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# ON THE STABILITY OF INTERPOLATION*

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**Abstract**

Some definitions on stability of interpolating process are given and then the sufficient and necessary conditions are obtained. On this basis, we conclude that the Lagrange interpolation is unstable, whereas several types of piecewise low order polynomial interpolation are stable. For high order approximation with data on isometric nodes, we recommend the Bernstein approximation owing to its high stability. Some ideas on the relationship between stability and convergence of interpolating process are also presented.

As indicated in practice, the high order polynomial interpolation is unstable, whereas the piecewise low order polynomial interpolation behaves very well. Some of the reasons have been given in [1, p. 12]. This paper will make a theoretical analysis of this problem in detail.

## 1. The Concept of Stability

By interpolation we mean: On an interval $[a, b]$, given the following infinite node triangle

$$\begin{array}{llll} x_0^0 & & & \\ x_0^1 & x_1^1 & & \\ x_0^2 & x_1^2 & x_2^2 & \\ & \cdots & & \\ x_0^n & x_1^n \cdots\cdots x_n^n & & \\ & \cdots & & \end{array} \tag{1.1}$$

where $a\leqslant x_i^n\leqslant b$ and $x_i^n\neq x_j^n (i\neq j)$, there exists a function set $S_n$ corresponding to each row of nodes $x_i^n (0\leqslant i\leqslant n)$ such that for the given data of a function $f(x)$ on nodes $x_i^n$ (function values $f(x_i^n)$ and possibly its derivatives $f^{m_i}(x_i^n)$), there exists a unique $\varphi\in S_n$ satisfying

$$f(x_i^n)=\varphi(x_i^n),$$

and possibly,

$$f^{m_i}(x_i^n)=\varphi^{m_i}(x_i^n), \quad m_i\geqslant 1.$$

Then $\varphi$ is called the interpolating function of $f$, denoted by

$$\varphi(x)=I_n[f; x].$$

In general, $S_n$ is a linear space and

$$1\in S_n, \ \forall n. \tag{1.2}$$

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Obviously, if $\varphi \in S_n$, then

$$\varphi(x) = I_n[\varphi; x]. \tag{1.3}$$

When the given data include $f(x_i^n)$ only and $S_n$ is a polynomial set with degree $\leqslant n$, this is the well-known Lagrange interpolation. When the given data include $f(x_i^n)$ and $f'(x_i^n)$, and $S_n$ is a polynomial set with degree $\leqslant 2n+1$, this is Hermite interpolation. When the given data include function values (or also derivative values) and $S_n$ is a set of piecewise polynomials with some smooth conditions on nodes, this is known as piecewise polynomial interpolation.

We now begin with the case that the given data include function values only. Due to the linearity of the interpolation operator, we give

**Definition 1.** *The interpolating process is said to be stable respect to the function, if* $\forall \varepsilon > 0$, $\exists \delta$ *such that*

$$\max_{0 \leqslant i \leqslant n} |f(x_i^n)| \leqslant \delta,$$

*implies*

$$\|I_n[f; x]\|_\infty \leqslant \varepsilon, \quad \forall n.$$

In Definition 1 and henceforward, the symbol $\|f\|_\infty$ stands for $\sup\limits_{a \leqslant x \leqslant b} |f(x)|$.

Let $\{l_i^n(x)\}_{0 \leqslant i \leqslant n}$ be the base of interpolation satisfying

$$\begin{cases} l_i^n(x) \in S_n \\ l_i^n(x_j^n) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases} \end{cases} \tag{1.4}$$

The expression of the basis functions may be simple (for example the Lagrange interpolation) or complex (for example the Hermite interpolation involving data of high order derivatives), or even may not have explicit form (for example the spline interpolation). From (1.4) each interpolating function can be written as

$$I_n[f; x] = \sum_{i=0}^{n} f(x_i^n) l_i^n(x). \tag{1.5}$$

Because of (1.2), (1.3) and (1.5),

$$\sum_{i=0}^{n} l_i^n(x) \equiv 1. \tag{1.6}$$

Denote

$$\lambda_n = \sup_{a \leqslant x \leqslant b} \sum_{i=0}^{n} |l_i^n(x)|. \tag{1.7}$$

Then we have

**Lemma 1.** *A necessary and sufficient condition for the stability of interpolation is that* $\lambda_n$ *is bounded for all* $n$.

Proof. From (1.5) and (1.7) we have

$$\|I_n[f; x]\|_\infty \leqslant \lambda_n \max_{0 \leqslant i \leqslant n} |f(x_i^n)|. \tag{1.8}$$

Thus the sufficient part is obtained.

