1}{2}$. The advantage of this scheme is that in order for the truncation error to be $O(\Delta t^2+\Delta x^2)$, only two systems of linear algebraic equations need to be solved. In what follows, we call the scheme with $\alpha_1=\frac{1}{2}$ Scheme D.

Assume $|q'|$ to be bounded again. For Scheme D we have the following result: it is convergent and its rate of convergence is $O\left(\Delta t^2\sqrt{1+\frac{\Delta x^4}{\Delta t^4}}\right)$ when $\Delta t=O(\Delta x^{\frac{1}{4}+\delta})$, $\delta\geq 0$.

It is clear that among ε_j^n, $\varepsilon_j^{n+\frac{1}{2}}$ and ε_j^{n+1}, the relations

$$\begin{aligned} & i\frac{\varepsilon_j^{n+\frac{1}{2}}-\varepsilon_j^n}{\Delta t/2}+\frac{1}{2}((A_{j+\frac{1}{2}}\varepsilon_{j\bar{x}}^{n+\frac{1}{2}})_x+(A_{j+\frac{1}{2}}\varepsilon_{j\bar{x}}^n)_x)+\frac{\beta_j}{2}(q(|U_j^n|^2)-q(|V_j^n|^2)) \\ &\quad \cdot(U_j^{n+\frac{1}{2}}+U_j^n)+\frac{\beta_j}{2}q(|V_j^n|^2)(\varepsilon_j^{n+\frac{1}{2}}+\varepsilon_j^n)+\frac{1}{2}F_j^{n+\frac{1}{2}}(\varepsilon_j^{n+\frac{1}{2}}+\varepsilon_j^n) \\ &\quad =O_j^n(\Delta t)+O_j^n(\Delta x^2) \end{aligned}$$

and

$$\begin{aligned} & i\frac{\varepsilon_j^{n+1}-\varepsilon_j^n}{\Delta t}+\frac{1}{2}((A_{j+\frac{1}{2}}\varepsilon_{j\bar{x}}^{n+1})_x+(A_{j+\frac{1}{2}}\varepsilon_{j\bar{x}}^n)_x)+\frac{\beta_j}{2}(q(|U_j^{n+\frac{1}{2}}|^2)-q(|V_j^{n+\frac{1}{2}}|^2)) \\ &\quad \cdot(U_j^{n+1}+U_j^n)+\frac{\beta_j}{2}q(|V_j^{n+\frac{1}{2}}|^2)(\varepsilon_j^{n+1}+\varepsilon_j^n)+\frac{1}{2}F_j^{n+\frac{1}{2}}(\varepsilon_j^{n+1}+\varepsilon_j^n) \\ &\quad =O_j^{n+\frac{1}{2}}(\Delta t^2)+O_j^{n+\frac{1}{2}}(\Delta x^2) \end{aligned}$$

hold.

Moreover, we can obtain the following relations similar to (7) using the method above:

$$\begin{aligned} & \frac{2}{\Delta t}\sum_{j=1}^{J-1}(|\varepsilon_j^{n+\frac{1}{2}}|^2-|\varepsilon_j^n|^2)+\sum_{j=1}^{J-1}\frac{\beta_j}{2}(q(|U_j^n|^2)-q(|V_j^n|^2))\operatorname{Im}(U_j^{n+\frac{1}{2}}+U_j^n,\ \varepsilon_j^{n+\frac{1}{2}}+\varepsilon_j^n) \\ &\quad =\sum_{j=1}^{J-1}\operatorname{Im}(O_j^n(\Delta t)+O_j^n(\Delta x^2),\ \varepsilon_j^{n+\frac{1}{2}}+\varepsilon_j^n) \end{aligned} \tag{25}$$

and

$$\frac{1}{\Delta t}\sum_{j=1}^{J-1}(|\varepsilon_j^{n+1}|^2-|\varepsilon_j^n|^2)+\sum_{j=1}^{J-1}\frac{\beta_j}{2}(q(|U_j^{n+\frac{1}{2}}|^2)-q(|V_j^{n+\frac{1}{2}}|^2))$$
$$\cdot\operatorname{Im}(U_j^{n+1}+U_j^n,\ \varepsilon_j^{n+1}+\varepsilon_j^n)=\sum_{j=1}^{J-1}\operatorname{Im}(O_j^{n+\frac{1}{2}}(\Delta t^2)+O_j^{n+\frac{1}{2}}(\Delta x^2),\ \varepsilon_j^{n+1}+\varepsilon_j^n). \tag{26}$$

