THE SPLITTING EXTRAPOLATION METHOD FOR MULTIDIMENSIONAL PROBLEMS*

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Abstract

This note presents a splitting extrapolation process, which uses successively one-dimensional extrapolation procedure along only one variable with other variables kept fixed. This splitting technique is applied to the numerical cubature of multiple integrals, multidimensional integral equations and the difference method for solving the Poisson equation. For each case, the corresponding error estimates are given. They show the advantage of this method over the isotropic extrapolation along all the variables.

1. Introduction

The extrapolation method is a simple and effective numerical method for computing integration and solving differential equations in the case of one dimension. For the multidimensional problems one can use extrapolation process along all variables homogeneously, but the effort will be high. This note presents the so called splitting extrapolation process, which uses the one-dimensional extrapolation process along only one variable, the other variables fixed. We hope this method is appropriate for the parallel algorithm and will save computational effort in comparison with the isotropic extrapolation.

2. Multiple Integrals

We are concerned with the s-dimensional integral in a cube:

$$I=\int_V f(x)dx \quad \text{with} \quad V=[-1, 1]^s.$$

Let us divide V in cuboids of length $h=(h_1, \cdots, h_s)$:

$$V=\bigcup_{j=1}^{n} V_j, \quad V_j=\prod_{i=1}^{s}\left[M_{ji}-\frac{h_i}{2}, \quad M_{ji}+\frac{h_i}{2}\right],$$

where $M_j=(M_{j1}, \cdots, M_{js})$ is the center of V_j. We define the rectangular cubature by

$$I_R(h_1, \cdots, h_s)=\sum_{j=1}^{n} \text{meas } (V_j)f(M_j)$$

and the trapezoidal cubature by

$$I_T(h_1, \cdots, h_s)=\sum_{j=1}^{n}\frac{1}{2s}\sum_{i=1}^{s}\text{meas}(V_j)\left(f(N_{ji}^{+})+f(N_{ji}^{-})\right),$$

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where $N_{ji}^{\pm}=(M_{j1}, \cdots, M_{ji}\pm\frac{h_i}{2}, \cdots, M_{js})$ are the centers on the faces of V_j.

The principle of splitting extrapolation bases on the following asymptotic expansion.

Theorem 1. *For* $f\in C^{(2m+2)}(V)$ *we have*

$$I-I_R(h_1, \cdots, h_s)=\sum_{\substack{|p|=k\\1\le k\le m}} c_{2p}h^{2p}+0(h_0^{2m+2}) \tag{1}$$

with the coefficients c_{2p} *independent of* h *and*

$$p=(p_1, \cdots, p_s),\ |p|=p_1+\cdots+p_s,\ h^q=h_1^{q_1}\cdots h_s^{q_s},\ h_0=\max\{h_1, \cdots, h_s\}.$$

Proof. Note first that

$$I-I_R(h_1, \cdots, h_s)=\sum_{j=1}^{n}\int_{V_j}(f(x)-f(M_j))dx. \tag{2}$$

Then insertion of the Taylor expansion

$$f(x)-f(M_j)=\sum_{\substack{|p|=k\\1\le k\le 2m+1}}\frac{1}{k!}f^{(p)}(M_j)(x-M_j)^p+0(h_0^{2m+2})$$

into (2) and use of

$$\int_{V_j}(x-M_j)^q dx=\begin{cases}0 \text{ if } q \text{ contains an odd component } q_i,\\ \dfrac{\operatorname{meas}(V_j)}{\prod_{i=1}^{s}(1+2p_i)}\left(\dfrac{h}{2}\right)^{2p} \text{ when } q=2p\end{cases}$$

imply

$$I-I_R(h_1, \cdots, h_s)=\sum_{j=1}^{n}\sum_{\substack{|p|=k\\1\le k\le m}}\frac{f^{(2p)}(M_j)}{(2k)!}\frac{\operatorname{meas}(V_j)}{\prod_{i=1}^{s}(1+2p_i)}\left(\frac{h}{2}\right)^{2p}+0(h_0^{2m+2}).$$