From (1.7), for any $n$ and arbitrary small $\varepsilon > 0$ there exists $\xi_n$ such that

$$\sum_{i=0}^{n} |l_i^n(\xi_n)| \geqslant \lambda_n - \varepsilon. \tag{1.9}$$

Take

$$f(x_i^n) = \delta \cdot \operatorname{sign} l_i^n(\xi_n), \tag{1.10}$$

where $\delta$ is a small positive number. Then $\max_i |f(x_i^n)| = \delta$. From (1.5), (1.10) and (1.9) we obtain

$$\|I_n[f; x]\|_\infty \geq |I_n[f; \xi_n]| = \sum_{i=0}^{n} \delta |l_i^n(\xi_n)| \geq \delta(\lambda_n - \varepsilon).$$

Then the boundlessness of $\|I_n[f; x]\|_\infty$ can be deduced from that of $\lambda_n$. So the necessary part is completed.

**Lemma 2.** *Assume that there exists a sequence $\varphi_n \in S_n$ satisfying*

$$\lim_{n\to\infty} \|\varphi_n - f\|_\infty = 0 \tag{1.11}$$

*and*

$$\lim_{n\to\infty} \lambda_n \|\varphi_n - f\|_\infty = 0. \tag{1.12}$$

*Then the interpolating sequence $I_n[f; x]$ converges uniformly to $f$, namely*

$$\lim_{n\to\infty} \|I_n[f; x] - f\|_\infty = 0. \tag{1.13}$$

Proof. From $I_n[\varphi_n; x] = \varphi_n$ and (1.8) we have

$$\|I_n[f; x] - f\|_\infty \leq \|I_n[f; x] - I_n[\varphi_n; x]\|_\infty + \|\varphi_n - f\|_\infty \leq (\lambda_n + 1)\|\varphi_n - f\|_\infty,$$

and then (1.13) follows.

The above lemma connects the concept of convergence with the concept of stability of interpolation: For a function $f$, if there exists a uniformly convergent sequence $\varphi_n$, stability implies convergence of interpolation. (From Lemma 1 $\lambda_n$ is bounded and then (1.12) follows from (1.11)) For the Lagrange interpolation, by the Weierstrass theorem, if the interpolation is stable, the interpolating sequence $I_n[f; x]$ must be uniformly convergent for any $f \in C^0[a, b]$. On account of this, one can conclude the instability by constructing an inconvergent interpolating sequence for a continuous function. As there is some available results about the $\lambda_n$ of the Lagrange interpolation, we can prove the conclusion directly from Lemma 1.

When $\lambda_n$ is unbounded, (1.12) is still possibly true. (An example will be given in the next section) So, the concept of stability is stronger than that of convergence for interpolation.

For some problems, not only the function but also the derivatives of the function of interpolation are required to be stable. It is known that in differential calculus derivatives are unstable in response to the perturbation of function values. In other words, for differentiable functions there does not exist a $\delta$ such that $\|f(x)\|_\infty \leq \delta$ implies $\|f'(x)\|_\infty \leq \varepsilon$ (we can see this simply from the function set $f_{n,\delta}(x) = \delta \sin(nx)$). So it is unreasonable to require the derivatives of an interpolating function to be stable in response to the perturbation of function values. We should give

**Definition 2.** *The interpolating process is said to be stable respect to the kth derivative, if $\forall \varepsilon > 0$, $\exists \delta$ such that*

$$\max_{0 \leq i \leq n-k} |f[x_i^n, x_{i+1}^n, \cdots, x_{i+k}^n]| \leq \delta$$

*implies*

$$\left\| \frac{d^k}{dx^k} I_n[f; x] \right\|_\infty \leq \varepsilon, \quad \forall n.$$

The above $f[x_i, x_{i+1}, \cdots, x_{i+k}]$ denotes the difference quotient of function $f$, namely

$$f[x_i]=f(x_i),$$

$$f[x_i,\ x_{i+1}]=\frac{f[x_{i+1}]-f[x_i]}{x_{i+1}-x_i}$$

$$\cdots$$

$$f[x_i,\ x_{i+1},\ \cdots,\ x_{i+k}]=\frac{f[x_{i+1}\cdots x_{i+k}]-f[x_i\cdots x_{i+k-1}]}{x_{i+k}-x_i}.$$

Now consider the case that the given data also include derivatives. For simplicity, we merely discuss the situation involving only the first derivative, the most general case in practice. Denote $h_n=\max\limits_{0<i<n-1}(x_{i+1}^n-x_i^n)$.

