AN ALGORITHM FOR REDUCING THE MATRIX NORM*

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Abstract

Based on the singular decomposition of 2×2 matrix an algorithm for reducing the matrix norm is presented. Under the optimal choice of the parameters the matrix B after transformation may be considered as "locally normal", that is, the four corresponding elements of BB^T-B^TB are zero.

1. Introduction

Eberlein [1, 2] proposed a Jacobi-like method to compute the eigenvalues and eigenvectors of arbitrary matrix A. At each step the similarity transformation $B=H^{-1}AH$ is needed to reduce the norm of the matrix, and in the real case only 4 of the elements h_{ij} of H differ from those of identity matrix and they are

$$h_{pp}=h_{qq}=\cosh\xi,\ h_{pq}=h_{qp}=\sinh\xi.$$

Later, the same problem was considered, but

$$h_{pp}=h_{qq}=1,\ \ h_{pq}=0,\ h_{qp}=\xi.$$

Since the singular decomposition of any 2×2 non-singular matrix is

$$\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\xi & 0\\ 0 & \eta\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$$

and the third matrix is useless for the norm reduction, hence in this paper we consider

$$h_{pp}=\xi\cos\theta,\ h_{pq}=-\eta\sin\theta,\ h_{qp}=\xi\sin\theta,\ h_{qq}=\eta\cos\theta.$$

In this way, we choose the suitable transformation among all possibilities, not only among the particular family depending on one parameter.

In § 3, excluding the easily-verified particular case where we may directly compute one or two eigenvalue, we prove that there exist the values of parameters θ, ξ and η to minimize the matrix norm. In § 4 we prove that under optimal choice of the parameters, the 4 corresponding elements of BB^T-B^TB are zero. In § 5 the unique problem is discussed. In § § 6—9 we consider how to determine the optimal values in different cases. Generally, the system, which the optimal values satisfy, can be reduced to an algebraic equation of order 8, and the interval including the root needed is located. In a particular case we may get the exact solution.

The speed of the norm reduction and the numerical stability are not to be discussed.

For simplicity, we consider the real matrix only, but obviously it can be generalized to the complex case. Further information will be presented in another paper.

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2. Notations

Let $A=(a_{ij})$ be any arbitrary $N\times N$ real matrix. Let the elements of $N\times N$ real matrix $H=(h_{ij})$ are defined as follows

$$h_{pp}=\xi\cos\frac{\theta}{2},\quad h_{pq}=-\eta\sin\frac{\theta}{2},$$

$$h_{qp}=\xi\sin\frac{\theta}{2},\quad h_{qq}=\eta\cos\frac{\theta}{2},$$

$$p<q\quad \text{fixed},\ \xi>0,\ \eta>0,$$

$$h_{ij}=\delta_{ij},\quad \text{otherwise}.$$

Obviously, $H^{-1}=(h'_{ij})$ exists, and

$$h'_{pp}=\xi^{-1}\cos\frac{\theta}{2},\ h'_{pq}=\xi^{-1}\sin\frac{\theta}{2},$$

$$h'_{qp}=-\eta^{-1}\sin\frac{\theta}{2},\ h'_{qq}=\eta^{-1}\cos\frac{\theta}{2},$$

$$h'_{ij}=\delta_{ij}.$$

Let $H^{-1}AH=B=(b_{ij})$. Obviously, $b_{ij}=a_{ij}$, when $i,\ j\neq p,\ q$.

Let

$$\tau(B)=\sum_j{}' b_{pj}^2+\sum_j{}' b_{qj}^2+\sum_i{}' b_{ip}^2+\sum_i{}' b_{iq}^2+b_{pp}^2+b_{qq}^2+b_{pq}^2+b_{qp}^2,$$

where $\sum'$ denotes that $i,\ j$ run through 1 to N except p and q.

Let

$$\begin{aligned}
&a_1=\frac{1}{2}(a_{pq}+a_{qp}),\ a_2=\frac{1}{2}(a_{pq}-a_{qp}),\ a_3=\frac{1}{2}(a_{qq}-a_{pp}),\\
&a_4=\sum_j{}' a_{pj}^2,\qquad a_5=\sum_j{}' a_{qj}^2,\qquad a_6=\sum_j{}' a_{pj}a_{qj},\\
&a_7=\sum_i{}' a_{ip}^2,\qquad a_8=\sum_i{}' a_{iq}^2,\qquad a_9=\sum_i{}' a_{ip}a_{iq},\\
&f_1=a_2+a_1\cos\theta+a_3\sin\theta,\\
&f_2=-a_2+a_1\cos\theta+a_3\sin\theta,\\
&f_3=a_3\cos\theta-a_1\sin\theta,\\
&f_4=\frac{1}{2}(a_4+a_5)+\frac{1}{2}(a_4-a_5)\cos\theta+a_6\sin\theta,\\
&f_5=\frac{1}{2}(a_4+a_5)-\frac{1}{2}(a_4-a_5)\cos\theta-a_6\sin\theta,\\
&f_6=a_6\cos\theta-\frac{1}{2}(a_4-a_5)\sin\theta,\\
&f_7=\frac{1}{2}(a_7+a_8)+\frac{1}{2}(a_7-a_8)\cos\theta+a_9\sin\theta,\\
&f_8=\frac{1}{2}(a_7+a_8)-\frac{1}{2}(a_7-a_8)\cos\theta-a_9\sin\theta,\\
&f_9=a_9\cos\theta-\frac{1}{2}(a_7-a_8)\sin\theta,
\end{aligned}\tag{2.1}$$

$$b_1=\frac{1}{2}(b_{pq}+b_{qp}),\quad b_2=\frac{1}{2}(b_{pq}-b_{qp}),\quad b_3=\frac{1}{2}(b_{qq}-b_{pp}),$$

$$b_4=\sum_j{}' b_{pj}^2,\qquad b_5=\sum_j{}' b_{qj}^2,\qquad b_6=\sum_j{}' b_{pj}b_{qj},$$
$$b_7=\sum_i{}' b_{ip}^2,\qquad b_8=\sum_i{}' b_{iq}^2,\qquad b_9=\sum_i{}' b_{ip}b_{iq}.$$

Therefore

$$\tau(B)=\xi^{-2}f_4+\eta^{-2}f_5+\xi^2 f_7+\eta^2 f_8+\frac{1}{2}(a_{pp}+a_{qq})^2+2f_3^2+\xi^{-2}\eta^2 f_1^2+\xi^2\eta^{-2}f_2^2. \tag{2.2}$$

We choose θ, ξ and η to minimize $\tau(B)$.