By induction we assume that (1) holds for $k\le m$ and come to prove it for $m+1$. In fact for $f\in C^{(2m+4)}(V)$,

$$\begin{aligned}I-I_R(h_1, \cdots, h_s)&=\sum_{j=1}^{n}\sum_{\substack{|p|=k\\1\le k\le m+1}}\frac{f^{(2p)}(M_j)}{(2k)!}\frac{\operatorname{meas}(V_j)}{\prod_{i=1}^{s}(1+2p_i)}\left(\frac{h}{2}\right)^{2p}+0(h_0^{2m+4})\\
&=\sum_{j=1}^{n}\sum_{\substack{|p|=k\\1\le k\le m+1}}\int_{V_j}f^{(2p)}(x)dx\frac{1}{(2k)!\prod_{i=1}^{s}(1+2p_i)}\left(\frac{h}{2}\right)^{2p}\\
&\quad-\sum_{j=1}^{n}\sum_{\substack{|p|=k\\1\le k\le m+1}}\int_{V_j}(f^{(2p)}(x)-f^{(2p)}(M_j))\frac{1}{(2k)!\prod_{i=1}^{s}(1+2p_i)}\left(\frac{h}{2}\right)^{2p}+0(h_0^{2m+4})\\
&=\sum_{\substack{|p|=k\\1\le k\le m+1}}\int_{V}f^{(2p)}(x)dx\frac{1}{(2k)!\prod_{i=1}^{s}(1+2p_i)}\left(\frac{h}{2}\right)^{2p}\\
&\quad-\sum_{\substack{|p|=k\\1\le k\le m+1}}\sum_{j=1}^{n}\int_{V_j}(f^{(2p)}(x)-f^{(2p)}(M_j))dx\frac{1}{(2k)!\prod_{i=1}^{s}(1+2p_i)}\left(\frac{h}{2}\right)^{2p}+0(h_0^{2m+4}).\end{aligned}$$

Then substitution of the asymptotic expansion (1) for $k\le m$ into

$$\sum_{j=1}^{n}\int_{V_j}(f^{(2p)}(x)-f^{(2p)}(M_j))dx,\ |p|\leqslant m+1$$

in the last term implies (1) for $m+1$.

Theorem 2. *For* $f\in C^{(2m+2)}(V)$,

$$I-I_T(h_1,\ \cdots,\ h_s)=\sum_{\substack{|p|=k\\1<k<m}}d_{2p}\,h^{2p}+0(h_0^{2m+2}).\tag{3}$$

Proof. Note first that

$$I-I_T(h_1,\ \cdots,\ h_s)=\sum_{j=1}\int_{V_j}(f(x)-\frac{1}{2s}\sum_{i=1}^{s}(f(N_{ji}^{+})+f(N_{ji}^{-})))dx=J_1+J_2,\tag{4}$$

where

$$J_1=\sum_{j=1}^{n}\int_{V_j}(f(x)-f(M_j))dx\tag{5}$$

and

$$\begin{aligned}J_2&=\sum_{j=1}^{n}\int_{V_j}(f(M_j)-\frac{1}{2s}\sum_{i=1}^{s}(f(N_{ji}^{+})+f(N_{ji}^{-}))dx\\&=-\sum_{j=1}^{n}\frac{\text{meas}(V_j)}{2s}\sum_{i=1}^{s}(f(N_{ji}^{+})-2f(M_j)+f(N_{ji}^{-}))\\&=-\sum_{j=1}^{n}\frac{\text{meas}(V_j)}{s}\sum_{i=1}^{s}\sum_{t=1}^{m}\frac{1}{(2t)!}D_{x_i}^{(2t)}f(M_j)\left(\frac{h_i}{2}\right)^{2t}+0(h_0^{2m+2})\\&=-\frac{1}{s}\sum_{j=1}^{n}\sum_{i=1}^{s}\sum_{t=1}^{m}\frac{1}{(2t)!}\int_{V_j}D_{x_i}^{(2t)}f(x)dx\left(\frac{h_i}{2}\right)^{2t}\\&\quad+\frac{1}{s}\sum_{j=1}^{n}\sum_{i=1}^{s}\sum_{t=1}^{m}\frac{1}{(2t)!}\int_{V_j}(D_{x_i}^{(2t)}f(x)-D_{x_i}^{(2t)}f(M_j))dx\left(\frac{h_j}{2}\right)^{2t}+0(h_0^{2m+2})\\&=-\frac{1}{s}\sum_{i=1}^{s}\sum_{t=1}^{m}\frac{1}{(2t)!}\int_{V}D_{x_i}^{(2t)}f(x)dx\left(\frac{h_i}{2}\right)^{2t}\\&\quad+\frac{1}{s}\sum_{i=1}^{s}\sum_{t=1}^{m}\frac{1}{(2t)!}\sum_{j=1}^{n}\int_{V_j}(D_{x_i}^{(2t)}f(x)-D_{x_i}^{(2t)}f(M_j))dx\left(\frac{h_j}{2}\right)^{2t}+0(h_0^{2m+2}).\end{aligned}\tag{6}$$

