NUMERICAL SOLUTIONS OF HARMONIC AND BIHARMONIC CANONICAL INTEGRAL EQUATIONS IN INTERIOR OR EXTERIOR CIRCULAR DOMAINS*

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Abstract

Elliptic boundary-value problems can be reduced to integral equations on the boundary by many different ways. The canonical reduction, suggested by Prof. Feng Kang[1-3], is a natural and direct approach of boundary reduction. This paper gives the numerical method for solving harmonic and biharmonic canonical integral equations in interior or exterior circular domains, together with their convergence and error estimates. Using the theory of distributions, the difficulty caused by the singularities of integral kernel is overcome. Results of several numerical calculations verify the theoretical estimates.

Introduction

In recent years, Feng Kang suggested a natural and direct method of redution, called canonical redution, of elliptic boundary-value problems over a domain to integral equations on the boundary, which preserves all the essential characteristics, including self-adjointness, coerciveness, variational functional, etc., of the original problem[1-3], The kernel of the resulting equation on the boundary, called the canonical integral equation, contains singularities of the type of the finite part of divergent integrals in the sense of the theory of distributions, which are of higher order than those of the usual Cauchy-type.

This paper gives the numerical method, based on the finite element approximations, for solving the canonical integral equations corresponding to the Neumann problems of harmonic and biharmonic equations over interior or exterior circular domains. Convergence and error estimates are also given. Several numerical experiments verify well the theoretical estimates and demonstrate the efficacy of the method.

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1. Harmonic Canonical Integral Equation

1.1. *Numerical solution*

Consider the harmonic equation in the interior or the exterior to the circle with radius R with Neumann condition

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$$\begin{cases} \Delta u = 0, & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = u_n(\theta) & \text{on } \Gamma = \partial\Omega, \end{cases} \tag{P1}$$

where $u_n(\theta) \in H^{-\frac{1}{2}}(\Gamma)$ and satisfies the consistency condition

$$\int_\Gamma u_n(\theta)\, d\theta = 0.$$

It is equivalent to the variational problem

$$\begin{cases} \text{Find } u \in H^1(\Omega) \text{ such that} \\ D(u, v) = \int_\Gamma u_n(\theta)\, v ds, \qquad \forall v \in H^1(\Omega), \\ D(u, v) = \iint\limits_\Omega \nabla u \cdot \nabla v\, dp \end{cases}$$

for the interior problem. For the exterior problem, the space $H^1(\Omega)$ is replaced by

$$W_0^1(\Omega) = \left\{ u \left| \frac{u}{\sqrt{x^2 + y^2} \ln(1 + x^2 + y^2)},\ u_x,\ u_y \in L^2(\Omega) \right. \right\}.$$

Moreover, the problem (P1) can be reduced into the canonical integral equation, which contains a singular kernel, as follows[1]

$$u_n(\theta) = -\frac{1}{4\pi R} \int_0^{2\pi} \frac{1}{\sin^2 \frac{\theta - \theta'}{2}} u_0(\theta')\, d\theta' \equiv -\frac{1}{4\pi R \sin^2 \frac{\theta}{2}} * u_0(\theta),$$

where $*$ denots the convolution, which can be defined through Fourier series in the sense of generalized functions[7,8]. Since

$$-\frac{1}{4\pi} \int_0^{2\pi} \frac{1}{\sin^2 \frac{\theta}{2}}\, d\theta = 0,$$

the solution of above-mentioned integral equation is unique up to an additive constant. It also corresponds to the variational problem

$$\begin{cases} \text{Find } u_0(\theta) \in H^{\frac{1}{2}}(\Gamma) \text{ such that} \\ \bar{D}(u_0, v_0) = \int_\Gamma u_n(\theta)\, v_0(\theta)\, ds, \qquad \forall v_0 \in H^{\frac{1}{2}}(\Gamma), \\ \bar{D}(u_0, v_0) = -\int_0^{2\pi} \int_0^{2\pi} \frac{1}{4\pi \sin^2 \frac{\theta - \theta'}{2}} u_0(\theta')\, v_0(\theta)\, d\theta'\, d\theta. \end{cases} \tag{1}$$

The original solution u of (P1) is obtained from u_0 by the Poisson formula

$$u(r, \theta) = \pm \frac{(R^2 - r^2)}{2\pi} \int_0^{2\pi} \frac{u(R, \theta')}{R^2 + r^2 - 2Rr\cos(\theta - \theta')}\, d\theta', \quad \text{for} \begin{cases} 0 \leqslant r < R, \\ \quad r > R. \end{cases}$$

We can easily prove

Proposition 1.1. $\bar{D}(\gamma u, \gamma v) = D(u, v)$, $\forall u, v \in H^1(\Omega)$, $\Delta u = 0$; $\bar{D}(u_0, v_0)$ is a positive definite symmetric bilinear form on $[H^{\frac{1}{2}}(\Gamma)/P_0] \times [H^{\frac{1}{2}}(\Gamma)/P_0]$, where P_0

is all constants, γ is the trace operator which maps $H^1(\Omega)$ onto $H^{\frac{1}{2}}(\Gamma)$.

