ON THE CONTRACTIVITY REGION OF RUNGE-KUTTA METHODS*

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Abstract

In this paper we first introduce the definition of contractivity region of Runge-Kutta methods and then examine the general features of the contractivity regions. We find that the intersections of the contractivity region and the axis plane in C^s are always either the whole axis plane or a generalized disk introduced by Dahlquist and Jeltsch. We also define the AN-contractivity and show that it is equivalent to the algebraic stability and can be determined locally in a neighborhood of the origin. However, many implicit methods are only r-circle contractive, but not AN-contractive. A simple bound for the radius r of the r-circle contractive methods is given.

1. Introduction

We shall consider the numerical solution of initial value problems

$$y'=f(x, y), \quad y(0) \quad \text{given} \tag{1.1}$$

where $y, f\in R^s$ or C^s. Assume that f satisfies the following monotonicity condition

$$\mathrm{Re}\,\langle f(x, y)-f(x, z), y-z\rangle\leqslant 0 \quad \text{for } y, z\in R^s \text{ or } C^s, \tag{1.2}$$

where $\langle\cdot, \cdot\rangle$ stands for an arbitrary inner product in C^s, and $\|\cdot\|$ is the corresponding norm. Let y and $\tilde{y}$ be two solutions to (1.1) corresponding to the initial values y_0 and $\tilde{y}_0$ respectively. By condition (1.2) we have

$$\frac{d}{dx}\|y(x)-\tilde{y}(x)\|^2\leqslant 0 \tag{1.3}$$

which shows that $\|y(x)-\tilde{y}(x)\|$ does not increase when x increases.

The general m-stage Runge-Kutta methods for system (1.1) have the form

$$\begin{cases} Y_i=y_{n-1}+h\sum_{j=1}^{m} a_{ij}f(x_{n-1}+hc_j, Y_j), \quad i=1, 2, \cdots, m, \\ y_n=y_{n-1}+h\sum_{j=1}^{m} b_j f(x_{n-1}+hc_j, Y_j), \quad n=1, 2, \cdots, \\ c_j=\sum_{k=1}^{m} a_{jk}. \end{cases} \tag{1.4}$$

Given $A=(a_{ij})_{m\times m}$ and $b=(b_1, b_2, \cdots, b_m)^T$, we shall denote the corresponding method (1.4) by $M(A, b)$. In terms of the Kronecker product symbol $\otimes$ it can be written as

$$\begin{cases} Y=\mathbb{1}\otimes y_{n-1}+hA\otimes I_sF_{n-1}(Y), \\ y_n=y_{n-1}+hb^T\otimes I_sF_{n-1}(Y), \end{cases} \tag{1.5}$$

where I_s is the $s\times s$ identity matrix and

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$$Y=\begin{Bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_m \end{Bmatrix}, \quad F_{n-1}(Y)=\begin{Bmatrix} f(x_{n-1}+hc_1,\ Y_1) \\ f(x_{n-1}+hc_2,\ Y_2) \\ \vdots \\ f(x_{n-1}+hc_m,\ Y_m) \end{Bmatrix}, \quad \mathbb{1}=\begin{Bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{Bmatrix}.$$

In applications it is expected that the numerical methods preserve the contractivity property (1.3) of the differential equation, namely if the computation starts with a slightly perturbed initial value $\tilde{y}_0$, instead of y_0, the obtained solution $\tilde{y}_n$ and the unperturbed solution y_n satisfy

$$\|y_n-\tilde{y}_n\|\leqslant\|y_{n-1}-\tilde{y}_{n-1}\| \quad \text{for } n=1,\ 2,\ \cdots. \tag{1.6}$$

Such requirement for the nonlinear problem (1.1) leads to the concept of BN-stability (B-stability for the autonomous problem: $y'=f(y)$, $y(0)=y_0$) introduced by Butcher in [1] and leads to the concept of AN-stability for the linear non-autonomous problem (A-stability for the linear autonomous problem). Another stability criterion named algebraic stability was developed by Butcher[2] and Crouzeix[3], which is significant in the study of BN- and B-stability properties of implicit Runge-Kutta methods. Dahlquist and Jeltsch introduced in [4] a concept of generalized disk contractivity for explicit and implicit Runge-Kutta methods, which is an extension of the AN-and BN-stability that are reasonable only for implicit methods.

In this paper we first introduce the definition of contractivity region of Runge-Kutta methods (implicit or explicit) and then examine the general features of the contractivity region. We find that the intersections of the contractivity region and the axis planes in C^s are always either the whole axis plane or a generalized disk introduced by Dahlquist and Jeltsch[4]. This fact gives some evidence to the concept of generalized disk contractivity. Set $C^-=\{z\in C;\ \mathrm{Re}\, z<0\}$. A method $M(A,\ b)$ is referred to as "AN-contractive" if its contractivity region contains $(C^-)^m$. We shall show that this property is equivalent to the algebraic stability and can be determined locally in a neighborhood of the origin. However, we shall see that many implicit methods are only r-circle contractive, but not AN-contractive. We shall provide a simple bound for the radius r of the r-circle contractive methods.