Then substitution of the asymptotic expansion (1) into J_1 and the last term of J_2 implies (3) with the coefficients d_{2p} independent of h. This completes the proof of Theorem 2.

Comparing (4)—(6) with (2) we can see the relation between the coefficients of (1) and (3):

$$d_{2p}=c_{2p}-\frac{3}{s}\,c_{2p},\ \ |p|=1.$$

Particularly, we have

Corollary 1. For $s=3$,

$$I-I_T(h_1,\ h_2,\ h_3)=0(h_0^4).$$

For $s\neq 3$, $$I-\frac{s}{3}(I_T(h_1,\ \cdots,\ h_s)-\left(1-\frac{3}{s}\right)I_R(h_1,\ \cdots,\ h_s))=0(h_0^4).$$

Therefore we can combine I_T with I_R to generate a cubature with the same accuracy as the Simpson rule.

We now turn to the extrapolation method. In the one dimensional case we divide the interval successively in 1, 2, 2^2, $\cdots$ equal subintervals and compute the corresponding rectangular (or trapezoidal) cubatures $T_0^{(0)}$, $T_0^{(1)}$, $T_0^{(2)}$, $\cdots$respectively. Then we define the Romberg sequence

$$T_m^{(k)}=\frac{4^m T_{m-1}^{(k+1)}-T_{m-1}^{(k)}}{4^m-1},\quad m=1,\ 2,\ \cdots.$$

Write the Romberg sequence

$$T_m^{(0)}=R(T_0^{(0)},\ \cdots,\ T_0^{(m)})$$

and introduce the notation

$$I(\cdot,\ x_i,\ \cdot)$$

in which only x_i is regarded as a variable but the others, x_j $(j\neq i)$, are fixed.

A splitting extrapolation process with accuracy of $0(h^{2m+1})$ is defined as follows. Start with

$$T_0^{(0)}=I^{(0)}(h_1,\ \cdots,\ h_s)=I_R(h_1,\ \cdots,\ h_s)$$

and continue with

$$I^{(r+1)}(h_1,\ \cdots,\ h_s)=\frac{1}{r+1}\sum_{i=1}^{s}T_{m-r,i}^{(r)}-\frac{1}{r+1}(s-r-1)I^{(r)}(h_1,\ \cdots,\ h_s),$$

where $\quad T_{m-r,i}^{(r)}=R(I^{(r)}\left(\cdot,\ \frac{h_i}{2^k},\ \cdot\right),\ 0\leqslant k\leqslant m-r),\ 0\leqslant r\leqslant m-1.$

Theorem 3. *For* $f\in C^{(2m+2)}(V)$,

$$I-I^{(m)}(h_1,\ \cdots,\ h_s)=0(h_0^{2m+2}).$$

Proof. Let $Q(k)$ be the set of vectors with only k components being non-zeros. Then the asymptotic expansion (1) can be rewritten as

$$I-I^{(0)}(h_1,\ \cdots,\ h_s)=\sum_{k=1}^{s}\sum_{1\leqslant|q|\leqslant m}\sum_{q\in Q(k)}c_{2q}h^{2q}+0(h_0^{2m+2}).\tag{7}$$