Because of

$$\left|\sum_{j=1}^{J-1}\frac{\beta_j}{2}(q(|U_j^n|^2)-q(|V_j^n|^2))\operatorname{Im}(U_j^{n+\frac{1}{2}}+U_j^n,\ \varepsilon_j^{n+\frac{1}{2}}+\varepsilon_j^n)\right|$$
$$\leqslant\sum_{j=1}^{J-1}\tilde{B}QW(2W+|\varepsilon_j^n|)|\varepsilon_j^n|(|\varepsilon_j^{n+\frac{1}{2}}|+|\varepsilon_j^n|)$$
$$\leqslant k\sum_{j=1}^{J-1}(|\varepsilon_j^n|^2+|\varepsilon_j^n|^4+|\varepsilon_j^{n+\frac{1}{2}}|^2)$$

and

$$\left|\sum_{j=1}^{J-1}\operatorname{Im}(O_j^n(\Delta t)+O_j^n(\Delta x^2),\ \varepsilon_j^{n+\frac{1}{2}}+\varepsilon_j^n)\right|\leqslant\sum_{j=1}^{J-1}c(\Delta t+\Delta x^2)(|\varepsilon_j^{n+\frac{1}{2}}|+|\varepsilon_j^n|)$$
$$\leqslant\sum_{j=1}^{J-1}\left[c^2(\Delta t^3+\Delta t\Delta x^4)+\frac{1}{\Delta t}(|\varepsilon_j^{n+\frac{1}{2}}|^2+|\varepsilon_j^n|^2)\right],$$

k and c being constants, it follows from (25) that

$$\|\varepsilon^{n+\frac{1}{2}}\|^2-\|\varepsilon^n\|^2\leqslant\frac{k\Delta t}{2}\left(\|\varepsilon^n\|^2+\frac{1}{\Delta x}\|\varepsilon^n\|^4+\|\varepsilon^{n+\frac{1}{2}}\|^2\right)$$
$$+\frac{1}{2}(\|\varepsilon^{n+\frac{1}{2}}\|^2+\|\varepsilon^n\|^2)+\frac{c^2}{2}(\Delta t^4+\Delta t^2\Delta x^4),$$

i. e.,

$$\|\varepsilon^{n+\frac{1}{2}}\|^2\leqslant\frac{\frac{3}{2}+\frac{k\Delta t}{2}}{\frac{1}{2}-\frac{k\Delta t}{2}}\|\varepsilon^n\|^2+\frac{\frac{k\Delta t}{2\Delta x}}{\frac{1}{2}-\frac{k\Delta t}{2}}\|\varepsilon^n\|^4+\frac{c^2}{2\cdot\left(\frac{1}{2}-\frac{k\Delta t}{2}\right)}(\Delta t^4+\Delta t^2\Delta x^4).$$

Consequently, if $\|\varepsilon^n\|^2=O(\Delta x)$ and Δt is small enough, we have

$$\|\varepsilon^{n+\frac{1}{2}}\|^2\leqslant 3.5\|\varepsilon^n\|^2+2c^2(\Delta t^4+\Delta t^2\Delta x^4). \tag{27}$$

Similarly, because of the following inequalities

$$\left|\sum_{j=1}^{J-1}\frac{\beta_j}{2}(q(|U_j^{n+\frac{1}{2}}|^2)-q(|V_j^{n+\frac{1}{2}}|^2))\operatorname{Im}(U_j^{n+1}+U_j^n,\ \varepsilon_j^{n+1}+\varepsilon_j^n)\right|$$
$$\leqslant\sum_{j=1}^{J-1}\tilde{B}QW(2W+|\varepsilon_j^{n+\frac{1}{2}}|)|\varepsilon_j^{n+\frac{1}{2}}|(|\varepsilon_j^{n+1}|+|\varepsilon_j^n|)$$
$$\leqslant k\sum_{j=1}^{J-1}(|\varepsilon_j^n|^2+|\varepsilon_j^{n+\frac{1}{2}}|^2+|\varepsilon_j^{n+1}|^2+|\varepsilon_j^{n+\frac{1}{2}}|^4)$$