**Definition 3.** *The interpolating process is said to be stable respect to the function, if* $\forall\varepsilon>0$, $\exists\delta$ *such that*

$$\max_i\{|f(x_i^n)|,\ h_n|f'(x_i^n)|\}\leqslant\delta$$

*implies* $$\|I_n[f;x]\|_\infty\leqslant\varepsilon,\quad \forall n.$$

**Definition 4.** *The interpolating process is said to be stable respect to the first derivative, if* $\forall\varepsilon>0$, $\exists\delta$ *such that*

$$\max_i\{|f[x_i^n,\ x_{i+1}^n]|,\ |f'(x_i^n)|\}\leqslant\delta$$

*implies* $$\|I_n'[f;x]\|_\infty\leqslant\varepsilon,\quad \forall n.$$

In the case of data including $f(x_i^n)$ and $f'(x_i^n)$, the base of interpolation, denoted by $l_i^n(x)$ and $\bar{l}_i^n(x)$, can be defined as follows:

$$\begin{aligned}&\text{(i)}\ \ l_i^n(x),\ \bar{l}_i^n(x)\in S_n,\\ &\text{(ii)}\ \ l_i^n(x_j^n)=\delta_{ij},\quad \frac{d}{dx}l_i^n(x_j^n)=0,\\ &\qquad \bar{l}_i^n(x_j^n)=0,\quad \frac{d}{dx}\bar{l}_i^n(x_j^n)=\delta_{ij}.\end{aligned}\tag{1.14}$$

For the base defined as (1.14), the interpolating function can be written in the following form

$$I_n[f;x]=\sum_{i=0}^n f(x_i^n)l_i^n(x)+\sum_{i=0}^n f'(x_i^n)\bar{l}_i^n(x).\tag{1.15}$$

Let
$$\bar{\lambda}_n=\sup_{a<x<b}\left\{\sum_{i=0}^n|l_i^n(x)|+\frac{1}{h_n}\sum_{i=0}^n|\bar{l}_i^n(x)|\right\}.\tag{1.16}$$

Then, by analysis similar to that in Lemma 1 we can prove

**Lemma 3.** *A necessary and sufficient condition of Hermite type interpolation, which merely involves data $f(x_i)$ and $f'(x_i)$, is that the $\bar{\lambda}_n$ defined in (1.16) is bounded.*

## 2. High Order Interpolation

For notational convenience, the superscript $n$ of $x_i^n$ and $l_i^n(x)$ will be dropped afterwards.

We first discuss the Lagrange interpolation. In this case the basis functions are

$$l_i(x)=\frac{w_{n+1}(x)}{(x-x_i)w_{n+1}'(x)}\quad (0\leqslant i\leqslant n)\tag{2.1a}$$

$$w_{n+1}(x)=\prod_{i=0}^{n}(x-x_i). \tag{2.1b}$$

No matter how the interpolating nodes $x_i\,(0\leqslant i\leqslant n)$ are selected, there is an estimation (see [2], p. 512)

$$\lambda_n=\max_{a<x<b}\sum_{i=0}^{n}|l_i(x)|>\frac{\ln(n+1)}{8\sqrt{\pi}}. \tag{2.2}$$

Thus, by Lemma 1 we obtain

**Theorem 1.** *Whatever the selection of the infinite node triangle* (1.1) *may be, the Lagrange interpolation is never stable.*

Actually, for usual isometric nodes, the increase of $\lambda_n$ is much greater than the increase of the lower bound given by (2.2).

Denote $x_i=a+ih$ and let $x=a+th$; then $l_i(x)$ can be transformed to

$$\varphi_i(t)=l_i(a+th)=\frac{\pi_{n+1}(t)}{(t-i)\pi'_{n+1}(i)}\quad(0\leqslant i\leqslant n),$$

$$\pi_{n+1}(t)=\prod_{i=0}^{n}(t-i)\quad(0\leqslant t\leqslant n).$$

It can be found that

$$\max_{0<t<n}|\varphi_i(t)|\leqslant|\varphi_i(n+1)|=\frac{(n+1)!}{(n+1-i)!\,i!}$$

$$\leqslant\begin{cases}(n+1)!\Big/\left[\left(\frac{n+1}{2}\right)!\right]^2 & n \text{ is odd},\\ (n+1)!\Big/\left(\frac{n+2}{2}\right)!\left(\frac{n}{2}\right)! & n \text{ is even}.\end{cases}$$