Proposition 1.2. The variational problem (1) has one and only one solution in $H^{\frac{1}{2}}(\Gamma)/P_0$.

In fact, using Green's formula, we get the former conclusion of proposition 1.1, from this and the positive definite symmetry, continuit y and $H^1(\Omega)/P_0$-ellipticity of $D(u, v)$ we can obtain corresponding properties of $\bar{D}(u_0, v_0)$. Then, using the Lax-Milgram theorem, we have the proposition 1.2.

Take the piecewise linear basis functions

$$L_i(\theta)=\begin{cases}\frac{N}{2\pi}(\theta-\theta_{i-1}), & \theta_{i-1}\leqslant\theta\leqslant\theta_i,\\ \frac{N}{2\pi}(\theta_{i+1}-\theta), & \theta_i\leqslant\theta\leqslant\theta_{i+1},\\ 0, & \text{otherwise,}\end{cases}$$

$$i=1, 2, \cdots, N,$$

where $\theta_i=\frac{i}{N}2\pi$. Let

$$u_0(\theta)\approx U_0(\theta)=\sum_{j=1}^{N}U_jL_j(\theta).$$

Obviously, $\{L_i(\theta)\}\subset H^1(\Gamma)\subset H^{\frac{1}{2}}(\Gamma)$. Using the formula[7]

$$-\frac{1}{4\pi\sin^2\frac{\theta}{2}}=\frac{1}{\pi}\sum_{n=1}^{\infty}n\cos n\theta,$$

from (1) we obtain equation

$$QU=b,$$

where $Q=[q_{ij}]_{N\times N}$, $U=[U_1, \cdots, U_N]^T$, $b=[b_1, \cdots, b_N]^T$,

$$b_i=\int_0^{2\pi}u_n(\theta)L_i(\theta)Rd\theta,$$

$$q_{ij}=q_{ji}=a_{|i-j|}, \tag{2}$$

$$a_k=\frac{4N^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin^4 n\frac{\pi}{N}\cos\frac{nk}{N}2\pi,\ k=0, 1, \cdots, N-1, \tag{3}$$

which is a convergent series,

$$Q=\begin{bmatrix}a_0 & a_1 & \cdots & a_{N-2} & a_{N-1}\\ a_{N-1} & a_0 & \cdots & a_{N-3} & a_{N-2}\\ \cdots & & \cdots & & \cdots\\ a_2 & a_3 & \cdots & a_0 & a_1\\ a_1 & a_2 & \cdots & a_{N-1} & a_0\end{bmatrix}\equiv(a_0, a_1, \cdots, a_{N-1}). \tag{4}$$

From now on the circulant matrix produced by $\alpha_1, \cdots, \alpha_N$ will be denoted by $(\alpha_1, \cdots, \alpha_N)$. Q is semi-positive definite and circulant with rank $N-1$. We can solve $QU=b$ by direct or iterative method, or by method provided in [6] and using FFT.

If we take the piecewise quadratic basis functions $\{\varphi_i(\theta)\}_{i=1,\cdots,2N}$, then we have $QU=b$, where

$$Q=[q_{ij}]_{2N\times 2N},\ U=[U_1, \cdots, U_{2N}]^T,\ b=[b_1, \cdots, b_{2N}]^T,$$

$$b_i=R\int_0^{2\pi} u_n(\theta)\varphi_i(\theta)d\theta,\ i=1,\ \cdots,\ 2N, \tag{5}$$

$$q_{2i,2j}=\frac{N^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}\left(\frac{2N}{n\pi}\sin\frac{2n\pi}{N}-\cos\frac{2n\pi}{N}-3\right)^2\cos\frac{i-j}{N}2n\pi,$$

$$q_{2i-1,2j-1}=\frac{16N^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}\left(\frac{N}{n\pi}\sin\frac{n\pi}{N}-\cos\frac{n\pi}{N}\right)^2\cos\frac{i-j}{N}2n\pi,$$

$$q_{2j,2i-1}=q_{2i-1,2j}=-\frac{4N^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}\left(\frac{N}{n\pi}\sin\frac{n\pi}{N}-\cos\frac{n\pi}{N}\right)$$

$$\times\left(\frac{2N}{n\pi}\sin\frac{2n\pi}{N}-\cos\frac{2n\pi}{N}-3\right)\cos\frac{2(i-j)-1}{N}n\pi,$$

$$i,\ j=1,\ \cdots,\ N. \tag{6}$$

All these series are convergent.

1.2. Convergence and error estimates

Let the linear space spanned by basis functions to be $S_N\subset H^{\frac{1}{2}}(\Gamma)$. Set $u_0(\theta)$ to be the solution of (1) and $U_0^{(N)}(\theta)$ to be its approximate solution. Then we have

Lemma 1.1. $\bar{D}(u_0-U_0^{(N)},\ V_0)=0,\ \forall V_0\in S_N,$

$$\|u_0-U_0^{(N)}\|_{\bar{D}}=\inf_{V_0\in S_N}\|u_0-V_0\|_{\bar{D}},$$

where $\|\cdot\|_{\bar{D}}$ *is the energy norm on* $H^{\frac{1}{2}}(\Gamma)/P_0$ *derived from* $\bar{D}(\ ,\ )$.