and
$$\left|\sum_{j=1}^{J-1}\operatorname{Im}(O_j^{n+\frac{1}{2}}(\Delta t^2)+O_j^{n+\frac{1}{2}}(\Delta x^2),\ \varepsilon_j^{n+1}+\varepsilon_j^n)\right|$$
$$\leqslant\sum_{j=1}^{J-1}[c^2(\Delta t^4+\Delta x^4)+|\varepsilon_j^{n+1}|^2+|\varepsilon_j^n|^2],$$

where k and c are constants, it follows from (26) that

$$\|\varepsilon^{n+1}\|^2-\|\varepsilon^n\|^2\leqslant(k+1)\Delta t\left(\|\varepsilon^n\|^2+\|\varepsilon^{n+1}\|^2+\|\varepsilon^{n+\frac{1}{2}}\|^2+\frac{1}{\Delta x}\|\varepsilon^{n+\frac{1}{2}}\|^4\right)+c^2\Delta t(\Delta t^4+\Delta x^4),$$

i. e.,

$$\|\varepsilon^{n+1}\|^2\leqslant\frac{1}{1-(k+1)\Delta t}\Big[(1+(k+1)\Delta t)\|\varepsilon^n\|^2$$

$$+\Big(1+\frac{1}{\Delta x}\|\varepsilon^{n+\frac{1}{2}}\|^2\Big)(k+1)\Delta t\|\varepsilon^{n+\frac{1}{2}}\|^2+c^2\Delta t(\Delta t^4+\Delta x^4)\Big]. \tag{28}$$

Moreover, it follows from (27) that $\|\varepsilon^{n+\frac{1}{2}}\|^2=O(\Delta x)$ when

$$\|\varepsilon^n\|^2=O(\Delta x),\ \Delta t=O(\Delta x^{\frac{1}{4}+\delta}),\ \delta\geqslant0. \tag{29}$$

Therefore, from (28) and (27) we can assert that there exist two constants α and $\tilde{c}$ such that

$$\|\varepsilon^{n+1}\|^2\leqslant(1+\alpha\Delta t)\|\varepsilon^n\|^2+\tilde{c}\Delta t(\Delta t^4+\Delta x^4). \tag{30}$$

From (30) we can prove the relation

$$\|\varepsilon^{n+1}\|^2\leqslant((1+\alpha\Delta t)^{n+1}-1)\frac{\tilde{c}}{\alpha}(\Delta t^4+\Delta x^4)\leqslant\frac{\tilde{c}}{\alpha}e^{\alpha T}(\Delta t^4+\Delta x^4),\ (n+1)\Delta t\leqslant T. \tag{31}$$

In fact, if $\Delta t=O(\Delta x^{\frac{1}{4}+\delta})$, $\delta\geqslant0$, and (31) holds, then the relation $\|\varepsilon^{n+1}\|^2=O(\Delta x)$ holds. Consequently, it is easy to derive (31) from (30) by induction.

We have therefore proved that Scheme D is convergent and its rate of convergence is $O\left(\Delta t^2\sqrt{1+\frac{\Delta x^4}{\Delta t^4}}\right)$ in L_2.

For the case $\alpha_1\neq\frac{1}{2}$, the convergence of scheme (23)—(24) can be proved by the same method. However, the convergence condition for $\alpha_1\neq\frac{1}{2}$ is $\Delta t=O(\Delta x^{\frac{1}{2}+\delta})$, $\delta\geqslant0$, and the rate of convergence is $O\left(\Delta t\sqrt{1+\frac{\Delta x^4}{\Delta t^2}}\right)$.

Remark. If $q(|U|^2)=|U|^{2p}$, p being an integer greater than 1, these results on Schemes C and D can still be proved by a similar method. However, the proof will become more tedious. Because the space here is limited, we omit the details. For Schemes A and B, the convergence can also be proved by the method in this paper if the condition is strengthened. For example, if $q(|U|^2)=|U|^4$, then the convergence can be proved by the method here under the condition $\Delta t=O(\Delta x^{2+\delta})$, $\delta>0$. It is likely that this kind of result can be improved, but to do so, a new method probably has to be found.

