TWO ALGORITHMS FOR SOLVING A KIND OF HEAT CONDUCTION EQUATIONS*

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1. Introduction

In this paper, a strategy is suggested for numerical solution of a kind of parabolic partial differential equations with nonlinear boundary conditions and discontinuous coefficients, which arise from practical engineering problems. First, a difference equation at the discontinuous point is established in which both the stability and the truncation error are consistent with the total difference equations. Then, on account of the fact that the coefficient matrix of the difference equations is tridiagonal and nonlinearity appears only in the first and the last equations, two algorithms are suggested: a mixed method combining the modified Gaussian elimination method with the successive recursion method, and a variant of the modified Gaussian elimination method. These algorithms are shown to be effective.

2. The Difference Equations of the Problem

Consider the parabolic partial differential equation

$$\frac{\partial u}{\partial t}=C\frac{\partial^2 u}{\partial x^2}, \tag{1}$$

where u, the required solution, satisfies the initial condition

$$u(x,\ 0)=\varphi(x), \tag{2}$$

and the boundary conditions

$$\left[\frac{\partial u}{\partial x}-\lambda_1 u\right]_{x=a}=v_1, \tag{3}$$

$$\left[\frac{\partial u}{\partial x}+\lambda_2 u\right]_{x=b}=v_2, \tag{4}$$

and the discontinuous condition is

$$k_1\frac{\partial u}{\partial x}\bigg|_{x_1-0}=k_2\frac{\partial u}{\partial x}\bigg|_{x_1+0},\qquad u|_{x_1-0}=u|_{x_1+0}, \tag{5}$$

where

$$\lambda_1=\lambda_1(t,\ u),\qquad \lambda_2=\lambda_2(t,\ u),$$
$$v_1=v_1(t,\ u),\qquad v_2=v_2(t,\ u),$$

and

$$C=\begin{cases} c_1 & a\leqslant x<x_1,\\ c_2 & x_1<x\leqslant b,\end{cases}$$

where c_1, c_2, k_1 and k_2 are constants.

For the region $[a\leqslant x\leqslant b,\ 0\leqslant t\leqslant T]$, there are two intervals in the x-direction. The intervals $[a,\ x_1]$ and $[x_1,\ b]$ are covered by the space increments $h_1=(x_1-a)/$

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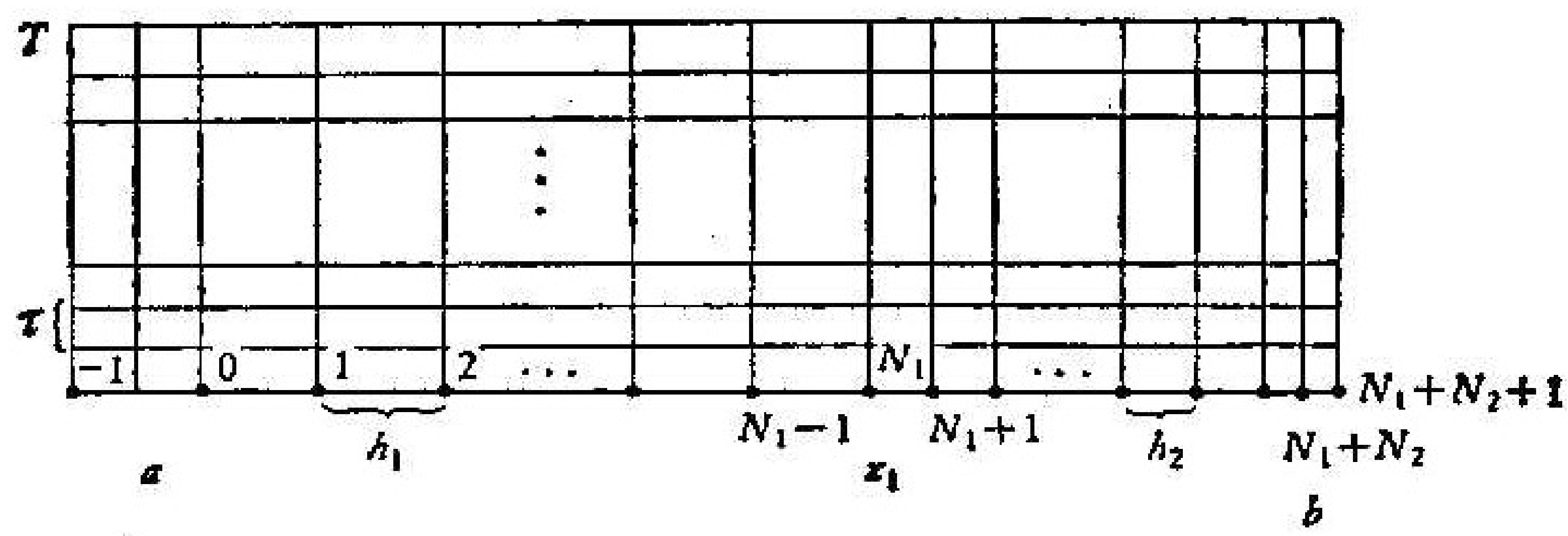
$\left(N_1+\frac{1}{2}\right)$ and $h_2=(b-x_1)\Big/\left(N_2+\frac{1}{2}\right)$ respectively, and time t is covered by $\tau=T/m$. A rectangular net is thus formed as follows.