Theorem 1.1. *If the interpolation operator* Π *satisfies* $\|v_0-\Pi v_0\|_{\frac{1}{2},\Gamma}\underset{h\to 0}{\longrightarrow}0,\ \forall v_0\in H^{\frac{1}{2}}(\Gamma)$, *and the solution* $u_0\in H^{\frac{1}{2}}(\Gamma)$ *of* (1) *exists, then*

$$\lim_{N\to\infty}\|u_0-U_0^{(N)}\|_{\bar{D}}=0.$$

Proof. Since $\|\cdot\|_{\bar{D}}$ and $\|\cdot\|_{H^{\frac{1}{2}}(\Gamma)/P_0}$ are equivalent, there exists a constant K, such that $\|v_0\|_{\bar{D}}\leqslant K\|v_0\|_{\frac{1}{2},\Gamma}\ \ \forall v_0\in H^{\frac{1}{2}}(\Gamma)$. And from the trace theorem, we have constant T, such that $\|v|_\Gamma\|_{\frac{1}{2},\Gamma}\leqslant T\|v\|_{H^1(\Omega)},\forall v\in H^1(\Omega)$. Because of $u_0\in H^{\frac{1}{2}}(\Gamma)$, there exists $u\in H^1(\Omega)$ such that $u|_\Gamma=u_0$. Then for arbitrary $\varepsilon>0$, we have $\tilde{u}\in C^\infty(\bar{\Omega})$, such that

$$\|u-\tilde{u}\|_{H^1(\Omega)}\leqslant\frac{\varepsilon}{2KT}.$$

Set $\tilde{u}|_\Gamma=\tilde{u}_0$, then $$\|u_0-\tilde{u}_0\|_{\bar{D}}\leqslant KT\|u-\tilde{u}\|_{H^1(\Omega)}\leqslant\frac{\varepsilon}{2}.$$

Moreover, for fixed $\tilde{u}$, there exists N_0, such that $\|\tilde{u}_0-\Pi\tilde{u}_0\|_{\frac{1}{2},\Gamma}<\frac{\varepsilon}{2K}$ when $N>N_0$. Then $\|\tilde{u}_0-\Pi\tilde{u}_0\|_{\bar{D}}<\frac{\varepsilon}{2}$. Using Lemma 1.1, we obtain

$$\|u_0-U_0^{(N)}\|_{\bar{D}}=\inf_{V_0\in S_N}\|u_0-V_0\|_{\bar{D}}\leqslant\inf_{V_0\in S_N}(\|u_0-\tilde{u}_0\|_{\bar{D}}+\|\tilde{u}_0-V_0\|_{\bar{D}})$$
$$\leqslant\|u_0-\tilde{u}_0\|_{\bar{D}}+\|\tilde{u}_0-\Pi\tilde{u}_0\|_{\bar{D}}<\varepsilon.$$

The proof is thus complete.

Theorem 1.2. *If* $u_0\in H^{K+1}(\Gamma)$, $k\geqslant 1$, *and the interpolation operator* Π *satisfies*

$$\|v_0-\Pi v_0\|_{H^s(\Gamma)}\leqslant Ch^{k+1-s}|v_0|_{k+1,\Gamma},\ \forall v_0\in H^{k+1}(\Gamma),\quad 0\leqslant s<k+1,$$

then $$\|u_0-U_0^{(N)}\|_{\bar{D}}\leqslant Ch^{k+\frac{1}{2}}\|u_0\|_{k+1,\Gamma},$$

where $$h=\frac{2\pi}{N}.$$

Proof. From lemma 1.1 and interpolation inequality[5], we get

$$\|u_0-U_0^{(N)}\|_D\leqslant C\|u_0-\Pi u_0\|_{\frac{1}{2},\Gamma}\leqslant C\|u_0-\Pi u_0\|^{\frac{1}{2}}_{L_2(\Gamma)}\|u_0-\Pi u_0\|^{\frac{1}{2}}_{H^1(\Gamma)}.$$

Using $$\|u_0-\Pi u_0\|_{L_2(\Gamma)}\leqslant Ch^{k+1}|u_0|_{k+1,\Gamma},\quad \|u_0-\Pi u_0\|_{H^1(\Gamma)}\leqslant Ch^k|u_0|_{k+1,\Gamma},$$

we have $$\|u_0-U_0^{(N)}\|_D\leqslant Ch^{k+\frac{1}{2}}\|u_0\|_{k+1,\Gamma}.$$

For the sake of simplicity, from now on we denote every arbitrary constant by C.

Lemma 1.2. *If* $u_n\in H^{\frac{1}{2}}(\Gamma)$ *and satisfies the consistency condition, then the solution of* (1) $u_0\in H^{\frac{3}{2}}(\Gamma)$ and $\|u_0\|_{\frac{3}{2},\Gamma}\leqslant C\|u_n\|_{\frac{1}{2},\Gamma}$ *where* C *is independent of* u_n.