Convergence of Iteration

When implicit schemes are used, a system of equations has to be solved in each step. For Schemes A and B, it is a non-linear system, so an iteration method is needed for its solution. In what follows, taking Scheme A as an example, we discuss some problems on iteration methods. The case where U is a scalar is first considered. We assume $a\geqslant|A(x)|$, and for the moment, we also suppose the sign of $A(x)$ does not change.

Obviously, Scheme A can be rewritten as

$$\begin{cases} a_jV_{j-1}+b_jV_j+c_jV_{j+1}=d_j+e_j(V_j), & j=1,2,\cdots,J-1,\\ V_0=V_J=0. \end{cases} \tag{32}$$

Here V_j stands for V_j^{n+1}, and

$$\begin{cases} a_j = \dfrac{\Delta t A_{j-\frac{1}{2}}}{2\Delta x^2}, \quad b_j = i - \dfrac{\Delta t\,(A_{j+\frac{1}{2}} + A_{j-\frac{1}{2}})}{2\Delta x^2} + \Delta t\Big(\dfrac{\beta_j}{4}\, q(|V_j^n|^2) + \dfrac{1}{2}\, F_j^{n+\frac{1}{2}}\Big), \\ c_j = \dfrac{\Delta t A_{j+\frac{1}{2}}}{2\Delta x^2}, \quad d_j = iV_j^n - \dfrac{\Delta t}{2}\,(A_{j+\frac{1}{2}} V_{jx}^n)_{\bar{x}} \\ \qquad\qquad - \Delta t\Big(\dfrac{\beta_j}{4}\, q(|V_j^n|^2) + \dfrac{1}{2}\, F_j^{n+\frac{1}{2}}\Big)V_j^n + \Delta t G_j^{n+\frac{1}{2}}, \\ e_j(V_j) = -\Delta t\, \dfrac{\beta_j}{4}\, q(|V_j|^2)\,(V_j + V_j^n). \end{cases} \tag{33}$$

This system can be solved by the following iteration method:

$$\begin{cases} a_j V_{j-1}^{(s)} + b_j V_j^{(s)} + c_j V_{j+1}^{(s)} = d_j + e_j(V_j^{(s-1)}), \quad J = 1, 2, \cdots, J-1, \\ V_0^{(s)} = V_J^{(s)} = 0, \quad s = 1, 2, \cdots, \end{cases} \tag{34}$$

where we assume $V_j^{(0)} = V_j^n$. From (34) we can easily obtain the following equations for $\varepsilon_j^{(s)} = V_j^{(s)} - V_j^{(s-1)}$:

$$\begin{cases} a_j \varepsilon_{j-1}^{(s)} + b_j \varepsilon_j^{(s)} + c_j \varepsilon_{j+1}^{(s)} = e_j(V_j^{(s-1)}) - e_j(V_j^{(s-2)}), \quad j = 1, 2, \cdots, J-1, \\ \varepsilon_0^{(s)} = \varepsilon_J^{(s)} = 0, \quad s = 2, 3, \cdots. \end{cases} \tag{35}$$

For convenience, we express the system in a matrix form

$$\varepsilon^{(s)} = B^{-1} E^{(s-1)}, \quad s = 2, 3, \cdots, \tag{36}$$

where

$$B = \begin{pmatrix} b_1 & c_1 & & \\ a_2 & b_2 & c_2 & \\ & \ddots & \ddots & \ddots \\ & & a_{J-1} & b_{J-1} \end{pmatrix}, \quad \varepsilon^{(s)} = \begin{pmatrix} \varepsilon_1^{(s)} \\ \varepsilon_2^{(s)} \\ \vdots \\ \varepsilon_{J-1}^{(s)} \end{pmatrix}, \quad E^{(s-1)} = \begin{pmatrix} e_1(V_1^{(s-1)}) - e_1(V_1^{(s-2)}) \\ \vdots \\ e_{J-1}(V_{J-1}^{(s-1)}) - e_{J-1}(V_{J-1}^{(s-2)}) \end{pmatrix}.$$