3. Existence

At first we exclude the case that the norm reduction is not required. Let

$$\begin{aligned} r&=a_4a_5-a_6^2, & s&=a_2(a_4+a_5)-a_1(a_4-a_5)-2a_3a_6,\\ r_1&=a_7a_8-a_9^2, & s_1&=a_2(a_7+a_8)+a_1(a_7-a_8)+2a_3a_9. \end{aligned} \tag{3.1}$$

Lemma 1. *If $a_4+a_5=0$ or $a_7+a_8=0$, the eigenvalues of $\begin{bmatrix} a_{pp} & a_{pq}\\ a_{qp} & a_{qq}\end{bmatrix}$ are those of A. If $(a_4+a_5)(a_7+a_8)\neq 0$, but $r=s=0$ or $r_1=s_1=0$, then by choosing θ and putting $\xi=\eta=1$, we may get one eigenvalue.*

Proof. The first part of the lemma is obvious. As the second part, if $r=s=0$, then we may get $f_2=f_5=0$, by taking $\sin\theta=\dfrac{2a_6}{a_4+a_5}$, $\cos\theta=\dfrac{a_4-a_5}{a_4+a_5}$. Consequently $b_{qj}=0$, $j\neq q$, and $b_{qq}=\dfrac{1}{2}(a_{pp}+a_{qq})+\dfrac{a_3(a_4-a_5)-2a_1a_6}{a_4+a_5}$ is an eigenvalue. If $r_1=s_1=0$, then taking $\sin\theta=\dfrac{2a_9}{a_7+a_8}$, $\cos\theta=\dfrac{a_7-a_8}{a_7+a_8}$, $b_{qq}=\dfrac{1}{2}(a_{pp}+a_{qq})+\dfrac{a_3(a_7-a_8)-2a_1a_9}{a_7+a_8}$ is an eigenvalue.

In the following we suppose

$$s\neq 0,\ \text{when } r=0;\ s_1\neq 0,\ \text{when } r_1=0. \tag{3.2}$$

Let

$$\tau=\tau(A)=a_4+a_5+a_7+a_8+a_{pp}^2+a_{qq}^2+a_{pq}^2+a_{qp}^2. \tag{3.3}$$

Theorem 2. *When (3.2) holds, the minimum of $\tau(B)$ exists.*

Proof. We prove it in 4 different cases.

Case 1. $r>0$ and $r_1>0$. Let

$$\alpha=\min f_4=\min f_5=\frac{1}{2}(a_4+a_5)-\left(\frac{1}{4}(a_4+a_5)^2-r\right)^{\frac{1}{2}}>0,$$

$$\alpha_1=\min f_7=\min f_8=\frac{1}{2}(a_7+a_8)-\left(\frac{1}{4}(a_7+a_8)^2-r_1\right)^{\frac{1}{2}}>0.$$

Then

$$\tau(B)\geqslant\xi^{-2}f_4\geqslant\xi^{-2}\alpha>\tau,\ \text{when } \xi^2<\frac{\alpha}{\tau},$$

$$\tau(B)\geqslant\xi^2 f_7\geqslant\xi^2\alpha_1>\tau,\quad \text{when } \xi^2>\frac{\tau}{\alpha_1}$$

$$\tau(B)\geqslant\eta^2 f_8\geqslant\eta^2\alpha_1>\tau,\quad \text{when } \eta^2>\frac{\tau}{\alpha_1},$$

$$\tau(B)\geqslant\eta^{-2}f_5\geqslant\eta^{-2}\alpha>\tau,\ \text{when } \eta^2<\frac{\alpha}{\tau}.$$

Therefore we may consider only the finite closed domain $0\leqslant\theta\leqslant2\pi$, $0<\frac{\alpha}{\tau}\leqslant\xi^2$, $\eta^2\leqslant\frac{\tau}{\alpha_1}$, and on this domain $\tau(B)$ is sufficiently smooth, so the minimum exists.

Case 2. $r=0$, $s\neq0$ and $r_1>0$.

Choose ε sufficient small and $0<\varepsilon<\frac{a_4+a_5}{2}$ in order that $\beta=\inf_{\Omega_1} f_1^2>0$, $\Omega_1=\{\theta\,|\,f_4<\varepsilon\}$, and $\gamma=\inf_{\Omega_2} f_2^2>0$, $\Omega_2=\{\theta\,|\,f_5<\varepsilon\}$. Then for any θ, when $\xi^2>\frac{\tau}{\alpha_1}$ or $\eta^2>\frac{\tau}{\alpha_1}$, we have $\tau(B)>\tau$ as case 1. When $\theta\overline{\in}(\Omega_1\cup\Omega_2)$ and $\xi^2<\frac{\varepsilon}{\tau}$ or $\eta^2<\frac{\varepsilon}{\tau}$, we have $\tau(B)>\tau$ also as case 1. When $\theta\in\Omega_1$, if $\eta^2<\frac{a_4+a_5}{2\tau}$, we have

$$\tau(\theta)\geqslant\eta^{-2}f_5=\eta^{-2}(a_4+a_5-f_4)>\eta^{-2}\,\frac{a_4+a_5}{2}>\tau,$$

and if

$$\eta^2\geqslant\frac{a_4+a_5}{2\tau},\quad \xi^2<\frac{a_4+a_5}{2\tau^2}\,\beta,$$

we have $\tau(B)\geqslant\xi^{-2}\eta^2f_1^2\geqslant\tau$. When $\theta\in\Omega_2$, if $\xi^2<\frac{a_4+a_5}{2\tau}$, we have

$$\tau(B)\geqslant\xi^{-2}f_4>\xi^{-2}\,\frac{a_4+a_5}{2}>\tau,$$

and if

$$\xi^2\geqslant\frac{a_4+a_5}{2\tau},\quad \eta^2<\frac{a_4+a_5}{2\tau^2}\gamma,$$

we have $\tau(B)\geqslant\xi^2\eta^{-2}f_2^2>\tau$. Therefore we may consider only the finite closed domain $0\leqslant\theta\leqslant2\pi$, $0<\min\left(\frac{\varepsilon}{\tau},\ \frac{\varepsilon}{\tau^2}\beta,\ \frac{\varepsilon}{\tau^2}\gamma\right)\leqslant\xi^2$, $\eta^2\leqslant\frac{\tau}{\alpha_1}$.

Case 3. $r>0$ and $r_1=0$, $s_1\neq0$.

Case 4. $r=0$, $s\neq0$ and $r_1=0$, $s_1\neq0$.

The theorem can be proved similarly for these two cases.