By use of the Crank-Nicolson formula to approximate equation (1) we obtain the difference equations

$$\begin{aligned}&-\gamma_1 u_{k-1,j+1}+(1+2\gamma_1)u_{k,j+1}-\gamma_1 u_{k+1,j+1}\\&\quad=\gamma_1 u_{k-1,j}+(1-2\gamma_1)u_{k,j}+\gamma_1 u_{k+1,j}\\&(k=0,\ 1,\ 2,\ \cdots,\ N_1-1;\ j=1,\ 2,\ \cdots,\ m),\end{aligned}\tag{6}$$

$$\begin{aligned}&-\gamma_2 u_{k-1,j+1}+(1+2\gamma_2)u_{k,j+1}-\gamma_2 u_{k+1,j+1}\\&\quad=\gamma_2 u_{k-1,j}+(1-2\gamma_2)u_{k,j}+\gamma_2 u_{k+1,j}\\&(k=N_1+1,\ N_1+2,\ \cdots,\ N_1+N_2;\ j=1,\ 2,\ \cdots,\ m),\end{aligned}\tag{7}$$

where $\gamma_1=c_1\tau/2h_1^2,\qquad \gamma_2=c_2\tau/2h_2^2.$

Let $h=\max\{h_1,\ h_2\}$. Obviously equation (1) is approximated by (6) and (7) with the truncation error $O(h^2+\tau^2)$. To obtain the truncation error $O(h^2)$ on the boundary conditions (3) and (4), we replace $\frac{\partial u}{\partial x}$ with the central difference operator, and u with the average of the two adjacent points:

$$\left\{\frac{\partial u}{\partial x}\bigg|_a\right\}_{j+1}=\frac{u_{0,j+1}-u_{-1,j+1}}{h_1},\quad \left\{\frac{\partial u}{\partial x}\bigg|_b\right\}_{j+1}=\frac{u_{N+1,j+1}-u_{N,j+1}}{h_2},$$

$$\{u|_a\}_{j+1}=\frac{u_{0,j+1}+u_{-1,j+1}}{2},\qquad \{u|_b\}_{j+1}=\frac{u_{N+1,j+1}+u_{N,j+1}}{2}.$$