In the following we use the L_∞ norm. It is easy to prove that if $\min_j\{|b_j| - |a_j| - |c_j|\} > 0$, then B^{-1} exists and

$$\|B^{-1}\| \leqslant \frac{1}{\min_j\{|b_j| - |a_j| - |c_j|\}}. \tag{37}$$

Clearly, for any two given positive numbers H and Δx, we can find a positive number Δt such that

$$2\,\frac{\Delta t^2}{\Delta x^2}\, aH + (H\Delta t)^2 \leqslant 1. \tag{38}$$

If (38) holds, from $a \geqslant |A_{j+\frac{1}{2}}|$, $j = 0, 1, \cdots, J-1$, it follows that

$$\begin{aligned} &\sqrt{1 + \Big(\frac{\Delta t}{2\Delta x^2}\,(A_{j+\frac{1}{2}} + A_{j-\frac{1}{2}})\Big)^2} - \frac{\Delta t}{2\Delta x^2}\,|A_{j+\frac{1}{2}} + A_{j-\frac{1}{2}}| \\ &\geqslant \sqrt{(H\Delta t)^2 + \frac{\Delta t^2}{\Delta x^2}\, H\,|A_{j+\frac{1}{2}} + A_{j-\frac{1}{2}}| + \Big(\frac{\Delta t}{2\Delta x^2}\,(A_{j+\frac{1}{2}} + A_{j-\frac{1}{2}})\Big)^2} \\ &\quad - \frac{\Delta t}{2\Delta x^2}\,|A_{j+\frac{1}{2}} + A_{j-\frac{1}{2}}| \geqslant H\Delta t. \end{aligned}$$

Therefore we have

$$|b_j|-|a_j|-|c_j|\geqslant\sqrt{1+\left(\frac{\Delta t}{2\Delta x^2}\,(A_{j+\frac{1}{2}}+A_{j-\frac{1}{2}})\right)^2}-\frac{\Delta t}{2\Delta x^2}\,|A_{j+\frac{1}{2}}+A_{j-\frac{1}{2}}|$$
$$-\Delta t\left|\frac{\beta_j}{4}\,q(|V_j^n|^2)+\frac{1}{2}\,F_j^{n+\frac{1}{2}}\right|\geqslant(H-\widetilde{Q})\Delta t.$$

Here we assume

$$\widetilde{Q}=\max_j\left\{\left|\frac{\beta_j}{4}\,q(|V_j^n|^2)+\frac{1}{2}\,F_j^{n+\frac{1}{2}}\right|\right\},$$

and assume that the sign of $A(x)$ does not change, which implies $|A_{j+\frac{1}{2}}|+|A_{j-\frac{1}{2}}|=|A_{j+\frac{1}{2}}+A_{j-\frac{1}{2}}|$. That is, if Δt satisfies (38) and H satisfies

$$H-\widetilde{Q}>0, \tag{39}$$

then B^{-1} exists and

$$\|B^{-1}\|\leqslant\frac{1}{(H-\widetilde{Q})\Delta t}. \tag{40}$$

We now turn to estimate the norm of $E^{(s-1)}$. Because of

$$|e_j(V_j^{(s-1)})-e_j(V_j^{(s-2)})|\leqslant\frac{\widetilde{B}\Delta t}{4}|q(|V_j^{(s-1)}|^2)(V_j^{(s-1)}+V_j^n)$$
$$-q(|V_j^{(s-2)}|^2)(V_j^{(s-2)}+V_j^n)|$$
$$\leqslant\frac{\widetilde{B}\Delta t}{4}(|q(|V_j^{(s-1)}|^2)|+|q'(\xi_j^{(s-1)})|$$
$$\cdot|V_j^{(s-2)}+V_j^n|(|V_j^{(s-1)}|+|V_j^{(s-2)}|))|\varepsilon_j^{(s-1)}|,$$

where $\xi_j^{(s-1)}=|V_j^{(s-1)}|^2+\theta(|V_j^{(s-2)}|^2-|V_j^{(s-1)}|^2)$, $0<\theta<1$, if there exists a constant $\tilde{c}_1$ independent of s, such that

$$|V_j^{(s-1)}|<\tilde{c}_1,\ |V_j^{(s-2)}|<\tilde{c}_1,\quad j=1,2,\cdots,J-1, \tag{41}$$

then there exists a constant $\tilde{c}_2$ dependent on $\tilde{c}_1$ but independent of s, such that

$$\|E^{(s-1)}\|\leqslant\tilde{c}_2\Delta t\|\varepsilon^{(s-1)}\|. \tag{42}$$

Consequently, when (38), (39) and (41) hold, we can derive

$$\|\varepsilon^{(s)}\|\leqslant\|B^{-1}\|\,\|E^{(s-1)}\|\leqslant\frac{\tilde{c}_2}{H-\widetilde{Q}}\|\varepsilon^{(s-1)}\|.$$