4. "Local Normality"

Since the minimum of $\tau(B)$ exists and $\tau(B)$ is a sufficiently smooth function of θ, ξ and η, the optimal values must satisfy

$$\frac{\partial}{\partial\theta}\tau(B)=0,\ \frac{\partial}{\partial\xi^2}\tau(B)=0,\ \frac{\partial}{\partial\eta^2}\tau(B)=0$$

or

$$(\xi^{-2}-\eta^{-2})(f_6-\xi^2\eta^2f_9+2\eta^2f_1f_3-2\xi^2f_2f_3)=0, \tag{4.1}$$

$$-\xi^{-4}f_4+f_7-\xi^{-4}\eta^2f_1^2+\eta^{-2}f_2^2=0, \tag{4.2}$$

$$-\eta^{-4}f_5+f_8+\xi^{-2}f_1^2-\xi^2\eta^{-4}f_2^2=0. \tag{4.3}$$

Let the Hessian matrix of $\tau(B)$ be $\begin{bmatrix}\tau_{11}&\tau_{12}&\tau_{13}\\ \tau_{12}&\tau_{22}&\tau_{23}\\ \tau_{13}&\tau_{23}&\tau_{33}\end{bmatrix}$, where

$$\tau_{11}=(\xi^{-2}-\eta^{-2})\Big(-\frac{1}{2}(f_4-f_5)+\frac{1}{2}\xi^2\eta^2(f_7-f_8)$$
$$+2\eta^2f_3^2-2\xi^2f_3^2+(f_1+f_2)(\xi^2f_2-\eta^2f_1)\Big).$$

If $\xi^2=\eta^2$, then $\tau_{11}=0$, and the Hessian matrix is not positive definite. Therefore θ, ξ and η satisfy (4.2), (4.3) and

$$f_6-\xi^2\eta^2 f_9+2\eta^2 f_1 f_3-2\xi^2 f_2 f_3=0. \tag{4.4}$$

Let $BB^T-B^TB=M=(m_{ij})$. Obviously

$$\begin{aligned} m_{pp}&=\xi^{-2}f_4-\xi^2 f_7+\xi^{-2}\eta^2 f_1^2-\xi^2\eta^{-2}f_2^2,\\ m_{qq}&=\eta^{-2}f_5-\eta^2 f_8-\xi^{-2}\eta^2 f_1^2+\xi^2\eta^{-2}f_2^2,\\ m_{pq}&=m_{qp}=\xi^{-1}\eta^{-1}f_6-\xi\eta f_9+2\xi^{-1}\eta f_1 f_3-2\xi\eta^{-1}f_2 f_3. \end{aligned} \tag{4.5}$$

By comparing (4.2)—(4.5), we get

Theorem 3. *To minimize $\tau(B)$ is equivalent to $m_{pp}=m_{qq}=m_{pq}=m_{qp}=0$.*

Let us then call the matrix B "locally normal".

5. Uniqueness

(4.2), (4.3) and (4.4) can be written as

$$\xi^4\eta^2 f_7+\xi^4 f_2^2-\eta^4 f_1^2-\eta^2 f_4=0, \tag{5.1}$$

$$\xi^2\eta^4 f_8+\eta^4 f_1^2-\xi^4 f_2^2-\xi^2 f_5=0, \tag{5.2}$$

$$\xi^2\eta^2 f_9+2\xi^2 f_2 f_3-2\eta^2 f_1 f_3-f_6=0. \tag{5.3}$$

Let $AA^T-A^TA=N=(n_{ij})$. Obviously

$$\begin{aligned} &n_{pp}=a_4-a_7+4a_1a_2,\quad n_{qq}=a_5-a_8-4a_1a_2,\\ &n_{pq}=n_{qp}=a_6-a_9+4a_2a_3. \end{aligned} \tag{5.4}$$

Theorem 4. *If $n_{pq}=n_{qq}=n_{pq}=0$, then the minimum of $\tau(B)$ is τ.*

Proof. Obviously

$$\begin{aligned} &a_4+a_5=a_7+a_8,\\ &f_4-f_7=\frac{1}{2}(a_4-a_5-a_7+a_8)\cos\theta+(a_6-a_9)\sin\theta\\ &\qquad=-4a_1a_2\cos\theta-4a_2a_3\sin\theta=f_2^2-f_1^2,\\ &f_5-f_8=f_1^2-f_2^2,\ f_6-f_9=2f_2f_3-2f_1f_3. \end{aligned}$$

Therefore for any θ, $\xi^2=\eta^2=1$ is always the solution of (5.1)—(5.3). On the other hand, for any fixed θ, from (5.1) we get

$$2\xi^2\frac{d\xi^2}{d\eta^2}=\frac{f_4f_2^2+2\eta^2f_1^2f_2^2+\eta^4f_7f_1^2}{(\eta^2f_7+f_2^2)^2}\geqslant 0,$$

$$\xi^4-\eta^4=\frac{\eta^2f_4+\eta^4f_1^2}{\eta^2f_7+f_2^2}-\eta^4=\frac{\eta^2(f_4+\eta^2f_7)(1-\eta^2)}{\eta^2f_7+f_2^2}\gtreqqless 0,\ \text{if}\ \eta^2\lesseqqgtr 1$$

and from (5.2)

$$2\eta^2\frac{d\eta^2}{d\xi^2}=\frac{f_5f_1^2+2\xi^2f_1^2f_2^2+\xi^4f_8f_2^2}{(\xi^2f_8+f_1^2)^2}\geqslant 0,$$

$$\eta^4-\xi^4=\frac{\xi^2f_5+\xi^4f_2^2}{\xi^2f_8+f_1^2}-\xi^4=\frac{\xi^2(f_5+\xi^2f_8)(1-\xi^2)}{\xi^2f_8+f_1^2}\gtreqqless 0,\ \text{if}\ \xi^2\lesseqqgtr 1.$$

Therefore for any θ, (5.1) and (5.2) have no solution other than $\xi^2=\eta^2=1$. Hence it is impossible to reduce the norm.

Theorem 5. *Suppose $-\frac{\pi}{2}<\theta\leqslant\frac{\pi}{2}$, then system (5.1)—(5.3) has unique solution.*

Proof. If there exist H_1 and H_2 such that both $B_1 = H_1^{-1} A H_1$ and $B_2 = H_2^{-1} A H_2$ are "local normal", then $B_2 = H_2^{-1} H_1 B_1 H_1^{-1} H_2$. From theorem 4, $H_1^{-1} H_2$ is unitary, and consequently $H_1^{-1} H_2 H_2^T H_1^T = I$, $H_1 H_1^T = H_2 H_2^T$. Denote the θ, ξ and η of H_j by θ_j, ξ_j and η_j respectively. Then

$$\begin{bmatrix} \cos\frac{1}{2}\theta_1 & -\sin\frac{1}{2}\theta_1 \\ \sin\frac{1}{2}\theta_1 & \cos\frac{1}{2}\theta_1 \end{bmatrix} \begin{bmatrix} \xi_1^2 & 0 \\ 0 & \eta_1^2 \end{bmatrix} \begin{bmatrix} \cos\frac{1}{2}\theta_1 & \sin\frac{1}{2}\theta_1 \\ -\sin\frac{1}{2}\theta_1 & \cos\frac{1}{2}\theta_1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\frac{1}{2}\theta_2 & -\sin\frac{1}{2}\theta_2 \\ \sin\frac{1}{2}\theta_2 & \cos\frac{1}{2}\theta_2 \end{bmatrix} \begin{bmatrix} \xi_2^2 & 0 \\ 0 & \eta_2^2 \end{bmatrix} \begin{bmatrix} \cos\frac{1}{2}\theta_2 & \sin\frac{1}{2}\theta_2 \\ -\sin\frac{1}{2}\theta_2 & \cos\frac{1}{2}\theta_2 \end{bmatrix}.$$