By the Stirling formula, we can obtain

$$\varphi_i(n+1)\leqslant 0(n^{-1/2}2^n)$$

and then

$$\lambda_n=\max_{0<t<n}\sum_{i=0}^{n}|\varphi_i(t)|\leqslant 0(n^{1/2}2^n). \tag{2.3}$$

In view of

$$\left|\varphi_{\left[\frac{n+1}{2}\right]}\left(\frac{1}{2}\right)\right|=\begin{cases}|\pi_{n+1}(1/2)|\Big/\left(\frac{n-1}{2}\right)\left[\left(\frac{n}{2}\right)!\right]^2 & n \text{ is even},\\ |\pi_{n+1}(1/2)|\Big/\frac{n}{2}\left(\frac{n+1}{2}\right)!\left(\frac{n-1}{2}\right)! & n \text{ is odd},\end{cases}$$

we obtain the lower bound

$$\lambda_n\geqslant\left|\varphi_{\left[\frac{n+1}{2}\right]}\left(\frac{1}{2}\right)\right|\geqslant 0(n^{-2}2^n). \tag{2.4}$$

Form (2.3) and (2.4) we can see that $\lambda_n$ is nearly an exponential growth. So the Lagrange interpolation with isometric nodes is extremely unstable.

Although Theorem 1 asserts that the Lagrange interpolation is unstable, the following case may still be considered. Select the interpolating nodes to be roots of the Chebyshev polynomial, namely

$$x_i=\frac{b+a}{2}-\frac{b-a}{2}\cos\frac{2i+1}{2(n+1)}\pi\quad(0\leqslant i\leqslant n). \tag{2.5}$$

In this case, we have (see [2], p. 540)

$$\lambda_n \leqslant \frac{4}{\pi}\ln(n+1)+8. \tag{2.6}$$

Comparing it with (2.2), we see that the increasing order of $\lambda_n$ attains the minimum.

Further, if a weaker $L^2_\rho$-norm is used instead of $L^\infty$, we have a stability result:

**Theorem 2.** *Assume that the interpolating nodes are selected according to* (2.5). *Then* $\forall\varepsilon$, $\exists\delta$ *such that if*

$$\max_{0\leqslant i\leqslant n}|f(x_i)|\leqslant\delta,\quad \forall n,$$

*then*

$$\|I_n[f;x]\|_{2,\rho}=\left[\int_a^b \rho(x)I_n^2[f;x]dx\right]^{1/2}\leqslant\varepsilon,$$

*where*

$$\rho(x)=(x-a)^{-1/2}(b-x)^{-1/2}.$$

Proof. In our case the $w_{n+1}(x)$ defined in (2.1b) is the Chebyshev polynomial of degree $(n+1)$ on the interval $[a, b]$. From orthogonality, if $Q(x)$ is a polynomial with degree $\leqslant n$, $\int_a^b \rho(x)w_{n+1}(x)Q(x)dx=0$ holds. By (2.1a) we have

$$l_i(x)l_j(x)=\frac{w_{n+1}(x)}{w'_{n+1}(x_i)}\cdot\frac{w_{n+1}(x)}{(x-x_i)(x-x_j)w'_{n+1}(x_j)}.$$

Because the second factor on the right side is a polynomial of degree $(n-1)$,

$$\int_a^b \rho(x)l_i(x)l_j(x)dx=0 \quad i\neq j. \tag{2.7}$$

From (2.7) and (1.6), we obtain

$$\begin{aligned}\int_a^b \rho(x)\Big(\sum_{i=0}^n f(x_i)l_i(x)\Big)^2dx &\leqslant \max_{0\leqslant i\leqslant n}|f(x_i)|^2\int_a^b\rho(x)\sum_{i=0}^n l_i^2(x)dx\\ &=\max_{0\leqslant i\leqslant n}|f(x_i)|^2\int_a^b\rho(x)\Big(\sum_{i=0}^n l_i(x)\Big)^2dx\\ &=\max_{0\leqslant i\leqslant n}|f(x_i)|^2\int_a^1\rho(x)dx=\pi\max_{0\leqslant i\leqslant n}|f(x_i)|^2,\end{aligned}$$

and then the proof is completed.