We regard h_i as variable but $h_j(j\neq i)$ as constants in (7), and compute the Romberg sequence along h_i:

$$T_{m,i}^{(0)}=R\left(I^{(0)}\left(\cdot,\ \frac{h_i}{2^k},\ \cdot\right),\ 0\leqslant k\leqslant m\right).$$

We can see that

$$I-T_{m,i}^{(0)}=\sum_{k=1}^{s}\sum_{1\leqslant|q|\leqslant m}\sum_{\substack{q\in Q(k)\\ q_i=0}}c_{2q}h^{2q}+0(h_0^{2m+2}),\tag{8}$$

since by extrapolation the terms containing h_i have been eliminated. Summation of (8) leads to

$$sI-\sum_{i=1}^{s}T_{m,i}^{(0)}=\sum_{k=1}^{s}(s-k)\sum_{1\leqslant|q|\leqslant m}\sum_{q\in Q(k)}c_{2q}h^{2q}+0(h_0^{2m+2}).$$

Combining this with (7) we eliminate the single-variable terms like h_k^2, h_k^4, $\cdots$ $(1\leqslant k\leqslant s)$ and get

$$I-\left(\sum_{i=1}^{s}T_{m,i}^{(0)}-(s-1)I^{(0)}(h_1,\ \cdots,\ h_s)\right)=I-I^{(1)}$$

$$=\sum_{k=2}^{s}\sum_{2\leqslant|q|\leqslant m}\sum_{q\in Q(k)}c_{2q}^{(1)}h^{2q}+0(h_0^{2m+2}).\tag{9}$$

Note that the largest exponent of h_i in (9) is $2(m-1)$ rather than $2m$ since the single-variable terms have been eliminated. Computing

$$T_{m,i}^{(1)}=R\Big(I^{(1)}\Big(\cdot,\ \frac{h_i}{2^k},\ \cdot\Big),\ 0\leqslant k\leqslant m-1\Big)$$

we get

$$I-T_{m,i}^{(1)}=\sum_{k=2}^{s}\ \sum_{2<|q|<m}\ \sum_{\substack{q\in Q(k)\\ q_i=0}} c_{2q}^{(1)}h^{2q}+O(h_0^{2m+2}). \tag{10}$$

Then summation of (10) leads to

$$sI-\sum_{i=1}^{s}T_{m,i}^{(1)}=\sum_{k=2}^{s}(s-k)\sum_{2<|q|<m}\ \sum_{q\in Q(k)} c_{2q}^{(1)}h^{2q}+O(h_0^{2m+2}).$$

Combining this with (9) we eliminate the twin-variable terms like $h_i^2h_j^2(i\neq j)$ and get

$$I-\frac{1}{2}\Big(\sum_{i=1}^{s}T_{m,i}^{(1)}-(s-2)I^{(1)}(h_1,\ \cdots,\ h_s)\Big)=I-I^{(2)}(h_1,\ \cdots,\ h_s)$$

$$=\sum_{k=3}^{s}\ \sum_{3<|q|<m}\ \sum_{q\in Q(k)} c_{2q}^{(2)}h^{2q}+O(h_0^{2m+2}).$$

Continuing in this way we can prove Theorem 3.

We remark that the above extrapolation process is not an optimal algorithm but a simpler one. In fact we need about $2^{m(m-1)/2}s^m$ meshpoints to get the accuracy of $O(h_0^{2m+1})$.

3. Multidimensional Integral Equations

Consider the integral equation

$$u(x)-\int_V K(x,\ y)u(y)dy=g(x) \tag{11}$$

with the smooth kernel $K(x,\ y)$ and the smooth right term g. It is well-known that the integral equation (11) can be solved approximately by the Nyström method defined as follows.