Consequently

$$\begin{bmatrix} \cos\frac{1}{2}(\theta_1-\theta_2) & -\sin\frac{1}{2}(\theta_1-\theta_2) \\ \sin\frac{1}{2}(\theta_1-\theta_2) & \cos\frac{1}{2}(\theta_1-\theta_2) \end{bmatrix} \begin{bmatrix} \xi_1^2 & 0 \\ 0 & \eta_1^2 \end{bmatrix}$$

$$= \begin{bmatrix} \xi_2^2 & 0 \\ 0 & \eta_2^2 \end{bmatrix} \begin{bmatrix} \cos\frac{1}{2}(\theta_1-\theta_2) & -\sin\frac{1}{2}(\theta_1-\theta_2) \\ \sin\frac{1}{2}(\theta_1-\theta_2) & \cos\frac{1}{2}(\theta_1-\theta_2) \end{bmatrix}.$$

Since $-\frac{\pi}{2} < \theta_j \leqslant \frac{\pi}{2}$, then $-\pi < \theta_1 - \theta_2 < \pi$, $\cos\frac{1}{2}(\theta_1-\theta_2) \neq 0$, hence $\xi_1^2 = \xi_2^2$, $\eta_1^2 = \eta_2^2$. On the other hand, from $\xi_1^2 \sin\frac{1}{2}(\theta_1-\theta_2) = \eta_2^2 \sin\frac{1}{2}(\theta_1-\theta_2)$ we get $\xi_1^2 \sin\frac{1}{2}(\theta_1-\theta_2) = \eta_1^2 \sin\frac{1}{2}(\theta_1-\theta_2)$. As pointed out in § 4 that $\xi_1^2 \neq \eta_1^2$, we obtain $\sin\frac{1}{2}(\theta_1-\theta_2) = 0$, and $\theta_1 = \theta_2$ follows.

6. The Analysis of System (5.1)—(5.3)

From (5.1) and (5.2) we get

$$\xi^4 \eta^2 f_7 + \xi^2 \eta^4 f_8 = \xi^2 f_5 + \eta^2 f_4$$

therefore

$$\xi^2 \eta^2 = \frac{\xi^2 f_5 + \eta^2 f_4}{\xi^2 f_7 + \eta^2 f_8}, \quad \frac{\xi^2}{\eta^2} = \frac{f_4 - \xi^2\eta^2 f_8}{\xi^2\eta^2 f_7 - f_5}. \tag{6.1}$$

Eliminating $\xi^2 f_2 - \eta^2 f_1$ we obtain from (5.2) and (5.3)

$$2f_3\left(\frac{\xi^2}{\eta^2} f_5 - \xi^2\eta^2 f_8\right) = (\xi^2\eta^2 f_9 - f_6)\left(\frac{\xi^2}{\eta^2} f_2 + f_1\right). \tag{6.2}$$

Eliminating $\frac{\xi^2}{\eta^2}$ from (6.1) and (6.2) we get

$$L\xi^4\eta^4 + M\xi^2\eta^2 + N = 0, \tag{6.3}$$

where

$$L = f_1 f_7 f_9 - f_2 f_8 f_9 + 2 f_3 f_7 f_8 = s_1 f_9 + 2 r_1 f_3,$$
$$M = f_2 f_4 f_9 + f_2 f_8 f_6 - f_1 f_5 f_9 - f_1 f_7 f_6,$$

$$N=f_1f_5f_6-f_2f_4f_6-2f_3f_4f_5=sf_6-2rf_3.$$

Eliminating $\xi^2\eta^2$ from (6.1) and (6.2) we get

$$P\frac{\xi^4}{\eta^4}+Q\frac{\xi^2}{\eta^2}+R=0, \tag{6.4}$$

where

$$P=f_2f_7f_6-f_2f_5f_9+2f_3f_5f_7,$$
$$Q=f_1f_7f_6+f_2f_8f_6-f_1f_5f_9-f_2f_4f_9,$$
$$R=f_1f_8f_6-f_1f_4f_9-2f_3f_4f_8.$$

Suppose

$$a_1(a_7+a_8)a_6-a_1(a_4+a_5)a_9+a_3(a_5a_7-a_4a_8)=0 \tag{6.5}$$

then (6.4) can be factored into

$$\left(\frac{\xi^2}{\eta^2}-1\right)\left(P\frac{\xi^2}{\eta^2}-R\right)=0.$$

Since the solution $\frac{\xi^2}{\eta^2}=1$ is useless, then we get $\frac{\xi^2}{\eta^2}=\frac{R}{P}$. Under the assumption (6.5), (6.4) can be factored into

$$((a_7+a_8)\xi^2\eta^2-(a_4+a_5))((a_4+a_5)L\xi^2\eta^2-(a_7+a_8)N)=0.$$

Suppose

$$a_4+a_5=a_7+a_8. \tag{6.6}$$

If $\xi^2\eta^2=1$, then (6.1) yields $\frac{\xi^2}{\eta^2}=1$. Therefore $\xi^2\eta^2=\frac{N}{L}$. From $\xi^2\eta^2=\frac{N}{L}$, $\frac{\xi^2}{\eta^2}=\frac{R}{P}$ and (5.3) we obtain

$$F\equiv(f_6L-f_9N)^2PR-4f_3^2(f_2R-f_1P)^2LN=0. \tag{6.7}$$

This is an equation of θ. Generally, we can determine θ from it. (6.7) is homogenous in $\sin\theta$ and $\cos\theta$ of order 8, or algebraic equation of $\tan\theta$ 8th order.

In the following we will discuss how to determine θ in the different cases mentioned above.