When interpolating nodes are selected according to (2.5), we can easily prove, by virtue of Theorem 2 and a similar analysis to that in Lemma 2, the $L^2_\rho$-convergence of interpolation for $f\in C^0[a, b]$, namely

$$\lim_{n\to\infty}\|I_n[f;x]-f\|_{2,\rho}=0.$$

For uniform convergence, the smoothness of function $f$ has to be stronger. For example, $f$ should be Hölder continuous, namely, there exist constants $M>0$ and $0<\alpha\leqslant1$ such that

$$|f(x_1)-f(x_2)|\leqslant M|x_1-x_2|^\alpha\ \forall x_1,\ x_2\in[a, b].$$

According to the Jackson theorem (see [3], p. 61), there exists a polynomial sequence $\{P_n\}$ satisfying

$$\|f-P_n\|_\infty\leqslant Cn^{-\alpha}, \tag{2.8}$$

where $C$ is a constant independent of $n$. Thus, from (2.8) and (2.6) we get

$$\lim_{n\to\infty}\lambda_n\|f-P_n\|_\infty=0.$$

Then

$$\lim_{n\to\infty}\|I_n[f;x]-f\|_\infty=0$$

follows from Lemma 2.

When data are given on isometric nodes and a high order polynomial approximation is required, it is appropriate to approach the problem with the Bernstein polynomial if it is not necessary for the approximating polynomial to coincide exactly with the data on the nodes. The Bernstein polynomial on the interval [0, 1] can be written as

$$\begin{cases} B_n[f; x] = \sum_{i=0}^{n} f\left(\frac{i}{n}\right) P_i^n(x), \\ P_i^n(x) = \frac{n(n-1)\cdots(n-i+1)}{i!} x^i (1-x)^{n-i}. \end{cases} \tag{2.9}$$

This approximation is extremely stable.

**Theorem 3.** *For all nonnegative integer $k$, there exists a constant $C_k$ dependent only on $k$ such that*

$$\left\| \frac{d^k}{dx^k} B_n[f; x] \right\|_\infty \leqslant C_k \max_{0 \leqslant i \leqslant n-k} \left| f\left[\frac{i}{n}, \cdots, \frac{i+k}{n}\right] \right| \quad \forall n \geqslant k. \tag{2.10}$$

This states that the Bernstein approximation is stable for all orders of derivatives.

Proof. Because $P_i^n(x) > 0$ and

$$\sum_{i=0}^{n} P_i^n(x) \equiv 1, \tag{2.11}$$

it is obvious from (2.9) that

$$\| B_n[f; x] \|_\infty \leqslant \max_i \left| f\left(\frac{i}{n}\right) \right|. \tag{2.12}$$

After calculation we obtain

$$\frac{d}{dx} B_n[f; x] = \sum_{i=0}^{n-1} f\left[\frac{i}{n}, \frac{i+1}{n}\right] P_i^{n-1}(x)$$

and then by induction we can prove

$$\frac{d^k}{dx^k} B_n[f; x] = \frac{n(n-1)\cdots(n-k+1)}{n^k} k! \sum_{i=0}^{n-k} f\left[\frac{i}{n}, \cdots, \frac{i+k}{n}\right] P_i^{n-k}(x). \tag{2.13}$$

From (2.11) and taking $C = k!$, we obtain (2.10) from (2.13).

## 3. Piecewise Low Order Interpolation

In this section, three types of interpolation will be discussed.

*A. Piecewise Second Order*

Assume that there are interpolating nodes of odd number: $a = x_0 < x_1 < \cdots < x_{2n} = b$, and function values are given. Now in each subinterval $[x_{2i}, x_{2i+2}]$ we set up a second order interpolation with the data $f(x_{2i})$, $f(x_{2i+1})$ and $f(x_{2i+2})$. With respect to this kind of interpolation, the interpolation base consists of

$$l_{2i}(x) = \begin{cases} \frac{(x - x_{2i-2})(x - x_{2i-1})}{(x_{2i} - x_{2i-2})(x_{2i} - x_{2i-1})} & x_{2i-2} \leqslant x \leqslant x_{2i} \quad \text{(deleted when } i = 0\text{)}, \\ \frac{(x - x_{2i+1})(x - x_{2i+2})}{(x_{2i} - x_{2i+1})(x_{2i} - x_{2i+2})} & x_{2i} \leqslant x \leqslant x_{2i+2} \quad \text{(deleted when } i = n\text{)}, \\ 0 & \text{elsewhere;} \end{cases}$$

$$l_{2i+1}(x)=\begin{cases}\dfrac{(x-x_{2i})(x-x_{2i+2})}{(x_{2i+1}-x_{2i})(x_{2i+1}-x_{2i+2})} & x_{2i}\leqslant x\leqslant x_{2i+2},\\ 0 & \text{elsewhere.}\end{cases}$$