Applying these formulas to (3) and (4) respectively, we have

$$u_{-1,j+1}=\frac{1-\frac{1}{2}h_1\lambda_{1,j+1}}{1+\frac{1}{2}h_1\lambda_{1,j+1}}u_{0,j+1}-\frac{h_1 v_{1,j+1}}{1+\frac{1}{2}h_1\lambda_{1,j+1}},\tag{8}$$

$$u_{N+1,j+1}=\frac{1-\frac{1}{2}h_2\lambda_{2,j+1}}{1+\frac{1}{2}h_2\lambda_{2,j+1}}u_{N,j+1}+\frac{h_2 v_{2,j+1}}{1+\frac{1}{2}h_2\lambda_{2,j+1}},\tag{9}$$

where

$$\begin{aligned}&N=N_1+N_2,\\&\lambda_{1,j+1}=\lambda_1(t_{j+1},\ (u_{-1,j+1}+u_{0,j+1})/2),\\&v_{1,j+1}=v_1(t_{j+1},\ (u_{-1,j+1}+u_{0,j+1})/2),\\&\lambda_{2,j+1}=\lambda_2(t_{j+1},\ (u_{N,j+1}+u_{N+1,j+1})/2),\\&v_{2,j+1}=v_2(t_{j+1},\ (u_{N,j+1}+u_{N+1,j+1})/2).\end{aligned}$$

Substituting (8) into (6) and (9) into (7), we obtain, for $k=0$ and $k=N$ respectively,

$$\left[1+2\gamma_1-\gamma_1\cdot\left(\frac{1-\frac{1}{2}h_1\lambda_{1,j+1}}{1+\frac{1}{2}h_1\lambda_{1,j+1}}\right)\right]u_{0,j+1}-\gamma_1 u_{1,j+1}$$

$$=\gamma_1\cdot u_{-1,j}+(1-2\gamma_1)u_{0,j}+\gamma_1\cdot u_{1,j}-\frac{\gamma_1 h_1 v_{1,j+1}}{1+\frac{1}{2}h_1\lambda_{1,j+1}}, \tag{10}$$

$$-\gamma_2 u_{N-1,j+1}+\left\{1+2\gamma_2-\gamma_2\cdot\left(\frac{1-\frac{1}{2}h_2\lambda_{2,j+1}}{1+\frac{1}{2}h_2\lambda_{2,j+1}}\right)\right\}u_{N,j+1}$$

$$=\gamma_2\cdot u_{N-1,j}+(1-2\gamma_2)u_{N,j}+\gamma_2\cdot u_{N+1,j}+\frac{\gamma_2 h_2 v_{2,j+1}}{1+\frac{1}{2}h_2\lambda_{2,j+1}}. \tag{11}$$

The difference equation at the discontinuous point with the truncation error $O(h^2+\tau^2)$ can be obtained from the viewpoint of the dummy point[2].

In fact, we can approximate the discontinuous condition (5) with the following equation

$$\frac{k_1}{h_1}(u^*_{N_1+1,j+1}-u_{N_1-1,j+1})=\frac{k_2}{h_2}(u_{N_1+1,j+1}-u^*_{N_1-1\,j+1}) \tag{12}$$

where $u^*_{N_1+1,j+1}$ is the value of u at the dummy point $(x_1+h_1,\ (j+1)\tau)$, and $u^*_{N_1-1,j+1}$ is the value of u at the dummy point $(x_1-h_2,\ (j+1)\tau)$.

To avoid the value at the dummy points in (12), we must establish supplementary equations at the points $(x_1-0,\ (j+1)\tau)$ and $(x_1+0,\ (j+1)\tau)$ by means of the Crank-Nicolson equations and the following conditions

$$u_{N_1-0,j+1}=u_{N_1+0,j+1}=u_{N_1,j+1},$$

$$u_{N_1-0,j}=u_{N_1+0,j}=u_{N_1,j}.$$

Then, $u^*_{N_1+1,j+1}$ and $u^*_{N_1-1,j+1}$ are obtained explicitly from the supplementary equations:

$$u^*_{N_1+1,j+1}=-u_{N_1-1,j+1}+\left(\frac{1}{\gamma_1}+2\right)u_{N_1,j+1}$$

$$-u_{N_1-1,j}-\left(\frac{1}{\gamma_1}-2\right)u_{N_1,j}-u^*_{N_1+1,j}, \tag{13}$$

$$u^*_{N_1-1,j+1}=-u_{N_1+1,j+1}+\left(\frac{1}{\gamma_2}+2\right)u_{N_1,j+1}$$

$$-u^*_{N_1-1,j}-\left(\frac{1}{\gamma_2}-2\right)u_{N_1,j}-u_{N_1+1,j}. \tag{14}$$