7. Case 1. $r>0$, $r_1>0$

At first we try to find θ, ξ and η in order that

$$b_4=b_7,\quad b_5=b_8,\quad b_6=b_9$$

or

$$\xi^4=\frac{f_4}{f_7},\quad \eta^4=\frac{f_5}{f_8},\quad \xi^2\eta^2=\frac{f_6}{f_9}.$$

Therefore

$$f_4f_5f_9^2=f_7f_8f_6^2.$$

Since

$$f_4f_5-f_6^2=a_4a_5-a_6^2=r,$$

$$f_7f_8-f_9^2=r_1,\quad \frac{f_6}{f_9}=\xi^2\eta^2>0$$

we get

$$r_1^{\frac{1}{2}}f_6=r^{\frac{1}{2}}f_9.$$

Let

$$\Delta=(r^{\frac{1}{2}}a_9-r_1^{\frac{1}{2}}a_6)^2+\left(\frac{1}{2}r^{\frac{1}{2}}(a_7-a_8)-\frac{1}{2}r_1^{\frac{1}{2}}(a_4-a_5)\right)^2. \tag{7.1}$$

If $\Delta\neq0$, then

$$\tan\theta = \frac{2(r^{\frac{1}{2}} a_9 - r_1^{\frac{1}{2}} a_6)}{r^{\frac{1}{2}}(a_7 - a_8) - r_1^{\frac{1}{2}}(a_4 - a_5)}. \tag{7.2}$$

If $\Delta = 0$, then θ may be arbitrary, and we take $\theta = 0$.

Carrying out the similarity transformation by H with θ, ξ and η defined above, despite of norm reduction, we may assume, without loss of generality,

$$a_4 = a_7,\ a_5 = a_8,\ a_6 = a_9.$$

Hence assumptions (6.5) and (6.6) hold, and the analysis in § 6 can be applied.

If $a_2 = 0$ or $a_1 = a_3 = 0$, then $n_{pp} = n_{qq} = n_{pq} = 0$, the norm reduction has been completed.

Suppose $a_2 \neq 0$ and $t = a_1^2 + a_3^2 \neq 0$.

If $k = a_3(a_4 - a_5) - 2a_1a_6 = a_3(a_7 - a_8) - 2a_1a_9 = 0$, then, $f_6 = f_9 = \frac{a_4 - a_5}{2a_1} f_3$, when $a_1 \neq 0$, and $f_6 = f_9 = \frac{a_6}{a_3} f_3$, when $a_3 \neq 0$. Therefore we may choose θ to satisfy $f_3 = 0$. Now, (5.3) holds, and we may determine ξ and η from (5.1) and (5.2). Eliminating η we get

$$G(\xi^2) = (\xi^2 f_5 + f_1^2)(\xi^2 f_2^2 + f_5)(\xi^4 - 1)^2 f_4^2 - \xi^2(\xi^4 f_2^2 - f_1^2)^2 f_5^2 = 0. \tag{7.3}$$

Choose the root $\xi \in (0, 1)$, when $f_1^2 > f_2^2$, and otherwise the root $\xi > 1$. As for η, we determine it by

$$\eta^2 = \frac{\xi^2(\xi^4 f_2^2 - f_1^2)}{(\xi^2 f_5 + f_1^2)(\xi^4 - 1) f_4}. \tag{7.4}$$

If $u = a_1(a_4 - a_5) + 2a_3a_6 = a_1(a_7 - a_8) + 2a_3a_9 = 0$, then,

$$\frac{1}{2}(a_4 - a_5)\cos\theta + a_6 \sin\theta = -\frac{a_6}{a_1} f_3, \qquad \text{when} \quad a_1 \neq 0$$

and

$$\frac{1}{2}(a_4 - a_5)\cos\theta + a_6 \sin\theta = \frac{a_4 - a_5}{2a_3} f_3, \qquad \text{when} \quad a_3 \neq 0.$$

Therefore we may choose θ to satisfy $f_3 = 0$ again and $\xi^2\eta^2 = 1$. Now (5.3) holds, and (5.1) and (5.2) are the same equation

$$G(\xi^2) = f_2^2\xi^8 + \frac{1}{2}(a_4 + a_5)\xi^6 - \frac{1}{2}(a_4 + a_5)\xi^2 - f_1^2 = 0. \tag{7.5}$$

It has only one positive root.

If $k \neq 0$, $u \neq 0$, then we consider (6.7).

If $a_2^2 > t$, when θ satisfies $f_3 = 0$, we get

$$F = \frac{1}{16} t^{-4} u^4 k^6 (a_2^2 - t) > 0.$$

If $a_2^2 < t$, then, let φ satisfy $\sin\varphi = \frac{a_1}{t^{\frac{1}{2}}}$, $\cos\varphi = \frac{a_3}{t^{\frac{1}{2}}}$, we get, by the identity

$$k^2 + u^2 = (a_4 + a_5)^2 t - 4rt$$

that $f_2R - f_1P = 0$ is equivalent to

$$ku(t - a_2^2)\tan^2(\theta + \varphi) - (4rt^2 + tu^2 - a_2^2u^2 + a_2^2k^2)\tan(\theta + \varphi) + a_2^2ku = 0.$$

For the root

$$\tan(\theta+\varphi)=\{4rt^2+(t-a_2^2)u^2+a_2^2k^2+[16r^2t^4+8rt^2(u^2t-a_2^2u^2+a_2^2k^2)+(u^2t-a_2^2u^2-a_2^2k^2)^2]^{\frac{1}{2}}\}/(2ku(t-a_2^2)) \tag{7.6}$$

its magnitude is greater than other's. Since the product of them equals $\frac{a_2^2}{t-a_2^2}$, hence we get

$$\tan^2(\theta+\varphi)>\frac{a_2^2}{t-a_2^2},\quad a_2^2<t\sin^2(\theta+\varphi).$$

Therefore $\quad f_1f_2=(a_1\cos\theta+a_3\sin\theta)^2-a_2^2=t\sin^2(\theta+\varphi)-a_2^2>0.$

It is easy to see that for the θ defined by (7.6)

$$P\neq 0,\quad R\neq 0,\quad f_6L-f_9N\neq 0,$$

$$F=(f_6L-f_9N)^2PR=(f_6L-f_9N)^2\frac{P^2}{f_2^2}f_1f_2>0.$$

If $a_2^2=t$, then one of f_1 and f_2 is zero, when θ satisfies $f_3=0$, subsequently, or $P=0$, or $R=0$, hence $F=0$. On the other hand, from (6.7) we get

$$\frac{dF}{d\theta}=f_6^2(L-N)^2\frac{d}{d\theta}(PR)+PR\frac{d}{d\theta}(f_6^2(L-N)^2)-4\frac{d}{d\theta}(f_3^2(f_2R-f_1P)^2LN),$$

$$\frac{d}{d\theta}(PR)=\frac{1}{4}((a_4+a_5)^2f_3-k(f_4-f_5))((a_4+a_5)^2(f_1+f_2)+kf_6)-a_2^2(f_4-f_5)f_6((f_4-f_5)^2-4f_6^2).$$

Therefore, if $a_2^2=t$ and θ satisfies $f_3=0$, then

$$\frac{dF}{d\theta}=-\frac{1}{8}t^{-3}u^3k^3(k^2+4rt),$$

Let θ_1 satisfy $L-N=0$, that is $uf_6+2rf_3=0$, then

$$F=-\frac{1}{4}f_3^2(f_2R-f_1P)^2L^2<0.$$

Now we get the conclusion that $F=0$ has a root between θ_1 and θ defined by $f_3=0$, if $a_2^2>t$; and a root between θ_1 and θ defined by (7.6), if $a_2^2<t$; and finally if $a_2^2=t$, since $\tan\theta_1=\frac{2ua_6+4ra_3}{(a_4-a_5)u+4ra_1}$, and $\tan\theta=\frac{a_3}{a_1}$ for θ satisfying $f_3=0$, we get

$$\tan(\theta_1-\theta)=-\frac{uk}{u^2+4rt},$$

and there is a root between θ_1 and θ solving $f_3=0$.