Let $h_i=x_{i+1}-x_i$ and assume that there is a constant $M$ for all $n$ satisfying

$$\max(h_{2i},\ h_{2i+1})/\min(h_{2i},\ h_{2i+1})\leqslant M. \tag{3.1}$$

Through calculation we obtain

$$\|l_{2i}(x)\|_\infty\leqslant\max\left(1,\ \frac{M}{4}\right), \tag{3.2a}$$

$$\|l_{2i+1}(x)\|_\infty\leqslant\frac{3}{4}+\frac{M}{4}. \tag{3.2b}$$

For any $x\in[x_{2i},\ x_{2i+2}]$ all $l_j(x)$ but $l_{2i}(x)$, $l_{2i+1}(x)$ and $l_{2i+2}(x)$ are identically equal to zero. So we get from (3.2a) and (3.2b) that

$$\lambda_n\leqslant\max\left(2,\ \frac{M}{2}\right)+\frac{3}{4}+\frac{M}{4}.$$

Thus from Lemma 1 we have

**Theorem 4**. *Assume that condition (3.1) is satisfied. Then the piecewise second order interpolation is stable respect to the function.*

*B. Piecewise Hermite Interpolation*

Assume that the data $f(x_i)$ and $f'(x_i)$ are given on each node $x_i$. In each subinterval $[x_i,\ x_{i+1}]$ we set up a third order interpolation with the given data on the end points. In this case, according to (1.14), the interpolation base consists of

$$l_i(x)=\begin{cases}\left(\dfrac{x-x_{i-1}}{x_i-x_{i-1}}\right)^2\left(1+2\dfrac{x-x_i}{x_{i-1}-x_i}\right) & x_{i-1}\leqslant x\leqslant x_i \quad (\text{deleted when } i=0),\\ \left(\dfrac{x-x_{i+1}}{x_i-x_{i+1}}\right)^2\left(1+2\dfrac{x-x_i}{x_{i+1}-x_i}\right) & x_i\leqslant x\leqslant x_{i+1} \quad (\text{deleted when } i=n),\\ 0 & \text{elsewhere;}\end{cases}$$

$$\bar{l}_i(x)=\begin{cases}\left(\dfrac{x-x_{i-1}}{x_i-x_{i-1}}\right)^2(x-x_i) & x_{i-1}\leqslant x\leqslant x_i \quad (\text{deleted when } i=0),\\ \left(\dfrac{x-x_{i+1}}{x_i-x_{i+1}}\right)^2(x-x_i) & x_i\leqslant x\leqslant x_{i+1} \quad (\text{deleted when } i=n),\\ 0 & \text{elsewhere.}\end{cases}$$

For any $x\in[x_i,\ x_{i+1}]$, all $l_j(x)$ and $\bar{l}_j(x)$ except $l_i(x)$, $l_{i+1}(x)$, $\bar{l}_i(x)$ and $\bar{l}_{i+1}(x)$ are identically equal to zero. So the interpolating function can be written as

$$I_n[f;x]=f(x_i)l_i(x)+f(x_{i+1})l_{i+1}(x)+f'(x_i)\bar{l}_i(x)+f'(x_{i+1})\bar{l}_{i+1}(x). \tag{3.3}$$

Through calculation, we get the following estimation for $x\in[x_i,\ x_{i+1}]$:

$$0\leqslant l_i(x)\leqslant 1,\quad l_i(x)+l_{i+1}(x)=1, \tag{3.4a}$$

$$|\bar{l}_i(x)|+|\bar{l}_{i+1}(x)|\leqslant\frac{1}{4}h_i, \tag{3.4b}$$

$$h_i|l'_j(x)|\leqslant\frac{3}{2}\quad j=i,\ i+1, \tag{3.4c}$$

$$|\bar{l}'_i(x)|+|\bar{l}'_{i+1}(x)|\leqslant 1. \tag{3.4d}$$

Let $h=\max_i h_i$, we obtain from (3.3), (3.4a) and (3.4b)

$$\bar{\lambda}_n \leqslant \frac{5}{4}, \tag{3.5}$$

or

$$\|I_n[f; x]\| \leqslant \frac{5}{4} \max_{0<i<n} \{|f(x_i)|, h|f'(x_i)|\}. \tag{3.5a}$$

From (3.4a), $l_i'(x)=-l_{i+1}'(x)$ holds on $[x_i, x_{i+1}]$. So we have

$$I_n'[f; x]=-l_i'(x)h_i f[x_i, x_{i+1}]+f'(x_i)\bar{l}_i'(x)+f'(x_{i+1})\bar{l}_{i+1}'(x).$$

On account of (3.4c) and (3.4d), we obtain

$$\|I_n'[f; x]\|_\infty \leqslant \frac{5}{2} \max_i \{|f[x_i, x_{i+1}]|, |f'(x_i)|\}. \tag{3.6}$$

Then, from Lemma 3, Definition 4, (3.5) and (3.6) we have proved

**Theorem 5**. *The piecewise Hermite interpolation are unconditionally stable respect to the function and the first derivative.*

*C. Spline Interpolation*

Here we only consider the problem with periodicity condition, for other kinds of boundary conditions can be discussed in a same manner.