It is easy to see that there exists a constant $\tilde{c}_1^*$ which depends on β_j^n, V_j^n, $A_{j+\frac{1}{2}}$, $F_j^{n+\frac{1}{2}}$, $G_j^{n+\frac{1}{2}}$, Δx but not on Δt, such that

$$\|\varepsilon^{(1)}\|\leqslant\frac{\tilde{c}_1^*}{2(H-\widetilde{Q})},\quad\|V^{(0)}\|\leqslant\frac{\tilde{c}_1^*}{2}.$$

Therefore, if

$$c\equiv\frac{\tilde{c}_2}{H-\widetilde{Q}}<1, \tag{43}$$

we have

$$|V_j^{(s-1)}|<|V_j^{(s-1)}-V_j^{(s-2)}|+|V_j^{(s-2)}-V_j^{(s-3)}|+\cdots+|V_j^{(1)}-V_j^{(0)}|+|V_j^{(0)}|$$
$$\leqslant\|\varepsilon^{(s-1)}\|+\|\varepsilon^{(s-2)}\|+\cdots+\|\varepsilon^{(1)}\|+\|V^{(0)}\|$$
$$\leqslant\frac{\|\varepsilon^{(1)}\|}{1-c}+\|V^{(0)}\|\leqslant\frac{\tilde{c}_1^*}{2}\left(\frac{1}{(1-c)(H-\widetilde{Q})}+1\right). \tag{44}$$

We can now complete our proof of convergence. The conclusion we shall prove is that if

$$\Delta t \leqslant \frac{\Delta x}{[(\tilde{c}_2^* + \tilde{Q} + 1)(2a + (\tilde{c}_2^* + \tilde{Q} + 1)\Delta x^2)]^{1/2}}, \tag{45}$$

the iteration (34) converges. Here the $\tilde{c}_2^*$ is the $\tilde{c}_2$ corresponding to $\tilde{c}_1 = \tilde{c}_1^*$. In fact, the condition (45) means that for $H = \tilde{c}_2^* + \tilde{Q} + 1$, (38) holds. Thus we have

$$\|B^{-1}\| \leqslant \frac{1}{(\tilde{c}_2^* + 1)\Delta t}.$$

Moreover, it is easy to see that $|V_j^{(1)}| \leqslant \tilde{c}_1^*$, $|V_j^{(0)}| < \tilde{c}_1^*$, and further

$$\|E^{(1)}\| \leqslant \tilde{c}_2^* \Delta t \|\varepsilon^{(1)}\|.$$

Consequently, the following inequality holds:

$$\|\varepsilon^{(2)}\| \leqslant \frac{\tilde{c}_2^*}{\tilde{c}_2^* + 1} \|\varepsilon^{(1)}\|. \tag{46}$$

In what follows, we prove by induction that for any s,

$$\|\varepsilon^{(s)}\| \leqslant c \|\varepsilon^{(s-1)}\|, \quad c = \frac{\tilde{c}_2^*}{\tilde{c}_2^* + 1} < 1. \tag{47}$$

(46) means that (47) holds for $s = 2$. We now suppose (47) holds for $s = 2, \cdots, m-1$, and prove that (47) also holds for $s = m$. Since (47) holds for $s = 2, \cdots, m-1$, (44) holds for $s = m$ and $m-1$, i. e.,

$$|V_j^{(m-1)}| \leqslant \frac{\tilde{c}_1^*}{2} \left(\frac{1}{\left(1 - \frac{\tilde{c}_2^*}{\tilde{c}_2^* + 1}\right)(\tilde{c}_2^* + 1)} + 1 \right) = \tilde{c}_1^*,$$

and

$$|V_j^{(m-2)}| \leqslant \tilde{c}_1^*.$$

Therefore for $s = m$, $\|E^{(s-1)}\| \leqslant \tilde{c}_2^* \Delta t \|\varepsilon^{(s-1)}\|$ and

$$\|\varepsilon^{(s)}\| \leqslant \|B^{-1}\| \|E^{(s-1)}\| \leqslant \frac{\tilde{c}_2^*}{\tilde{c}_2^* + 1} \|\varepsilon^{(s-1)}\|,$$

i. e., for $s = m$, (47) holds. Consequently, (47) holds for any s, which implies that the iteration is convergent.