Approximating the integral operator

$$Ku=\int_V K(x,\ y)u(y)dy$$

with the rectangular cubature

$$K_Ru=\sum_{j=1}^{n}\mathrm{meas}(V_j)K(x,\ M_j)u(M_j)$$

or the trapezoidal cubature

$$K_Tu=\sum_{j=1}^{n}\frac{1}{2s}\sum_{i=1}^{s}\mathrm{meas}(V_j)\left(K(x,\ N_{ji}^{+})+K(x,\ N_{ji}^{-})\right),$$

one gets the approximate solution

$$u_R-K_Ru_R=g \quad \text{or} \quad u_T-K_Tu_T=g$$

with the solution u_R or u_T. The result is

$$u-u_R=O(h_0^2), \qquad u-u_T=O(h_0^2).$$

However, by a principle about the combination of approximate solutions proposed by the authors[1], it follows from Corollary 1 that

Corollary 2. For $s=3$,

$$u-u_T=0(h_0^4).$$

For $s\neq 3$,
$$u-\frac{s}{3}\left[u_T-\left(1-\frac{3}{s}\right)u_R\right]=0(h_0^4).$$

4. Partial Differential Equations

Our attention will be restricted to the Poisson equation

$$\Delta u=g \quad \text{in } V=(-1,\ 1)^s,$$
$$u=0 \quad \text{on } \partial V,$$

where the solution u is assumed to be sufficiently smooth.

It is well-known that the Poisson equation can be solved approximately by the finite difference method defined as follows.

Approximating the Laplace operator Δ with the central difference quotient

$$\Delta^h u=h^{-2}\sum_{k=1}^{s}(u(x_1,\ \cdots,\ x_k-h,\ \cdots,\ x_s)-2u(x_1,\ \cdots,\ x_k,\ \cdots,\ x_s)$$
$$+u(x_1,\ \cdots,\ x_k+h,\ \cdots,\ x_s))$$

defined on the mesh

$$V^h=\{(x_1,\ \cdots,\ x_s) \text{ with } x_k=m_k h,\ m_k=0,\ \pm 1,\ \cdots,\ \pm(n-1),\ nh=1\},$$

one gets the finite difference equation

$$\Delta^h u^h=g \quad \text{in } V^h,$$
$$u^h=0 \quad \text{on } \partial V^h$$

with the solution u^h defined on V^h. The result is

$$u-u^h=0(h_0^2) \quad \text{on } V^h.$$

However, by the splitting extrapolation method, we fine the mesh V^h along only one variable x_i, which is denoted by

$$V_i^h=\{(x_1,\ \cdots,\ x_s) \text{ with } x_i=\left(\frac{1}{2}+m_i\right)h,\ x_k=m_k h(k\neq i)\}\cup V^h$$

and define the corresponding difference quotient

$$\Delta_i^h u=\left(\frac{h}{2}\right)^{-2}\left[u\left(x_1,\ \cdots,\ x_i-\frac{h}{2},\ \cdots,\ x_s\right)-2u(x_1,\ \cdots,\ x_i,\ \cdots,\ x_s)\right.$$
$$\left.+u\left(x_1,\ \cdots,\ x_i+\frac{h}{2},\ \cdots,\ x_s\right)\right]+h^{-2}\sum_{k\neq i}(u(x_1,\ \cdots,\ x_k-h,\ \cdots,\ x_s)$$
$$-2u(x_1,\ \cdots,\ x_k,\ \cdots,\ x_s)+u(x_1,\ \cdots,\ x_k+h,\ \cdots,\ x_s))$$

on V_i^h. Then we compute the solutions $u_i^h(1\leqslant i\leqslant s)$ of the corresponding finite difference equations

$$\Delta_i^h u_i^h=g \quad \text{in } V_i^h,$$
$$u_i^h=0 \quad \text{on } \partial V^h.$$

In [1] the authors proved that

$$u-\frac{1}{3}\left(4\sum_{i=1}^{s}u_i^h-(4s-3)u^h\right)=0(h_0^4)$$

and that the necessary computational effort of the isotropic extrapolation method is about $2^{3(s-1)}$ times that of the splitting extrapolation method. What we can add here is that this result can be carried on to accuracy of higher order by using the Romberg process.

Reference

[1] Lin Qun and Lü Tao, The combination of approximate solutions for accelerating the convergence, submitted to *RAIRO Numer. Anal.*