Substituting (13) and (14) into (12), we have

$$-u_{N_1-1,j+1}+\frac{1}{2}\left[\left(\frac{1}{\gamma_1}+2\right)+\frac{k_2h_1}{k_1h_2}\left(\frac{1}{\gamma_2}+2\right)\right]u_{N_1,j+1}-\frac{k_2h_1}{k_1h_2}u_{N_1+1,j+1}$$

$$=\frac{1}{2}\left\{u_{N_1-1,j}+\left[\left(\frac{1}{\gamma_1}-2\right)+\frac{k_2h_1}{k_1h_2}\left(\frac{1}{\gamma_2}-2\right)\right]u_{N_1,j}+\frac{k_2h_1}{k_1h_2}u_{N_1+1,j}\right.$$

$$\left.+u^*_{N_1+1,j}+\frac{k_2h_1}{k_1h_2}u^*_{N_1-1,j}\right\}, \tag{15}$$

where $u^*_{N_1+1,j}$ and $u^*_{N_1-1,j}$ can be obtained from (13) and (14), and when $j=0$, we take their initial values as φ_{N_1+1} and φ_{N_1-1} respectively. The nonlinear equations of the problem are thus constructed in matrix form

$$A(U_{j+1})U_{j+1}=F(U_{j+1}), \tag{16}$$

where

$$A(U_{j+1})=\begin{pmatrix}
\left(\frac{1}{\gamma_1}+2-\alpha_{1,j+1}\right) & -1 & & & & & & & \\
-1 & \left(\frac{1}{\gamma_1}+2\right) & -1 & & & & & & \\
 & -1 & \left(\frac{1}{\gamma_1}+2\right) & -1 & & & & & \\
 & & \ddots & \ddots & \ddots & & & & \\
 & & & -1 & \frac{1}{2}\Delta_1 & -\frac{k_2h_1}{k_1h_2} & & & \\
 & & & & -1 & \left(\frac{1}{\gamma_2}+2\right) & -1 & & \\
 & & & & & \ddots & \ddots & \ddots & \\
 & & & & & & -1 & \left(\frac{1}{\gamma_2}+2\right) & -1 \\
 & & & & & & & -1 & \left(\frac{1}{\gamma_2}+2-\alpha_{2,j+1}\right)
\end{pmatrix},$$

$$\Delta_1=\left(\frac{1}{\gamma_1}+2\right)+\frac{k_2h_1}{k_1h_2}\left(\frac{1}{\gamma_2}+2\right),$$

$$U^T_{j+1}=(u_{0,j+1},\ u_{1,j+1},\ \cdots,\ u_{N_1,j+1},\ u_{N_1+1,j+1},\ \cdots,\ u_{N,j+1}),$$

and

$$F(U_{j+1})=B(U_j)U_j+Q(U_{j+1}),$$

where

$$B(U_j)=\begin{pmatrix}
\left(\frac{1}{\gamma_1}-2+\alpha_{1,j}\right) & 1 & & & & \\
1 & \left(\frac{1}{\gamma_1}-2\right) & 1 & & & \\
\ddots & \ddots & \ddots & & & \\
 & \frac{1}{2} & \frac{1}{2}\Delta_2 & \frac{1}{2}\frac{k_2h_1}{k_1h_2} & & \\
 & & 1 & \left(\frac{1}{\gamma_2}-2\right) & 1 & \\
 & & & \ddots & \ddots & \ddots \\
 & & & & 1 & \left(\frac{1}{\gamma_2}-2+\alpha_{2,j}\right)
\end{pmatrix},$$

and

$$Q^T(U_{j+1})=\Big[-(\beta_{1,j}+\beta_{1,j+1}),\ 0,\ \cdots,\ 0,\ \frac{1}{2}\Big(u^*_{N_1+1,j}+\frac{k_2h_1}{k_1h_2}u^*_{N_1-1,j}\Big),\ 0,\ \cdots,\ 0,\ (\beta_{2,j}+\beta_{2,j+1})\Big],$$