When $A(x)$ changes its sign somewhere, the relation $|A_{j+\frac{1}{2}}| + |A_{j-\frac{1}{2}}| = |A_{j+\frac{1}{2}} + A_{j-\frac{1}{2}}|$ will not hold at the places where $A(x)$ changes its sign, which means that (40) cannot be derived from (38) alone. But at these places, $|A_{j+\frac{1}{2}}| \leqslant a'\theta\Delta x$ and $|A_{j-\frac{1}{2}}| \leqslant a'(1-\theta)\Delta x$, $0 \leqslant \theta \leqslant 1$ if the derivative of $A(x)$ satisfies the relation $|A'(x)| \leqslant a'$. In this case we have $|A_{j+\frac{1}{2}}| + |A_{j-\frac{1}{2}}| \leqslant a'\Delta x$, and further

$$|b_j| - |a_j| - |c_j| \geqslant 1 - \frac{a'\Delta t}{2\Delta x} - \tilde{Q}\Delta t.$$

Therefore when (38) holds and

$$1 - \frac{a'\Delta t}{2\Delta x} \geqslant H\Delta t, \quad \text{i. e.,} \quad \Delta t \leqslant \frac{2\Delta x}{a' + 2H\Delta x}, \tag{48}$$

we again have $\|B^{-1}\| \leqslant \frac{1}{(H - \tilde{Q})\Delta t}$. From this conclusion, we can see that if

$$\Delta t \leqslant \min\left\{ \frac{\Delta x}{[(\tilde{c}_2^* + \tilde{Q} + 1)(2a + (\tilde{c}_2^* + \tilde{Q} + 1)\Delta x^2)]^{\frac{1}{2}}}, \frac{2\Delta x}{a' + 2(\tilde{c}_2^* + \tilde{Q} + 1)\Delta x} \right\}, \tag{49}$$

the iteration is convergent no matter whether the sign of $A(x)$ is constant or not.

If U is a vector, and $A(x)$ is a real diagonal matrix whose elements and the derivatives of whose elements are bounded, the same result can be obtained by quite a similar method. The main difference is that in this case the matrix is not tridiagonal. However, the number of elements in each row is finite and except for at most three elements, all elements are quantities of $O(\Delta t)$, so we can obtain the following estimates similar to (37) and (40):

$$\|B^{-1}\|\leqslant\frac{1}{\min\limits_{j}\{|b_j|-|a_j|-|c_j|-\tilde{c}\Delta t\}}$$

and

$$\|B^{-1}\|\leqslant\frac{1}{(H-\tilde{Q}_1)\Delta t},$$

where $\tilde{c}$ and $\tilde{Q}_1$ are constants. Moreover, in this case we can also find a constant $\tilde{c}_2$ such that (42) holds. Therefore we can find a constant c, such that the iteration is convergent when $\Delta t\leqslant c\Delta x$.

If $A(x)$ is a general real symmetric matrix, the convergence of iteration can be proved by a similar method. Suppose $A(x)=P(x)^{-1}\Lambda(x)P(x)$, where Λ is the Jordan form of A. Let $\tilde{V}_j=P_jV_j$. According to (32), we know that $\tilde{V}_j$ satisfies

$$P_ja_jP_{j-1}^{-1}\tilde{V}_{j-1}+P_jb_jP_j^{-1}\tilde{V}_j+P_jc_jP_{j+1}^{-1}\tilde{V}_{j+1}=P_jd_j+P_je_j(V_j),$$
$$j=1,\ 2,\ \cdots,\ J-1.$$