2. Contractivity Region

To motivate the definition we consider the following test problem

$$y'=\lambda(x)y, \quad y(0)=y_0, \tag{2.1}$$

where λ: $R^+\Rightarrow C$ is a given function and $\mathrm{Re}\,\lambda(x)\leqslant 0$ for $x\in R^+$. Set

$$z_i=h\lambda(x_{n-1}+hc_i), \quad i=1,\ 2,\ \cdots,\ m,$$
$$\zeta=(z_1,\ z_2,\ \cdots,\ z_m),$$
$$Z=\mathrm{diag}(z_1,\ z_2,\ \cdots,\ z_m).$$

For this problem, (1.4) takes the form

$$\begin{cases} Y=y_{n-1}\mathbb{1}+AZY, \\ y_n=y_{n-1}+b^TZY, \end{cases} \tag{2.2}$$

and by substitution $Y=(I_m-AZ)^{-1}(y_{n-1}\mathbb{1})$ we have

$$y_n = K(\zeta) y_{n-1}, \tag{2.3}$$

where $K(\zeta)$ is a rational function of complex variables $z_1, z_2, \cdots, z_m$. If the $\lambda(x)$ in (2.1) is constant, we have, with $z = h\lambda$,

$$K(\zeta) = R(z) = 1 + z b^T (I_m - zA)^{-1} \mathbb{1}. \tag{2.4}$$

$K(\zeta)$ and $R(z)$ are named the "K-function" and "R-function" of the method $M(A, b)$ respectively.

Definition 1. *Given Runge-Kutta method $M(A, b)$, the following subset of $(\bar{C})^m$ is called the contractivity region of $M(A, b)$*

$$\Omega(A, b) = \{\zeta \in (\bar{C})^m;\ |K(\zeta)| \leqslant 1\},$$

where $\bar{C}$ denotes the complex plane closed by the point ∞.

Definition 2. *Given subset $D \subset (\bar{C})^m$, method $M(A, b)$ is called D-contractive if $\Omega(A, b) \supset D$. Particularly, it is called AN-contractive if $\Omega(A, b) \supset (C^-)^m$.*

It can be easily seen that the AN-stability introduced by Butcher[1] is equivalent to

$$\Omega(A, b) \supset \{\zeta \in C^m;\ \mathrm{Re}\, z_i \leqslant 0,\ z_i = z_j \text{ if } c_i = c_j,\ i, j = 1, 2, \cdots, m\}$$

and the A-stability is equivalent to

$$\Omega(A, b) \supset \{\zeta \in C^m;\ z_i = z,\ i = 1, 2, \cdots, m,\ \mathrm{Re}\, z \leqslant 0\}.$$

Obviously AN-contractivity implies AN-stability and A-stability.

We introduce the matrix

$$A^* = (a^*_{ij})_{m\times m},\quad a^*_{ij} = b_j - a_{ij}.$$

By applying the Cramer rule to the linear system

$$(I_m - A\cdot Z) Y = y_{n-1} \mathbb{1},$$

the function $K(\zeta)$ can be expressed as a quotient of two determinants[6]

$$K(\zeta) = \frac{\det(I_m + A^* Z)}{\det(I_m - AZ)}. \tag{2.5}$$

Thus by the maximum modulus theorem of complex function with several complex variables we have

Theorem 1. *A Runge-Kutta method $M(A, b)$ is AN-contractive if and only if $\det(I_m - AZ)$ has no zero in $(C^-)^m$ and*

$$|K(\zeta)| \leqslant 1$$

for all $\zeta = (z_1, z_2, \cdots, z_m) \in C^m$ such that $\mathrm{Re}\, z_i = 0,\ i = 1, 2, \cdots, m$.

In the next section we need the following lemmas.

Lemma 1 (Burrage and Butcher[2]). *Let D be the set of all ζ in C^m such that* $\det(I_m - AZ) \neq 0$, *and let $u = (I_m - AZ)^{-1}\mathbb{1}$. Then for $\zeta \in D$,*

$$|K(\zeta)|^2 - 1 = 2\sum_{i=1}^m b_i \mathrm{Re}(z_i)\, |u_i|^2 - \sum_{i,j=1}^m q_{ij} \bar{z}_i \bar{u}_i z_j u_j, \tag{2.6}$$

where $q_{ij} = b_i a_{ij} + b_j a_{ji} - b_i b_j$.

Obviously D contains a neighborhood of the origin in C^m. We see from this lemma that the surface $S(A, b)$ of the contractivity region $\Omega(A, b)$ is determined by the equation

$$2\sum_{i=1}^{m} b_i|u_i|^2\mathrm{Re}(z_i) - \sum_{i,j=1}^{m} q_{ij}\bar{z}_i\bar{u}_i z_j u_j = 0. \tag{2.7}$$

Lemma 2. *For any Runge-Kutta method $M(A, b)$,*

(i) *$S(A, b)$ passes through the origin $\zeta=0$;*

(ii) *$\Omega(A, b)$ is symmetric with respect to the real axis plane in C^m;*

(iii) *$\Omega(A, b)$ is closed.*

Proof. (i) holds obviously by (2.7). Since the coefficients of the method, a_{ij} and b_j, are real numbers, in addition to $u=(I_m-AZ)^{-1}\mathbb{1}$ we have $\bar{u}=(I_m-A\bar{Z})^{-1}\mathbb{1}$; hence $|K(\zeta)|^2=|K(\bar{\zeta})|^2$ and (ii) follows. (iii) can be shown by the continuity of $|K(\zeta)|$ and the definition of $\Omega(A, b)$.