It can be obtained from the differentiability result of solution of harmonic problem and the trace theorem.

Theorem 1.3. *If the condition of theorem 1.2 is satisfied and*

$$\int_0^{2\pi}[u_0(\theta)-U_0^{(N)}(\theta)]d\theta=0,\ \text{then}\ \|u_0-U_0^{(N)}\|_{L_2(\Gamma)}\leqslant Ch^{k+1}\|u_0\|_{k+1,\Gamma}.$$

Proof. Let W_0 is the solution of $-\frac{1}{4\pi\sin^2\frac{\theta}{2}}*W_0=u_0-U_0^{(N)}$, we have $\overline{D}(W_0,v_0)=(u_0-U_0^{(N)},v_0)$, $\forall v_0\in H^{\frac{1}{2}}(\Gamma)$. Then $\overline{D}(W_0,u_0-U_0^{(N)})=(u_0-U_0^{(N)},u_0-U_0^{(N)})$. Using lemma 1.1, we get

$$\|u_0-U_0^{(N)}\|^2_{L_2(\Gamma)}=\overline{D}(W_0-\Pi W_0,u_0-U_0^{(N)})\leqslant C\|W_0-\Pi W_0\|_{H^{1/2}(\Gamma)/P_0}\|u_0-U_0^{(N)}\|_D.$$

Since $u_0-U_0^{(N)}\in H^{\frac{1}{2}}(\Gamma)$, from lemma 1.2, we obtain $\|W_0\|_{H^{3/2}(\Gamma)/P_0}\leqslant C\|u_0-U_0^{(N)}\|_D$, then using theorem 1.2, we have

$$\|u_0-U_0^{(N)}\|^2_{L_2(\Gamma)}\leqslant Ch\|u_0-U_0^{(N)}\|^2_D\leqslant Ch^{2k+2}\|u_0\|^2_{k+1,\Gamma},$$

i. e. $$\|u_0-U_0^{(N)}\|_{L_2(\Gamma)}\leqslant Ch^{k+1}\|u_0\|_{k+1,\Gamma}.$$

We can also easily prove

Lemma 1.3. *If* $v_0\in S_N$ *and* S_N *is the space spanned by piecewise linear basis functions, then*

$$\max_{[\theta_i,\theta_{i+1}]}|v_0|\leqslant\sqrt{\frac{6}{h}}\|v_0\|_{L_2[\theta_i,\theta_{i+1}]}.$$

From this we obtain

Theorem 1.4. *If* S_N *is such as in lemma 1.3 and* $u_0\in H^2(\Gamma)$ *satisfies*

$$\int_0^{2\pi}[u_0(\theta)-U_0^{(N)}(\theta)]d\theta=0,\quad \text{then}$$

$$\max_{[0,2\pi]}|u_0(\theta)-U_0^{(N)}(\theta)|\leqslant Ch^{\frac{3}{2}}\|u_0\|_{H^2(\Gamma)}.$$

This estimate is not the best to be expected.

1.3. Numerical example

$$\begin{cases}\Delta u=0,\ \text{in}\ \Omega=\text{interior or exterior to the unit circle},\\ \left.\dfrac{\partial u}{\partial n}\right|_{r=1}=\cos\theta.\end{cases}$$

Taking piecewise linear basis functions, we have

N	Number of nodes	$\max_i \lvert U_i - u_0(\theta_i) \rvert$	Ratio	Remark
16	16	0.01235024	15.540953	$\left(\frac{64}{16}\right)^2=16$
64	64	0.00079469		
			3.9806151	$\left(\frac{128}{64}\right)^2=4$
128	128	0.00019964		

Taking piecewise quadratic basis functions, we have

N	Number of nodes	$\max_i \lvert U_i - u_0(\theta_i) \rvert$	Ratio	Remark
4	8	0.02008343	11.870902	$\left(\frac{8}{4}\right)^3=8$
8	16	0.00169182		
			11.986822	$\left(\frac{16}{8}\right)^3=8$
16	32	0.00014114		

2. Biharmonic Canonical Integral Equation

2.1. Numerical solution

Consider the biharmonic boundary-value problem in the interior to the unit circle

$$\begin{cases} \Delta^2 u=0, & \text{in } \Omega, \\ Mu(\theta)=m(\theta),\ Qu(\theta)=q(\theta), & \text{on } \Gamma, \end{cases} \tag{P2}$$

where $$Mu=\left[\nu\Delta u+(1-\nu)\left(\frac{\partial^2 u}{\partial x^2}n_x^2+\frac{\partial^2 u}{\partial y^2}n_y^2+2\frac{\partial^2 u}{\partial x\,\partial y}n_x n_y\right)\right]_\Gamma,$$

$$Qu=\left\{-\frac{\partial}{\partial n}\Delta u+(1-\nu)\frac{\partial}{\partial S}\left[\left(\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}\right)n_x n_y+\frac{\partial^2 u}{\partial x\,\partial y}(n_y^2-n_x^2)\right]\right\}_\Gamma,$$