where

$$\Delta_2=\left(\frac{1}{\gamma_1}-2\right)+\frac{k_2h_1}{k_1h_2}\left(\frac{1}{\gamma_2}-2\right),$$

$$\alpha_{1,j+1}=\frac{1-\frac{1}{2}h_1\lambda_{1,j+1}}{1+\frac{1}{2}h_1\lambda_{1,j+1}},\qquad \alpha_{2,j+1}=\frac{1-\frac{1}{2}h_2\lambda_{2,j+1}}{1+\frac{1}{2}h_2\lambda_{2,j+1}},$$

$$\beta_{1,j+1}=\frac{h_1v_{1,j+1}}{1+\frac{1}{2}h_1\lambda_{1,j+1}},\qquad \beta_{2,j+1}=\frac{h_2v_{2,j+1}}{1+\frac{1}{2}h_2\lambda_{2,j+1}}.$$

3. Solution of the Nonlinear Difference Equations

Because the boundary conditions are nonlinear, the difference equations obtained are also nonlinear. And to solve this kind of nonlinear equations an iterative method will be applied as a rule in the following steps:

① Given an initial vector of the solution u, calculate the coefficient terms and the right-side terms of the equations, which are nonlinear functions of u.

② Solve the equations by means of the modified Gaussian elimination method.

③ Check if the required error of the two successive solutions is satisfied. If not, go to step ① to repeat the above process. If the process is convergent and the error can be satisfied, then the solution is obtained.

We can see that such an algorithm costs much computing time because the linear equations are solved always during the iterative process. In order to reduce the computing effort, we give two algorithms according to the sole appearance of the nonlinear terms in the first and the last of the difference equations.

For simplicity we rewrite equation (16) as

$$A_{N+1}X_{N+1}=F_{N+1}, \tag{17}$$

where

$$A_{N+1}=\begin{pmatrix} b_0 & c_0 & & & \\ a_1 & b_1 & c_1 & & \\ & \ddots & \ddots & \ddots & \\ & & a_{N-1} & b_{N-1} & c_{N-1} \\ & & & a_N & b_N \end{pmatrix},\quad X_{N+1}=\begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-1} \\ x_N \end{pmatrix},\quad F_{N+1}=\begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_{N-1} \\ f_N \end{pmatrix}.$$

Here A_{N+1}, X_{N+1} and F_{N+1} are equal to $A(U_{j+1})$, U_{j+1} and $F(U_{j+1})$ in equation (16), respectively. By equation (16), b_0 and b_N in A_{N+1} are related to X_{N+1}; so are f_0 and f_N.

3.1. The mixed algorithm

For the convenience of investigation assume that the last equation in equation (17) is nonlinear. Our algorithm is described as follows.

Let

$$A_{N+1}=\begin{pmatrix} A_N & u_N \\ v_N & b_N \end{pmatrix},\quad X_{N+1}=\begin{pmatrix} X_N \\ x_N \end{pmatrix},\quad F_{N+1}=\begin{pmatrix} F_N \\ f_N \end{pmatrix},$$

where A_N is an $N\times N$ matrix obtained by deleting the last row and column from A_{N+1}, and

$$v_N=(0,\ 0,\ \cdots,\ 0,\ a_N),$$
$$u_N=(0,\ 0,\ \cdots,\ 0,\ C_{N-1})^T,$$
$$X_N=(x_0,\ x_1,\ \cdots,\ x_{N-1})^T,$$
$$F_N=(f_0,\ f_1,\ \cdots,\ f_{N-1})^T.$$