If $P(x)$ and $P^{-1}(x)$ are smooth functions, then $P_ja_jP_{j-1}^{-1}$, $P_jb_jP_j^{-1}$, $P_jc_jP_{j+1}^{-1}$ can be expressed as a sum of a diagonal matrix and a matrix whose every element is a quantity of $O(\Delta x)$, and the relation among the three diagonal matrices is similar to that in the case where $A(x)$ is a real diagonal matrix. It is clear that in the present case we can derive

$$\|B^{-1}\|\leqslant\frac{1}{\left(H-\tilde{Q}_1-Q_1\dfrac{\Delta x}{\Delta t}\right)\Delta t},$$

where Q_1 is a constant. The reason for the appearance of the term $Q_1\dfrac{\Delta x}{\Delta t}$ is that there exist certain matrices whose every element is a quantity of $O(\Delta x)$. Therefore we know from (38) and (48) that in order for the iteration to converge, Δt is required to satisfy the following inequalities:

$$2\frac{\Delta t^2}{\Delta x^2}a\left(\tilde{c}_2^*+\tilde{Q}_1+Q_1\frac{\Delta x}{\Delta t}+1\right)+\left(\tilde{c}_2^*+\tilde{Q}_1+Q_1\frac{\Delta x}{\Delta t}+1\right)^2\Delta t^2\leqslant1 \tag{50}$$

and

$$\left(\tilde{c}_2^*+\tilde{Q}_1+Q_1\frac{\Delta x}{\Delta t}+1\right)\Delta t+\frac{a'\Delta t}{2\Delta x}\leqslant1. \tag{51}$$

That is, if $\Delta x\leqslant\dfrac{1}{Q_1}$ and Δt satisfies

$$\Delta t\leqslant\min\left\{\frac{-Q_1(a+H^*\Delta x^2)+[Q_1^2(a+H^*\Delta x^2)^2+(1-Q_1^2\Delta x^2)H^*(2a+H^*\Delta x^2)]^{\frac{1}{2}}}{H^*(2a+H^*\Delta x^2)}\Delta x,\right.$$
$$\left.\frac{(1-Q_1\Delta x)\Delta x}{\dfrac{a'}{2}+H^*\Delta x}\right\}, \tag{52}$$

where $H^*=\tilde{c}_2^*+\tilde{Q}_1+1$, then the iteration converges.

Consequently, we have proved the following conclusion: if the coefficients of partial differential equations and their derivatives are bounded, and Δx is sufficiently small, then there exists a constant c, such that the iteration converges when $\Delta t\leqslant c\Delta x$.

When Scheme B is used, an iteration similar to (34) may be adopted, and the convergence of iteration can be discussed by the same method. The convergence condition is also of the form of (49) or (52). [1] has discussed the convergence of iteration, and for the case where $A(x)$ is a diagonal matrix, proved that if $\Delta t \leqslant c\Delta x^2$, c being a certain constant, then the iteration converges. It is clear that this condition is much stronger than (47).

Conclusion

For two classes of implicit schemes for the generalized non-linear Schrödinger system, we have discussed their accuracy, the convergence conditions of these schemes and the convergence condition of an iteration for solving the corresponding nonlinear difference equations. In the following table, we list these results for four typical schemes.

Table

Schemes	Sufficient convergence conditions of schemes	Sufficient convergence conditions of iteration	Accuracy of schemes
A	$\Delta t = O(\Delta x^{\frac{1}{2}+\delta})$, $\delta > 0$	$\Delta t \leqslant c\Delta x$	$O(\Delta t^2 + \Delta x^2)$
B	$\Delta t = O(\Delta x^{\frac{1}{2}+\delta})$, $\delta > 0$	$\Delta t \leqslant c\Delta x$	$O(\Delta t + \Delta x^2)$
C	$\Delta t = O(\Delta x^{\frac{1}{2}+\delta})$, $\delta \geqslant 0$	no iteration	$O(\Delta t + \Delta x^2)$
D	$\Delta t = O(\Delta x^{\frac{1}{2}+\delta})$, $\delta \geqslant 0$	no iteration	$O(\Delta t^2 + \Delta x^2)$

According to this table, we should say that Scheme D is better than others because it possesses a second order accuracy and a weak convergence condition, and needs no iteration. Scheme B has first order accuracy and a strong convergence condition, and needs iterations, so it seems that this scheme is not as good as the others.

Reference

[1] Chang Qian-shun, Conservative difference schemes for the generalized nonlinear Schrödinger system, *Scientia Sinica*, Series **A**, 1983, No. 3, 202—214. (in Chinese)