ν is Poisson ratio when $0<\nu<\frac{1}{2}$, $m(\theta)\in H^{-\frac{1}{2}}(\Gamma)$, $q(\theta)\in H^{-\frac{3}{2}}(\Gamma)$ and they satisfy the condition

$$\int_\Gamma\left(m\frac{\partial p}{\partial n}+qp\right)ds=0,\quad \forall p\in P_1(\Omega),$$

where for biharmonic problem $(\nu=1)$

$$P_1(\Omega)=\{u\in H^2(\Omega)\mid \Delta u=0\},$$

and for plate-bending problem $\left(0<\nu<\frac{1}{2}\right)$

$$P_1(\Omega)=\{\text{polynomial which degree}\leqslant 1\}.$$

From [4] we know that the boundary-value problem (P2) can be reduced into the canonical integral equation

$$\begin{cases} m(\theta)=(1+\nu)u_n(\theta)-\int_0^{2\pi}\frac{1}{2\pi\sin^2\frac{\theta-\theta'}{2}}u_n(\theta')d\theta' \\ \qquad +(1+\nu)u_0''(\theta)+\int_0^{2\pi}\frac{1}{2\pi\sin^2\frac{\theta-\theta'}{2}}u_0(\theta')d\theta', \\ q(\theta)=(1+\nu)u_n''(\theta)+\int_0^{2\pi}\frac{1}{2\pi\sin^2\frac{\theta-\theta'}{2}}u_n(\theta')d\theta' \\ \qquad -(1+\nu)u_0''(\theta)+\int_0^{2\pi}\frac{1}{2\pi\sin^2\frac{\theta-\theta'}{2}}u_0''(\theta')d\theta'. \end{cases}$$

However, it is to be understood in the sense of generalized functions, the differential and the singular integral all can be defined through generalized Fourier series. The original solution u of (P2) is obtained from (u_n, u_0) by the Poisson formula[4].

Consider the variational problem

$$\begin{cases} \text{Find } (u_n, u_0)\in H^{\frac{1}{2}}(\Gamma)\times H^{\frac{3}{2}}(\Gamma) \text{ such that} \\ \bar{D}(u_n, u_0; v_n, v_0)=\int_0^{2\pi}(mv_n+qv_0)ds,\ \forall (v_n, v_0)\in H^{\frac{1}{2}}(\Gamma)\times H^{\frac{3}{2}}(\Gamma), \\ \bar{D}(u_n, u_0; v_n, v_0)=\int_0^{2\pi}\Big\{v_n(\theta)\Big[(1+\nu)u_n(\theta)-\int_0^{2\pi}\frac{u_n(\theta')}{2\pi\sin^2\frac{\theta-\theta'}{2}}d\theta' \\ \qquad +(1+\nu)u_0''(\theta)+\int_0^{2\pi}\frac{u_0(\theta')}{2\pi\sin^2\frac{\theta-\theta'}{2}}d\theta'\Big] \\ \qquad +v_0(\theta)\Big[(1+\nu)u_n''(\theta)+\int_0^{2\pi}\frac{u_n(\theta')}{2\pi\sin^2\frac{\theta-\theta'}{2}}d\theta' \\ \qquad -(1+\nu)u_0''(\theta)+\int_0^{2\pi}\frac{u_0''(\theta')}{2\pi\sin^2\frac{\theta-\theta'}{2}}d\theta'\Big]\Big\}d\theta. \end{cases} \tag{7}$$

We can easily prove[4]

Proposition 2.1. $\bar{D}(\partial_n u, \gamma u; \partial_n v, \gamma v)=D(u, v)$, $\forall u, v\in H^2(\Omega)$ and $\Delta^2 u=0$; $\bar{D}(u_n, u_0, v_n, v_0)$ is a positive definite symmetric bilinear form on $V(\Gamma)\times V(\Gamma)$, where

$$D(u, v)=\iint_\Omega\Big\{\Delta u\,\Delta v-(1-\nu)\times\Big[\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial x^2}\frac{\partial^2 u}{\partial y^2}-2\frac{\partial^2 u}{\partial x\partial y}\frac{\partial^2 v}{\partial x\partial y}\Big]\Big\}dx\,dy,$$

$$V(\Gamma)=[H^{\frac{1}{2}}(\Gamma)\times H^{\frac{3}{2}}(\Gamma)]/\bar{P}_1(\Gamma),$$

$$\bar{P}_1(\Gamma)=\Big\{\Big(\frac{\partial p}{\partial n}, p\Big)_\Gamma\in H^{\frac{1}{2}}(\Gamma)\times H^{\frac{3}{2}}(\Gamma)\Big|_{p\in P_1(\Omega)}\Big\}.$$

Proposition 2.2. The variational problem (7) has one and only one solution in $V(\Gamma)$.