We may locate this root by any method and determine ξ and η from

$$\xi^2\eta^2=\frac{N}{L}\quad\text{and}\quad\frac{\xi^2}{\eta^2}=\frac{R}{P}.$$

It can be proved that the values of θ, ξ and η thus determined solve the system (5.1) —(5.3), but we omit it here.

8. Case 2. $r>0,\ r_1=0,\ s_1\neq 0$

In this case we try to find θ, ξ and η in order that

$$b_4=b_7,\quad b_5=b_8,\quad b_1=0$$

or
$$\xi^4=\frac{f_4}{f_7},\quad \eta^4=\frac{f_5}{f_8},\quad \xi^2 f_2+\eta^2 f_1=0.$$

Therefore
$$f_1^2 f_5 f_7=f_2^2 f_4 f_8.$$

Let
$$\begin{aligned}h\equiv f_1^2 f_5 f_7-f_2^2 f_4 f_8 &= (a_2^2+(a_1\cos\theta+a_3\sin\theta)^2)\,((a_5a_7-a_4a_8)\cos\theta)\\ &+((a_4+a_5)a_9-(a_7+a_8)a_6\sin\theta)+2a_2(a_1\cos\theta+a_3\sin\theta)\\ &\times\Big[\Big(\frac{1}{2}(a_4+a_5)(a_7+a_8)-2a_6a_9\Big)\sin^2\theta+(a_4a_8+a_5a_7)\cos^2\theta\\ &-((a_7-a_8)a_6+(a_4-a_5)a_9)\sin\theta\cos\theta\Big]=0. \end{aligned}\tag{8.1}$$

It is a homogeneous equation in $\sin\theta$ and $\cos\theta$ of order 3. Therefore there exists at least a real root θ. Obviously, only those θ which satisfy $f_1f_2<0$ or $f_1=f_2=0$ are needed. When $a_2=0$, choose θ satisfying $a_1\cos\theta+a_3\sin\theta=0$. Now, $f_1=f_2=0$, $f_4f_5f_7f_8\neq 0$, and we can determine ξ and η.

When $a_2\neq 0$, if $a_2^2>t$, we have
$$f_1f_2=(a_1\cos\theta+a_3\sin\theta)^2-a_2^2\leqslant t-a_2^2<0.$$

If $a_2^2\leqslant t$, since h may be written as
$$h=(a_2^2+t\sin^2(\theta+\varphi))(f_5f_7-f_4f_8)+2a_2t^{\frac{1}{2}}\sin(\theta+\varphi)(f_5f_7+f_4f_8),$$
where $\sin\varphi=\frac{a_1}{\sqrt{t}}$, $\cos\varphi=\frac{a_3}{\sqrt{t}}$, then
$$h=4a_2^2f_5f_7>0,\quad \text{when } \sin(\theta+\varphi)=\frac{a_2}{\sqrt{t}} \text{ or } f_2=0,$$
$$h=-4a_2^2f_4f_8<0, \text{when } \sin(\theta+\varphi)=-\frac{a_2}{\sqrt{t}} \text{ or } f_1=0.$$

Therefore there exists a root of $h=0$ between $\sin^{-1}\frac{a_2}{\sqrt{t}}-\varphi$ and $\sin^{-1}\left(-\frac{a_2}{\sqrt{t}}\right)-\varphi$, Obviously this root satisfies $\sin^2(\theta+\varphi)<\frac{a_2^2}{t}$, that is $f_1f_2<0$.

Carrying out the similarity transformation with θ, ξ and η thus determined, as in § 7, we may assume without loss of generality that
$$a_4=a_7,\ a_5=a_8,\ a_1=0.$$

Hence assumptions (6.5) and (6.6) hold.

If $a_4=a_5$, we choose $\theta=\frac{\pi}{2}$, then (5.3) is satisfied. When $a_4=a_9$, adding (5.1) and (5.2), we get
$$\eta^2=\frac{(a_4-a_6)\xi^2}{2a_4\xi^4-(a_4+a_6)}.$$

Substituting it into (5.2), we obtain
$$\begin{aligned}G(\xi^2)&=4a_4^2(a_3-a_2)^2\xi^{10}+4a_4^2(a_4-a_6)\xi^8-4a_4(a_4+a_6)(a_3-a_2)^2\xi^6-4a_4(a_4-a_6)^2\xi^4\\ &+[(a_3-a_2)^2(a_4+a_6)^2-(a_3+a_2)^2(a_4-a_6)^2]\xi^2+(a_4-a_6)(a_4+a_6)^2=0.\end{aligned}\tag{8.2}$$

Since $G\left(\left(\frac{a_4+a_6}{2a_4}\right)^{\frac{1}{2}}\right)=-(a_4-a_6)^2(a_3+a_2)^2\left(\frac{a_4+a_6}{2a_4}\right)^{\frac{1}{2}}<0$, hence there exists a root of

$G(\xi^2)=0$ which is greater than $\left(\frac{a_4+a_6}{2a_4}\right)^{\frac{1}{2}}$. Determine this root and η. Similarly, when $a_4=-a_9$, we have

$$\xi^2=\frac{(a_4+a_6)\eta^2}{2a_4\eta^4-(a_4-a_6)}$$

and

$$G(\eta^2)=4a_4^2(a_3+a_2)^2\eta^{10}+4a_4^2(a_4+a_6)\eta^8-4a_4(a_4-a_6)(a_3+a_2)^2\eta^6-4a_4(a_4^2-a_6^2)\eta^4$$
$$+[(a_3+a_2)^2(a_4-a_6)^2-(a_3-a_2)^2(a_4+a_6)^2]\eta^2+(a_4+a_6)(a_4-a_6)^2=0. \tag{8.3}$$

Since $G\left(\left(\frac{a_4-a_6}{2a_4}\right)^{\frac{1}{2}}\right)=-(a_3-a_2)^2(a_4+a_6)^2\left(\frac{a_4-a_6}{2a_4}\right)^{\frac{1}{2}}<0$, hence there exists a root of $G(\eta^2)=0$ which is greater than $\left(\frac{a_4-a_6}{2a_4}\right)^{\frac{1}{2}}$.