By [3], we have

$$A_{N+1}^{-1}=\begin{pmatrix} A_N^{-1}+\dfrac{A_N^{-1}\cdot u_N\cdot v_N\cdot A_N^{-1}}{\alpha_N} & -\dfrac{A_N^{-1}\cdot u_N}{\alpha_N} \\ -\dfrac{v_N\cdot A_N^{-1}}{\alpha_N} & \dfrac{1}{\alpha_N} \end{pmatrix},$$

where

$$\alpha_N=b_N-v_N A_N^{-1}u_N.$$

Then we have

$$X_{N+1}=A_{N+1}^{-1}F_{N+1}=\begin{pmatrix} X_N \\ x_N \end{pmatrix}=\begin{pmatrix} X_N^* \\ 0 \end{pmatrix}-\frac{f_N-v_N\cdot X_N^*}{b_N-v_N Z_N}\begin{pmatrix} Z_N \\ -1 \end{pmatrix}, \tag{18}$$

where

$$X_N^*=A_N^{-1}F_N, \tag{19}$$
$$Z_N=A_N^{-1}u_N. \tag{20}$$

Substituting $v_N=(0,\ 0,\ \cdots,\ 0,\ a_N)$ into (18), we obtain

$$X_{N+1}=\begin{pmatrix} X_N^* \\ 0 \end{pmatrix}-\frac{f_N-a_N x_N^*}{b_N-a_N z_N}\begin{pmatrix} Z_N \\ -1 \end{pmatrix}, \tag{21}$$

where x_N^* and z_N are respectively the Nth components of X_N^* and Z_N.

Now we may replace b_N and f_N in (21) with $b_N(x_N,\ x_{N+1})$ and $f_N(x_N,\ x_{N+1})$. An iterative method can be constructed as follows:

$$X_{N+1}^{(m+1)}=\begin{pmatrix} X_N^* \\ 0 \end{pmatrix}-\frac{f_N(x_N^{(m)},\ x_{N+1}^{(m)})-a_N x_N^*}{b_N(x_N^{(m)},\ x_{N+1}^{(m)})-a_N z_N}\cdot\begin{pmatrix} Z_N \\ -1 \end{pmatrix}, \tag{22}$$

where $m=0,\ 1,\ 2,\ \cdots$.

In the iterative process, the components $x_N^{(m+1)}$ and $x_{N+1}^{(m+1)}$ can be solved from (22). If the iteration is convergent and the required error is satisfied, the rest of X_{N+1}, x_1, x_2, $\cdots$, x_{N-1}, can be obtained by (22). That is, the total solution X_{N+1} is obtained.

The executive steps of our algorithm which only takes the boundary condition (4) as nonlinear may be summarized as follows.

① Calculate X_N and Z_N in (19) and (20) by use of the modified Gaussian elimination method.

② Calculate b_N and f_N which contain the nonlinear terms.

③ Evaluate $x_N^{(m+1)}$ and $x_{N+1}^{(m+1)}$ by (22).

④ Check if the required error of the two successive iterative solutions is satisfied. If not, go to step ②; otherwise use (22) again to get $x_1^{(m+1)}$, $x_2^{(m+1)}$, $\cdots$, $x_{N-1}^{(m+1)}$ and thus obtain the required solution X_{N+1}.

We see that only two nonlinear equations have to be solved in the above iterative process. Hence a great amount of computing time is saved.

When the boundary conditions (3) and (4) are both nonlinear, the first and the last equations in (17) are nonlinear. Cutting off the first and the last rows and columns from A_{N+1}, we get an $(N-1)\times(N-1)$ matrix A_{N-1}. Similarly for X^*_{N-1} and Z_{N-1}. X_N and Z_N can be obtained recurrently by means of the following formulas

$$X^*_N=\begin{pmatrix}0\\ X^*_{N-1}\end{pmatrix}-\frac{f_0-c_0x^*_1}{b_0-c_0z_1}\begin{pmatrix}-1\\ Z_{N-1}\end{pmatrix}, \tag{23}$$

$$Z_N=\frac{a_1}{b_0-c_0z_1}\begin{pmatrix}-1\\ Z_{N-1}\end{pmatrix}, \tag{24}$$

where x^*_1 and z_1 are the first components of X^*_{N-1} and Z_{N-1} respectively.

Finally, using (21) we obtain X_{N+1}.