Take the piecewise Hermite cubic basis functions

$$F_j(\theta)=\begin{cases}-2\left(\frac{N}{2\pi}\right)^3(\theta-\theta_{j-1})^3+3\left(\frac{N}{2\pi}\right)^2(\theta-\theta_{j-1})^2, & \theta_{j-1}\leqslant\theta\leqslant\theta_j,\\ 2\left(\frac{N}{2\pi}\right)^3(\theta-\theta_j)^3-3\left(\frac{N}{2\pi}\right)^2(\theta-\theta_j)^2+1, & \theta_j\leqslant\theta\leqslant\theta_{j+1},\\ 0, & \text{otherwise},\end{cases}$$

$$G_j(\theta)=\begin{cases}\left(\frac{N}{2\pi}\right)^2(\theta-\theta_{j-1})^3-\frac{N}{2\pi}(\theta-\theta_{j-1})^2, & \theta_{j-1}\leqslant\theta\leqslant\theta_j,\\ \left(\frac{N}{2\pi}\right)^2(\theta-\theta_j)^3-2\left(\frac{N}{2\pi}\right)(\theta-\theta_j)^2+(\theta-\theta_j), & \theta_j\leqslant\theta\leqslant\theta_{j+1},\\ 0, & \text{otherwise},\end{cases}$$

$$j=1,\ 2,\ \cdots,\ N.$$

Obviously, $\{F_i(\theta)\}\cup\{G_i(\theta)\}\subset H^{\frac{3}{2}}(\Gamma)$. Let

$$u_n(\theta)\approx U_n(\theta)=\sum_{j=1}^{N}(X_jF_j(\theta)+Y_jG_j(\theta)),$$

$$u_0(\theta)\approx U_0(\theta)=\sum_{j=1}^{N}(U_jF_j(\theta)+V_jG_j(\theta)),$$

then from (7) we obtain

$$Q\begin{bmatrix}X\\Y\\U\\V\end{bmatrix}=\begin{bmatrix}Q_{11}&Q_{12}&Q_{13}&Q_{14}\\Q_{21}&Q_{22}&Q_{23}&Q_{24}\\Q_{31}&Q_{32}&Q_{33}&Q_{34}\\Q_{41}&Q_{42}&Q_{43}&Q_{44}\end{bmatrix}\begin{bmatrix}X\\Y\\U\\V\end{bmatrix}=\begin{bmatrix}\alpha\\\beta\\\gamma\\\delta\end{bmatrix}, \tag{8}$$

where

$$\begin{cases}\alpha_i=\int_0^{2\pi}m(\theta)F_i(\theta)d\theta, & \beta_i=\int_0^{2\pi}m(\theta)G_i(\theta)d\theta,\\ \gamma_i=\int_0^{2\pi}q(\theta)F_i(\theta)d\theta, & \delta_i=\int_0^{2\pi}q(\theta)G_i(\theta)d\theta;\end{cases} \tag{9}$$

$$Q_{11}=(1+\nu)\left(\frac{52\pi}{35N},\ \frac{9\pi}{35N},\ 0,\ \cdots,\ 0,\ \frac{9\pi}{35N}\right)+(a_0,\ a_1,\ \cdots,\ a_{N-1}),$$

$$Q_{12}=Q_{21}^T=(1+\nu)\left(0,\ -\frac{13\pi^2}{105N^2},\ 0,\ \cdots,\ 0,\ \frac{13\pi^2}{105N^2}\right)+(e_0,\ e_{N-1},\ \cdots,\ e_1),$$

$$Q_{13}=Q_{31}^T=-(1+\nu)\left(\frac{6N}{5\pi},\ -\frac{3N}{5\pi},\ 0,\ \cdots,\ 0,\ -\frac{3N}{5\pi}\right)-(a_0,\ a_1,\ \cdots,\ a_{N-1}),$$

$$Q_{14}=Q_{41}^T=(1+\nu)\left(0,\ -\frac{1}{10},\ 0,\ \cdots,\ 0,\ \frac{1}{10}\right)+(e_0,\ e_1,\ \cdots,\ e_{N-1}),$$

$$Q_{22}=(1+\nu)\left(\frac{16\pi^3}{105N^3},\ -\frac{2\pi^3}{35N^3},\ 0,\ \cdots,\ 0,\ -\frac{2\pi^3}{35N^3}\right)+(d_0,\ d_1,\ \cdots,\ d_{N-1}),$$

$$Q_{23}=Q_{32}^T=(1+\nu)\left(0,\ \frac{1}{10},\ 0,\ \cdots,\ 0,\ -\frac{1}{10}\right)+(e_0,\ e_{N-1},\ \cdots,\ e_1),$$

$$Q_{24}=Q_{42}^T=-(1+\nu)\left(\frac{8\pi}{15N},\ -\frac{\pi}{15N},\ 0,\ \cdots,\ 0,\ -\frac{\pi}{15N}\right)-(d_0,\ d_1,\ \cdots,\ d_{N-1}),$$

$$Q_{33}=(1+\nu)\left(\frac{6N}{5\pi},\ -\frac{3N}{5\pi},\ 0,\ \cdots,\ 0,\ -\frac{3N}{5\pi}\right)+(b_0,\ b_1,\ \cdots,\ b_{N-1}),$$