If $a_3=0$, then (6.7) is an identity. But (5.3) becomes $f_9\xi^2\eta^2=f_6$. From it, (5.1) and (5.2), eliminating ξ and η, we get

$$\tilde{F}=r^2(f_5f_9-f_7f_6)(f_8f_6-f_4f_9)f_9^4-a_2^4f_6f_9(a_4+a_5)^2(f_6-f_9)^2$$
$$\times((f_4-f_5)f_9+(f_7-f_8)f_6)^2=0 \tag{8.4}$$

which is equivalent to (6.7) in many cases. Obviously,

$$\tilde{F}=\frac{1}{2^6}r^2(a_4-a_5)^6(a_6+a_9)^2>0, \quad \text{when} \quad \theta=\pm\frac{\pi}{2}.$$

When θ satisfies $f_9=0$, then

$$\tilde{F}=0 \quad \text{and} \quad \frac{d\tilde{F}}{d\theta}=\frac{1}{2}a_2^4(a_6-a_9)^5(a_4-a_5)^5.$$

Therefore if $(a_4-a_5)(a_6-a_9)>0$, then $\tilde{F}=0$ has a root θ which is less than $\tan^{-1}\frac{2a_9}{a_4-a_5}$, and if $(a_4-a_5)(a_6-a_9)<0$, then there is a root greater than $\tan^{-1}\frac{2a_9}{a_4-a_5}$. Compute this θ and determine ξ and η from $\xi^2\eta^2=\frac{N}{L}$ and $\frac{\xi^2}{\eta^2}=\frac{R}{P}$.

In the case $K=a_3(a_4-a_5)\neq 0$, if θ satisfies $f_9=0$, then

$$L=s_1f_9=0, \quad F=0,$$

$$\frac{dF}{d\theta}=2f_3^2(f_2R-f_1P)^2s_1^2(a_4-a_5)(a_6-a_9).$$

If $a_2^2>t$, when θ satisfies $f_8=0$, i. e. $\cos\theta=0$, then

$$F=\frac{1}{16}t(a_4-a_5)^6(a_6+a_9)^2(a_2^2-t)>0.$$

Therefore when $(a_4-a_5)(a_6-a_9)>0$ (or <0), there exists a root of $F=0$ which is less (or greater resp.) than $\tan^{-1}\frac{2a_9}{a_4-a_5}$. If $a_2^2=t$, when θ satisfies $f_8=0$, $F=0$, but by comparing the signs of $\frac{dF}{d\theta}$ at this θ and at the θ satisfying $f_9=0$, we may get the same conclusion as above. If $a_2^2<t$, when θ satisfies $f_2R-f_1P=0$, i. e.

$$(t-a_2^2)(a_4-a_5)(a_6+a_9)\tan^2\theta-[a_2^2((a_4-a_5)^2-4a_6a_9)$$
$$-2a_2a_3(a_4+a_5)(a_6-a_9)+4a_3^2a_4a_5]\tan\theta+a_2^2(a_4-a_5)(a_6+a_9)=0 \qquad (8.5)$$

we may prove

$$a_2^2((a_4-a_5)^2-4a_6a_9)-2a_2a_3(a_4+a_5)(a_6-a_9)+4a_3^2a_4a_5=ss_1+4a_9(a_6+a_9)(t-a_2^2)>0$$

for the root

$$\tan\theta=\frac{ss_1+4a_9(a_6+a_9)(t-a_2^2)+\sqrt{s_1^2(s^2+4r(t-a_2^2))}}{2(t-a_2^2)(a_4-a_5)(a_6+a_9)} \qquad (8.6)$$

its magnitude is greater than other's, then $\tan^2\theta>\frac{a_2^2}{t-a_2^2}$. Now

$$F=(f_6L-f_9N)^2\frac{P^2}{f_2^2}f_1f_2=\frac{1}{f_2^2}(f_6L-f_9N)^2P^2(a_3^2\sin^2\theta-a_2^2)>0.$$

Denote the θ satisfying $f_9=0$ by θ_1. Since

$$\tan\theta-\tan\theta_1=(a_4-a_5)(a_9-a_6)\frac{ss_1+\sqrt{s_1^2(s^2+4r(t-a_2^2))}}{2(t-a_2^2)(a_4-a_5)^2r}$$

we get there is a root between θ and θ_1.

We may treat the case 3 similarly.

9. Case 4. $r=r_1=0,\ s\neq0\neq s_1$

Let

$$\Delta\equiv(a_5a_8-a_4a_7)^2+((a_4+a_5)a_9+(a_7+a_8)a_6)^2$$
$$=(a_4a_7+a_5a_8+2a_6a_9)^2+((a_4-a_5)a_9-(a_7-a_8)a_6)^2. \qquad (9.1)$$

When $\Delta\neq0$, we try to θ, ξ and η in order that

$$b_4=b_7,\ b_5=b_8,\ b_4=b_5$$

or

$$\xi^4=\frac{f_4}{f_7},\quad \eta^4=\frac{f_5}{f_8},\quad \frac{\xi^2}{\eta^2}=\frac{f_4}{f_5}.$$

Therefore

$$f_4f_7=f_5f_8$$

and from it we get

$$\tan\theta=\frac{a_5a_8-a_4a_7}{(a_4+a_5)a_9+(a_7+a_8)a_6}. \qquad (9.2)$$

Take

$$\sin\theta=(a_5a_8-a_4a_7)\Delta^{-\frac{1}{2}},$$

then $b_4=b_5=b_6=b_7=b_8=b_9=\frac{1}{2}(a_4+a_5)^{\frac{1}{2}}(a_7+a_8)^{\frac{1}{2}}\Delta^{-\frac{1}{2}}(a_5a_8+a_4a_7+2a_6a_9)$.

Carry out the similarity transformation with the θ, ξ and η so defined, as above we may assume without loss of generality

$$a_4=a_5=a_6=a_7=a_8=a_9.$$

So

$$L=2a_4^2(a_2+a_3)\cos\theta,\qquad N=2a_4^2(a_2-a_3)\cos\theta,$$
$$P=2a_4^2\cos\theta(a_3-a_2\sin\theta),\qquad R=-2a_4^2\cos\theta(a_3+a_2\sin\theta). \qquad (9.3)$$

If $a_2=0$, then the norm reduction has been completed. Suppose $a_2\neq0$.