(23) and (24) can be obtained similarly to (21). The algorithm in this case is similar to the above description; we will not repeat it.

3.2. The new modified Gaussian elimination method

We rewrite (17)as

$$\begin{cases} x_0=-\dfrac{c_0}{b_0}x_1+\dfrac{f_0}{b_0}, & (25)\\ x_k=A_{1,k}x_0+B_{1,k}x_{k+1}+D_{1,k}, \quad k=1,\ 2,\ \cdots,\ N-1, & (26)\\ x_N=\dfrac{a_N}{b_N}x_{N-1}+\dfrac{f_N}{b_N}. & (27)\end{cases}$$

Let $A_{1,0}=1$, $B_{1,0}=D_{1,0}=0$. Then we have

$$\begin{aligned} A_{1,k}&=-\frac{a_kA_{1,k-1}}{b_k+a_kB_{1,k-1}},\\ B_{1,k}&=-\frac{c_k}{b_k+a_kB_{1,k-1}}, \qquad k=1,\ 2,\ \cdots,\ N-1.\\ D_{1,k}&=\frac{f_k-a_kD_{1,k-1}}{b_k+a_kB_{1,k-1}}.\end{aligned} \tag{28}$$

The set of $N-2$ equations in (17) can be denoted by

$$x_k=A_{2,k}x_1+B_{2,k}x_{k+1}+D_{2,k}, \qquad k=2,\ 3,\ \cdots,\ N-1. \tag{29}$$

Let $A_{2,1}=1$, $B_{2,1}=D_{2,1}=0$. Then

$$\begin{aligned} A_{2,k}&=-\frac{a_kA_{2,k-1}}{b_k+a_kB_{2,k-1}},\\ B_{2,k}&=-\frac{c_k}{b_k+a_kB_{2,k-1}}, \qquad k=2,\ 3,\ \cdots,\ N-1.\\ D_{2,k}&=\frac{f_k-a_kD_{2,k-1}}{b_k+a_kB_{2,k-1}},\end{aligned} \tag{30}$$

Now we combine the last equations of (26) and (29) with (25) and (27):

$$\begin{cases} x_0=-\dfrac{c_0}{b_0}x_1+\dfrac{f_0}{b_0},\\ x_{N-1}=A_{1,N-1}x_0+B_{1,N-1}x_N+D_{1,N-1},\\ x_{N-1}=A_{2,N-1}x_1+B_{2,N-1}x_N+D_{2,N-1},\\ x_N=-\dfrac{a_N}{b_N}x_{N-1}+\dfrac{f_N}{b_N},\end{cases} \tag{31}$$

where $A_{1,k}$, $B_{1,k}$, $D_{1,k}$, $A_{2,k}$, $B_{2,k}$ and $D_{2,k}$ ($k=1, 2, \cdots, N-1$) may also be deduced from (28) and (30). Next we get x_0, x_1, x_{N-1} and x_N. Substituting them into (26) and (29), we obtain the solution X_{N+1}. Because the calculation of the nonlinear terms f_0, f_N, a_0 and b_N are restricted in (31), the computing process is greatly simplified and the computing time is saved. In application of this method to solving (17) $A_{i,k}$ and $B_{i,k}$($i=1, 2$) are only calculated once in the process because they are independent of the time increment.

4. The Stability of the Algorithm

We now discuss the stability of (26), (28), (29) and (30).

From (17) we have

$$\begin{cases} a_k=-1 \quad (k=1, 2, \cdots, N), \\ c_k=-1 \quad (k=0, 1, \cdots, N-1; k\neq N_1), \\ c_{N_1}=-\dfrac{k_2h_1}{k_1h_2}, \\ b_k=\begin{cases} \dfrac{1}{\gamma_1}+2 \quad (k=1, 2, \cdots, N_1-1), \\ \dfrac{1}{\gamma_2}+2 \quad (k=N_1+1, \cdots, N-1), \end{cases} \\ b_0=\left(\dfrac{1}{\gamma_1}+2-\alpha_{1,j+1}\right), \\ b_{N_1}=\dfrac{1}{2}\Delta_1, \\ b_N=\left(\dfrac{1}{\gamma_2}+2-\alpha_{2,j+1}\right). \end{cases} \tag{32}$$