Assume that the data $f(x_i)$ are given on nodes $a=x_0<x_1<\cdots<x_n=b$, and $f(x_0)=f(x_n)$. we seek a periodic function $\varphi\in C^2$ $(-\infty, +\infty)$ which is a third order polynomial within each subinterval $[x_i, x_{i+1}]$ and coincides with the data on nodes and has a cycle of $T=b-a$. As well known, in this case a cubic spline is uniquely determined, which we denote by $I_n[f; x]$.

Let

$$m_i=I_n'[f; x_i], \quad 0\leqslant i\leqslant n-1. \tag{3.7}$$

On account of the continuity of $I_n''[f; x]$ at nodes $x_i$ $(1\leqslant i\leqslant n-1)$ and the periodicity, we can get a linear system for $m=[m_0, m_1, \cdots, m_{n-1}]^T$:

$$Am=g, \tag{3.8}$$

where

$$A=\begin{bmatrix} 2 & \mu_0 & & & \lambda_0 \\ \lambda_1 & 2 & \mu_1 & & \\ & \ddots & \ddots & \ddots & \\ & & \lambda_{n-2} & 2 & \mu_{n-2} \\ \mu_{n-1} & & & \lambda_{n-1} & 2 \end{bmatrix}, \tag{3.8a}$$

$$\lambda_i=\frac{h_i}{h_{i-1}+h_i}, \quad \mu_i=\frac{h_{i-1}}{h_{i-1}+h_i}, \tag{3.8b}$$

$$g_i=3\lambda_i f[x_{i-1}, x_i]+3\mu_i f[x_i, x_{i+1}] \tag{3.8c}$$

and $f(x_{-1})=f(x_{n-1})$, $h_{-1}=h_{n-1}$ by periodicity.

If matrix $A$ satisfies the following condition

$$|a_{ii}|>\sum_{j\neq i}|a_{ij}|, \quad 1\leqslant i\leqslant n, \tag{3.9}$$

it can be proved that (cf. e. g. [4], p. 134)

$$\|A^{-1}\|_\infty \leqslant [\min_i(|a_{ii}|-\sum_{j\neq i}|a_{ij}|)]^{-1}, \tag{3.10}$$

where $\|A\|_\infty$ stands for the norm $\max_{x\neq 0}\frac{\|Ax\|_\infty}{\|x\|_\infty}$ with $\|x\|_\infty=\max_i|x_i|$. The inverse of matrix $A$ defined by (3.8a), (3.8b) satisfies condition (3.9). So $\|A^{-1}\|_\infty \leqslant 1$ follows from (3.10) and then we obtain

$$\|m\|_\infty = \|A^{-1}g\|_\infty \leqslant \|g\|_\infty. \tag{3.11}$$

From (3.8c) and (3.8b) we have

$$\|g\|_\infty \leqslant 3 \max_{0 \leqslant i \leqslant n-1} |f[x_i, \ x_{i+1}]| \leqslant \frac{6}{\min\limits_i h_i} \max_i |f(x_i)|. \tag{3.12}$$

For each subinterval $[x_i, x_{i+1}]$, the cubic spline can be expressed in a Hermite interpolating form (3.3), namely

$$I_n[f; x] = f(x_i) l_i(x) + f(x_{i+1}) l_{i+1}(x) + m_i \bar{l}_i(x) + m_{i+1} \bar{l}_{i+1}(x).$$

From (3.5a), (3.11) and (3.12) we get

$$\|I_n[f; x]\|_\infty \leqslant \frac{5}{4} \max_{0 \leqslant i \leqslant n-1} \{|f(x_i)|, \ h|m_i|\} \leqslant \frac{15}{2} \frac{h}{\Delta} \max_{0 \leqslant i \leqslant n-1} |f(x_i)|, \tag{3.13}$$

where $h = \max\limits_i h_i$, $\Delta = \min\limits_i h_i$. From (3.6), (3.11), (3.12) we get

$$\|I_n'[f; x]\|_\infty \leqslant \frac{5}{2} \max_{0 \leqslant i \leqslant n-1} \{|f(x_i, \ x_{i+1})|, \ |m_i|\} \leqslant \frac{15}{2} \max_{0 \leqslant i \leqslant n-1} |f[x_i, \ x_{i+1}]|. \tag{3.14}$$