If $a_2^2<a_3^2$, then the only possibility is $\cos\theta=0$. Take $\sin\theta=1$. From (5.2) we get

$$\frac{\xi^2}{\eta^2}=\frac{a_3+a_2}{a_3-a_2}.$$

Substituting it into (5.1), we get

$$\xi^4=1,\ \eta^2=\frac{a_3-a_2}{a_3+a_2}.$$

As for (5.3), it obviously holds. Furthermore we have

$$\tau(A)=\tau(B)+4a_2^2.$$

If $a_2^2>a_3^2$, then (6.7) becomes

$$F=a_4^2a_3^2\cos^2\theta(a_2^2\sin^2\theta-a_3^2)-4(a_2^2-a_3^2)(a_3\cos\theta-a_1\sin\theta)^2 \times((a_2^2-a_3^2)\sin\theta-a_1a_3\cos\theta)^2=0. \quad (9.4)$$

If $a_1=0$, then $\cos\theta=0$. Taking $\sin\theta=1$, we get

$$\xi^2\eta^2=\frac{a_2-a_3}{a_2+a_3}\quad\text{and}\quad\frac{\xi^2}{\eta^2}=\frac{a_2+a_3}{a_2-a_3},$$

subsequently

$$\xi^2=1\quad\text{and}\quad\eta^2=\frac{a_2-a_3}{a_2+a_3}.$$

If $a_3=0$, then $\sin\theta=0$. Taking $\cos\theta=1$, we get $\xi^2\eta^2=1$, (5.3) obviously holds, and (5.1) and (5.2) are the same equation

$$G(\xi^2)=(a_1-a_2)^2\xi^8+a_4\xi^6-a_4\xi^2-(a_1+a_2)^2=0. \quad (9.5)$$

It has only one positive root.

In the case $a_1a_3\neq0$, $F=0$ can be written as

$$f(\tan\theta)=a_4^2a_3^2(a_2^2\tan^2\theta-a_3^2\tan^2\theta-a_3^2)-4(a_2^2-a_3^2)(a_1\tan\theta-a_3)^2 \times((a_2^2-a_3^2)\tan\theta-a_1a_3)^2=0. \quad (9.6)$$

If $a_2^2=a_1^2+a_3^2$, then

$$f(\tan\theta)=a_4^2a_3^2(a_1^2\tan^2\theta-a_3^2)-4a_1^4(a_1\tan\theta-a_3)^4;$$

$$f\left(\frac{a_3}{a_1}\right)=0,\qquad f'\left(\frac{a_3}{a_1}\right)=2a_1a_4^2a_3^3.$$

Therefore if $a_1a_3>0$, then $f=0$ has a root $\tan\theta>\frac{a_3}{a_1}$. And if $a_1a_3<0$, then there is a root less than $\frac{a_3}{a_1}$. If $a_2^2>a_1^2+a_3^2$, then

$$f\left(\frac{a_3}{a_1}\right)=\frac{a_4^2a_3^4}{a_1^2}(a_2^2-a_3^2-a_1^2)>0.$$

And in order to gaurantee $\frac{\xi^2}{\eta^2}>0$, when $a_1a_3>0$, we take the root $\tan\theta$ of $f(\tan\theta)=0$ which is greater than $\frac{a_3}{a_1}$, otherwise less than $\frac{a_3}{a_1}$. If $a_2^2<a_1^2+a_3^2$, then

$$f\left(\frac{a_1a_3}{a_2^2-a_3^2}\right)=\frac{a_4^2a_3^4(a_1^2+a_3^2-a_2^2)}{a_2^2(a_2^2-a_3^2)}>0.$$

And $f<0$, as $\tan\theta\to\pm\infty$. Therefore, if $a_1a_3>0$, we take the root $\tan\theta$ of $f(\tan\theta)=0$ which is greater than $\frac{a_1a_3}{a_2^2-a_3^2}$, and otherwise less than it.

When $\Delta=0$, as in § 8, we try to find θ, ξ and η in order that

$$b_4=a_7, \quad b_5=b_8, \quad b_1=0.$$

Subsequently, θ is defined by $h=f_1^2 f_5 f_7-f_2^2 f_4 f_8=0$. Let $\theta=\omega+\varphi$, where

$$\sin\varphi=\frac{2a_6}{a_4+a_5}, \quad \cos\varphi=\frac{a_4-a_5}{a_4+a_5}.$$

Using $$a_5a_8=a_4a_7=-a_6a_9, \quad a_4a_9+a_8a_6=a_5a_9+a_7a_6=0$$

we get $$f_4=\frac{1}{2}(a_4+a_5)(1+\cos\omega), \quad f_5=\frac{1}{2}(a_4+a_5)(1-\cos\omega),$$

$$f_7=\frac{1}{2}(a_7+a_8)(1-\cos\omega), \quad f_8=\frac{1}{2}(a_7+a_8)(1+\cos\omega).$$

Therefore

$$h=\frac{1}{4}(a_4+a_5)(a_7+a_8)\left(f_1^2(1-\cos\omega)^2-f_2^2(1+\cos\omega)^2\right)=0. \tag{9.7}$$

From $f_1(1-\cos\omega)+f_2(1+\cos\omega)=0$, we get

$$\sin\omega=\frac{s}{(k^2+s^2)^{\frac{1}{2}}}, \quad \cos\omega=\frac{k}{(k^2+s^2)^{\frac{1}{2}}}.$$

Furthermore, it is easy to see that $b_4=-b_6=b_9$.

Carrying out the similarity transformation with θ, ξ and η so defined, we may assume

$$a_4=a_5=-a_6=a_7=a_8=a_9, \quad a_1=0.$$

So $$s=s_1=2a_4(a_2-a_3), \quad L=-sa_4\cos\theta, \quad N=sa_4\cos\theta.$$

Therefore the only possibility is $\cos\theta=0$. Taking $\sin\theta=1$, (5.1) and (5.2) become

$$\xi^4(a_3-a_2)^2-\eta^4(a_3+a_2)^2-2a_4\eta^2=0,$$

$$2\xi^2\eta^4a_4+\eta^4(a_2+a_3)^2-\xi^4(a_3-a_2)^2=0$$

respectively, and (5.3) obviously holds. Therefore $\xi^2\eta^2=1$ and

$$(a_3-a_2)^2\xi^8-2a_4\xi^2-(a_2+a_3)^2=0.$$

It has one and only one positive root $\xi^2>\left(\frac{2a_4}{(a_3-a_2)^2}\right)^{\frac{1}{3}}$.

In conclusion, in some particular cases we find the optimal values of the parameters θ, ξ and η, and in the general case we find the interval within which the optimal value lies.

Since there is a 1–1 correspondence between $\xi^2\eta^2$ and $\tan\theta$, we may transform the equation of θ into that of $\xi^2\eta^2$, and the latter is simpler, but we omit it here.

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