$$Q_{34}=Q_{43}^T=(1+\nu)\left(0,\ \frac{1}{10},\ 0,\ \cdots,\ 0,\ -\frac{1}{10}\right)+(f_0,\ f_{N-1},\ \cdots,\ f_1),$$

$$Q_{44}=(1+\nu)\left(\frac{8\pi}{15N},\ -\frac{\pi}{15N},\ 0,\ \cdots,\ 0,\ -\frac{\pi}{15N}\right)+(e_0,\ e_1,\ \cdots,\ e_{N-1}); \tag{10}$$

$$b_i=\frac{18N^4}{\pi^5}\sum_{j=1}^{\infty}\frac{1}{j^3}\left(\frac{4N^2}{\pi^2j^2}\sin^4\frac{j\pi}{N}-\frac{4N}{\pi j}\sin^2\frac{j\pi}{N}\sin\frac{j}{N}2\pi+\sin^2\frac{j}{N}2\pi\right)\cos\frac{ji}{N}2\pi,$$

$$a_i=\frac{18N^4}{\pi^5}\sum_{j=1}^{\infty}\frac{1}{j^5}\left(\frac{4N^2}{\pi^2j^2}\sin^4\frac{j\pi}{N}-\frac{4N}{\pi j}\sin^2\frac{j\pi}{N}\sin\frac{j}{N}2\pi+\sin^2\frac{j}{N}2\pi\right)\cos\frac{ji}{N}2\pi,$$

$$c_i=\frac{N^2}{\pi^3}\sum_{j=1}^{\infty}\frac{1}{j^3}\left[\frac{18N^2}{\pi^2j^2}\sin^2\frac{j}{N}2\pi-\frac{N}{\pi j}\left(48\sin\frac{j}{N}2\pi+12\sin\frac{j}{N}4\pi\right)\right.$$
$$\left.+72\cos^2\frac{j}{N}\pi-8\sin\frac{j}{N}\pi\sin\frac{j}{N}3\pi\right]\cos\frac{ji}{N}2\pi,$$

$$d_i=\frac{N^2}{\pi^3}\sum_{j=1}^{\infty}\frac{1}{j^5}\left[\frac{18N^2}{\pi^2j^2}\sin^2\frac{j}{N}2\pi-\frac{N}{\pi j}\left(48\sin\frac{j}{N}2\pi+12\sin\frac{j}{N}4\pi\right)\right.$$
$$\left.+72\cos^2\frac{j}{N}\pi-8\sin\frac{j}{N}\pi\sin\frac{j}{N}3\pi\right]\cos\frac{ji}{N}2\pi,$$

$$f_i=\frac{N^3}{\pi^4}\sum_{j=1}^{\infty}\frac{1}{j^3}\left[-\frac{36N^2}{\pi^2j^2}\sin^2\frac{j\pi}{N}\sin\frac{j}{N}2\pi+\frac{12N}{\pi j}\sin^2\frac{j\pi}{N}\left(5\cos\frac{j}{N}2\pi+7\right)\right.$$
$$\left.-6\sin\frac{j}{N}4\pi-24\sin\frac{j}{N}2\pi\right]\sin\frac{ji}{N}2\pi,$$

$$e_i=\frac{N^3}{\pi^4}\sum_{j=1}^{\infty}\frac{1}{j^5}\left[-\frac{36N^2}{\pi^2j^2}\sin^2\frac{j\pi}{N}\sin\frac{j}{N}2\pi+\frac{12N}{\pi j}\sin^2\frac{j\pi}{N}\left(5\cos\frac{j}{N}2\pi+7\right)\right.$$
$$\left.-6\sin\frac{j}{N}4\pi-24\sin\frac{j}{N}2\pi\right]\sin\frac{ji}{N}2\pi,$$

$$i=0,\ 1,\ \cdots,\ N-1. \tag{11}$$

All these series are convergent. Q is semi-positive definite.

For biharmonic problem in the exterior to the unit circle we have similar result.

2.2. Convergence and error estimates

Set the product space spanned by basis functions to be $S_N\subset H^{\frac{1}{2}}(\Gamma)\times H^{\frac{3}{2}}(\Gamma)$, $(u_n,\ u_0)$ to be the solution of (7), $(U_n^{(N)},\ U_0^{(N)})$ to be its approximate solution. $\|\cdot\|_{\bar D}$ is the energy norm on $[H^{\frac{1}{2}}(\Gamma)\times H^{\frac{3}{2}}(\Gamma)]/\bar P_1(\Gamma)$ derived from $\bar D(\cdot,\ \cdot;\ \cdot,\ \cdot)$. Using such method as used in section 1.2, we can obtain following results.