We now discuss the second and third order derivatives. Let $M_i = I_n''[f; x_i]$. On account of the continuity of $I_n'[f; x]$ at nodes $x_i$ and periodicity of the end points, we get a system for $M = [M_0, M_1, \cdots, M_{n-1}]^T$:

$$BM = d, \tag{3.15}$$

where

$$B = \begin{bmatrix} 2 & \lambda_0 & & & \mu_0 \\ \mu_1 & 2 & \lambda_1 \ddots & & \\ & & \ddots & & \\ & & \mu_{n-2} & \ddots 2 & \lambda_{n-2} \\ \lambda_{n-1} & & & \mu_{n-1} & 2 \end{bmatrix}, \tag{3.15a}$$

$$d_i = 6f[x_{i-1}, \ x_i, \ x_{i+1}]. \tag{3.15b}$$

Here $\lambda_i$ and $\mu_i$ are the same as defined in (3.8b).

Because $\|B^{-1}\|_\infty \leqslant 1$, we obtain from (3.15), (3.15b) that

$$\|M\|_\infty \leqslant 6 \max_{0 \leqslant i \leqslant n-1} |f[x_{i-1}, \ x_i, \ x_{i+1}]|. \tag{3.16}$$

$I_n''[f; x]$ is a linear function within each $[x_i, x_{i+1}]$, so it has the form

$$I_n''[f; x] = \frac{M_i(x_{i+1} - x)}{h_i} + \frac{M_{i+1}(x - x_i)}{h_i} \quad x_i \leqslant x \leqslant x_{i+1}. \tag{3.17}$$

From (3.17), (3.16) we get

$$\|I_n''[f; x]\|_\infty = 6 \max_{0 \leqslant i \leqslant n-1} |f[x_{i-1}, \ x_i, \ x_{i+1}]|. \tag{3.18}$$

Let

$$\widetilde{M}_i = 2f[x_{i-1}, \ x_i, \ x_{i+1}]. \tag{3.19}$$

Then

$$B\widetilde{M} = \tilde{d}, \tag{3.20}$$

where $\quad \tilde{d}_i = \mu_i \widetilde{M}_{i-1} + 2\widetilde{M}_i + \lambda_i \widetilde{M}_{i+1}.$

Substracting (3.15) from (3.20), we get

$$B(\tilde{M}-M)=\tilde{d}-d,$$

where

$$\tilde{d}_i-d_i=-2\mu_i(x_{i+1}-x_{i-2})f[x_{i-2},\ x_{i-1},\ x_i,\ x_{i+1}]+2\lambda_i(x_{i+2}-x_{i-1})f[x_{i-1},\ x_i,\ x_{i+1},\ x_{i+2}].$$

Then

$$\|\tilde{M}-M\|_\infty\leqslant\|\tilde{d}-d\|_\infty\leqslant 6h\max_{0\leqslant i\leqslant n-1}|f[x_i,\ x_{i+1},\ x_{i+2},\ x_{i+3}]|. \tag{3.21}$$

The right side of (3.21) involves $f(x_{n+1})$, $f(x_{n+2})$; we take them as $f(x_1)$ and $f(x_2)$ according to the periodic condition.

From (3.17), we have on $[x_i,\ x_{i+1}]$

$$I_n'''[f;x]=\frac{1}{h_i}(M_{i+1}-M_i)=\frac{1}{h_i}[(M_{i+1}-\tilde{M}_{i+1})+(\tilde{M}_{i+1}-\tilde{M}_i)+(\tilde{M}_i-M_i)],$$

and from (3.19), (3.21) we get

$$\|I_n'''[f;x]\|\leqslant 18\frac{h}{\Delta}\max_{0\leqslant i\leqslant n-1}|f[x_i,\ x_{i+1},\ x_{i+2},\ x_{i+3}]|. \tag{3.22}$$

Now we conclude from (3.13), (3.14), (3.18) and (3.22) the following

**Theorem 6.** *Assume that there exists a constant $M$ for all $n$ satisfying*

$$\max_i h_i/\min_i h_i\leqslant M.$$

*Then the periodic spline interpolation is stable respect to the function and the derivatives up to the third order.*

The same result can be generalized to other kinds of boundaries.

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