Substituting (32) into (28) we have

$$A_{1,k}=\frac{A_{1,k-1}}{\left(\dfrac{1}{\gamma}+2\right)-B_{1,k-1}},$$

$$B_{1,k}=\frac{1}{\left(\dfrac{1}{\gamma}+2\right)-B_{1,k-1}}, \quad \gamma=\begin{cases}\gamma_1, \ k=1, 2, \cdots, N_1-1, \\ \gamma_2, \ k=N_1+1, \cdots, N-1,\end{cases}$$

$$D_{1,k}=\frac{f_k-D_{1,k-1}}{\left(\dfrac{1}{\gamma}+2\right)-B_{1,k-1}},$$

$$B_{1,N_1}=\frac{\dfrac{k_2h_1}{k_1h_2}}{\dfrac{1}{2}\left[\left(\dfrac{1}{\gamma_1}+2\right)+\dfrac{k_2h_1}{k_1h_2}\left(\dfrac{1}{\gamma_2}+2\right)\right]-B_{1,N_1-1}}.$$

Since $B_{1,0}=0$, $\gamma_1>0$, $\gamma_2>0$, we have, by induction,

$$0<B_{1,k}<1, \quad k=1, 2, \cdots, N-1.$$

Let
$$A_{1,k}+\varepsilon_{A_1,k}=\frac{A_{1,k-1}+\varepsilon_{A_1,k-1}}{\left(\frac{1}{\gamma}+2\right)-B_{1,k-1}}.$$

Then
$$|\varepsilon_{A_1,k}|=\left|\frac{\varepsilon_{A_1,k-1}}{\left(\frac{1}{\gamma}+2\right)-B_{1,k-1}}\right|<|\varepsilon_{A_1,k-1}|.$$

Similarly,

$$|\varepsilon_{B_1,k}|=\left|\frac{1}{\left(\frac{1}{\gamma}+2\right)-(B_{1,k-1}+\varepsilon_{B_1,k-1})}-\frac{1}{\left(\frac{1}{\gamma}+2\right)-B_{1,k-1}}\right|$$

$$=\left|\frac{\varepsilon_{B_1,k-1}}{\left[\left(\frac{1}{\gamma}+2\right)-(B_{1,k-1}+\varepsilon_{B_1,k-1})\right]\left[\left(\frac{1}{\gamma}+2\right)-B_{1,k-1}\right]}\right|<|\varepsilon_{B_1,k-1}|,$$

$$\varepsilon_{D_1,k}=\left|\frac{\varepsilon_{D_1,k-1}}{\left(\frac{1}{\gamma}+2\right)-B_{1,k-1}}\right|<|\varepsilon_{D_1,k-1}|.$$

Hence (28) is stable, and so is (30).

The above results are due to restrictions on h_1, h_2 or τ_1, τ_2 at the discontinuous point.

Now we can prove that (26) and (29) are stable.

In fact, since $A_{1,0}=1$, $B_{1,0}=0$ we have

$$A_{1,1}=\frac{\gamma}{1+2\gamma},\qquad B_{1,1}=\frac{\gamma}{1+2\gamma}.$$

Further, we have

$$A_{1,1}+B_{1,1}=\frac{2\gamma}{1+2\gamma}<1,\quad \gamma>0.$$

Then induction leads to

$$A_{1,k}+B_{1,k}=\frac{A_{1,k-1}+1}{\left(\frac{1}{\gamma}+2\right)-B_{1,k-1}}<\frac{2-B_{1,k-1}}{\left(\frac{1}{\gamma}+2\right)-B_{1,k-1}}<1.$$

Similarly,

$$A_{2,k}+B_{2,k}<1.$$

Hence (26) and (29) are stable.

The algorithms presented have been applied to actual calculation with satisfactory result. The second algorithm may be applied also to the case that the difference equations contain several nonlinear equations at the top and bottom.

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