Theorem 2.1. *If the interpolation operator Π which constructs S_N satisfies $\|v_0-\Pi v_0\|_{H^{3/2}(\Gamma)}\xrightarrow[h\to0]{}0$, $\forall v_0\in H^{\frac{3}{2}}(\Gamma)$, and the solution $(u_n,\ u_0)\in H^{\frac{1}{2}}(\Gamma)\times H^{\frac{3}{2}}(\Gamma)$ of (7) exists, then*

$$\lim_{N\to\infty}\|(u_n-U_n^{(N)},\ u_0-U_0^{(N)})\|_{\bar D}=0.$$

Theorem 2.2. *If $(u_n,\ u_0)\in H^{k+1}(\Gamma)\times H^{k+2}(\Gamma)$, and the interpolation operator Π satisfies*

$$\|v_0-\Pi v_0\|_{H^s(\Gamma)}\leqslant Ch^{k+1-s}|v_0|_{k+1,\Gamma},\quad \forall v_0\in H^{k+1}(\Gamma),\ 0\leqslant s<k+1,$$

then
$$\|(u_n,\ u_0)-(U_n^{(N)},\ U_0^{(N)})\|_{\bar D}\leqslant Ch^{k+\frac{1}{2}}\|(u_n,\ u_0)\|_{H^{k+1}(\Gamma)\times H^{k+2}(\Gamma)}.$$

Theorem 2.3. *If the condition of theorem 2.2 is satisfied and*

$$\int_\Gamma\Big[(u_n-U_n^{(N)})\frac{\partial p}{\partial n}+(u_0-U_0^{(N)})p\Big]dS=0,\quad \forall p\in P_1(\Omega),$$

then $$\|(u_n-U_n^{(N)},\ u_0-U_0^{(N)})\|_{L_2(\Gamma)\times L_2(\Gamma)}\leqslant Ch^{k+1}\|(u_n,\ u_0)\|_{H^{k+1}(\Gamma)\times H^{k+2}(\Gamma)}.$$

Theorem 2.4. *If* $(u_n,\ u_0)\in H^4(\Gamma)\times H^5(\Gamma)$,

$$\int_\Gamma\Big[(u_n-U_n^{(N)})\frac{\partial p}{\partial n}+(u_0-U_0^{(N)})p\Big]dS=0,\quad \forall p\in P_1(\Omega),$$

and Π is the piecewise Hermite cubic interpolation operator, then

$$\max\big[\max_{[0,2\pi]}|u_n(\theta)-U_n^{(N)}(\theta)|,\ \max_{[0,2\pi]}|u_0(\theta)-U_0^{(N)}(\theta)|\big]\leqslant Ch^{\frac{7}{2}}\|(u_n,\ u_0)\|_{H^4(\Gamma)\times H^5(\Gamma)}.$$

2.3. Numerical examples (Take $\nu=0.5$)

(a) $$\begin{cases}\Delta^2u=0, \text{ in } \Omega=\text{interior to the unit circle,}\\ Mu=-12\cos 3\theta,\ Qu=48\cos 3\theta, \text{ on } \partial\Omega.\end{cases}$$

N	$\max_i\|U_n(\theta_i)-u_n(\theta_i)\|$	$\max_i\|U'_n(\theta_i)-u'_n(\theta_i)\|$	$\max_i\|U_0(\theta_i)-u_0(\theta_i)\|$	$\max_i\|U'_0(\theta_i)-u'_0(\theta_i)\|$
24	0.1238714×10^{-2}	0.8178915×10^{-1}	0.3420155×10^{-3}	0.1917666×10^{-2}
48	0.6614398×10^{-4}	0.1347160×10^{-1}	0.1950099×10^{-4}	0.1346814×10^{-3}
Ratio	18.72754	6.0712276	17.538366	14.238536

N	R	0.1	0.3	0.5	0.7
48	$U(R,\ 0)$	0.1990031×10^{-2}	0.5157094×10^{-1}	0.2187542	0.5179423
	Error	0.3160356×10^{-7}	0.9467044×10^{-6}	0.4218880×10^{-5}	0.1237349×10^{-4}
	Relative error	0.1588118×10^{-4}	0.1835765×10^{-4}	0.1928630×10^{-4}	0.2389027×10^{-4}

(b) $$\begin{cases}\Delta^2u=0, \text{ in } \Omega=\text{exterior to the unit circle,}\\ Mu=-3\cos 3\theta,\ Qu=33\cos 3\theta, \text{ on } \partial\Omega.\end{cases}$$

N	$\max_i\|U_n(\theta_i)-u_n(\theta_i)\|$	$\max_i\|U'_n(\theta_i)-u'_n(\theta_i)\|$	$\max_i\|U_0(\theta_i)-u_0(\theta_i)\|$	$\max_i\|U'_0(\theta_i)-u'_0(\theta_i)\|$
24	0.1293942×10^{-2}	0.6520362×10^{-1}	0.4034118×10^{-3}	0.2612500×10^{-2}
48	0.8534865×10^{-4}	0.1199005×10^{-1}	0.2986027×10^{-4}	0.1632207×10^{-3}
Ratio	15.16066	5.4381441	13.509985	16.005935

N	R	1.5	5	20	100
48	$U(R,\ 0)$	0.6666793	0.2000020	0.5000057×10^{-1}	0.9999837×10^{-2}
	Error	0.1271495×10^{-4}	0.2079648×10^{-5}	0.5751819×10^{-6}	0.1627392×10^{-6}
	Relative error	0.1907242×10^{-4}	0.1039824×10^{-4}	0.1150363×10^{-4}	0.1627392×10^{-4}

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