We shall denote the i-th real axis and imaginary axis of C^m by R_i and I_i respectively, and use the notations

$$R_{i,-\varepsilon}=\{\zeta\in C^m;\ z_i\in(-\varepsilon, 0),\ z_j=0 \text{ for } j\neq i,\ j=1, 2, \cdots, m\}$$

$$I_{i,\varepsilon}=\{\zeta\in C^m;\ z_i\in[-\sqrt{-1}\,\varepsilon, \sqrt{-1}\,\varepsilon],\ z_j=0 \text{ for } j\neq i,\ j=1, 2, \cdots, m\}$$

$$Q=(q_{ij})_{m\times m},\ q_{ij}=b_i a_{ij}+b_j a_{ji}-b_i b_j. \tag{2.8}$$

Lemma 3. *For any Runge-Kutta method $M(A, b)$,*

(i) *$R_{i,-\varepsilon}\subset\Omega(A, b)$ for some $\varepsilon>0$ if and only if $b_i\geqslant 0$, for $i=1, 2, \cdots, m$;*

(ii) *Let $\{i_1, i_2, \cdots, i_r\}$ be a subset of $I=\{1, 2, \cdots, m\}$. Then $I_{i_1,\varepsilon}\times I_{i_2,\varepsilon}\times\cdots\times I_{i_r,\varepsilon}\subset\Omega(A, b)$ for some $\varepsilon>0$ if and only if*

$$Q_r=\begin{vmatrix} q_{i_1i_1} & q_{i_1i_2} & \cdots & q_{i_1i_r} \\ q_{i_2i_1} & q_{i_2i_2} & \cdots & q_{i_2i_r} \\ \cdots & \cdots & \cdots & \cdots \\ q_{i_ri_1} & q_{i_ri_2} & \cdots & q_{i_ri_r} \end{vmatrix}\geqslant 0.$$

Proof. Let $\zeta^*=(z_1^*, z_2^*, \cdots, z_m^*)$ be such that $z_i^*=-\varepsilon$ and $z_j^*=0$ for $j\neq i$. When ε is sufficiently small we have by (2.6)

$$|K(\zeta^*)|^2-1=-2b_i|u_i|^2\varepsilon+0(\varepsilon^2), \tag{2.9}$$

which shows b_i must be nonnegative if $R_{i,-\varepsilon}\subset\Omega(A, b)$ for some $\varepsilon>0$. Notice that if $b_i=0$, then $q_{ii}=0$, and $|K(\zeta^*)|^2-1=0$, so that the condition $b_i\geqslant 0$ is also sufficient for the inclusion $R_{i,-\varepsilon}\subset\Omega(A, b)$.

To show (ii) we observe that for sufficiently small $\varepsilon>0$,

$$u_j(\zeta)=1+\psi_j(\zeta),\ j=1, 2, \cdots, m, \text{ for } |\zeta|<\varepsilon, \tag{2.10}$$

where $\quad \psi_j(\zeta)\to 0$ as $\varepsilon\to 0$.

Now we choose $\zeta^*=(z_1^*, z_2^*, \cdots, z_m^*)$ such that

$$\begin{cases} z_{i_1}^*=\sqrt{-1}\,\eta_{i_1}\varepsilon,\ \cdots,\ z_{i_r}^*=\sqrt{-1}\,\eta_{i_r}\varepsilon, \\ z_j^*=0 \text{ for } j\overline{\in}\{i_1, i_2, \cdots, i_r\}. \end{cases} \tag{2.11}$$

Thus for $\varepsilon>0$ sufficiently small we have by (2.7)

$$|K(\zeta^*)|^2-1=-\varepsilon^2(Q_r\eta, \eta)+\varepsilon^2C(\varepsilon), \tag{2.12}$$

where $\eta=(\eta_{i_1}, \eta_{i_2}, \cdots, \eta_{i_r})$ and $C(\varepsilon)\to 0$ as $\varepsilon\to 0$. Assume $I_{i_1,\varepsilon}\times\cdots\times I_{i_r,\varepsilon}\subset\Omega(A, b)$ for some $\varepsilon>0$. If Q_r were negative, there would exist an $\eta_0\in R^r$ such that $(Q_r\eta_0, \eta_0)<0$. Thus by (2.11) and (2.12), for any small $\varepsilon>0$ we could find such $\zeta\in I_{i_1,\varepsilon}\times\cdots\times I_{i_r,\varepsilon}$ that

$$|K(\zeta)|^2-1>0.$$

This is a contradiction with the assumption. Hence $Q_r\geqslant 0$ is a necessary condition for the inclusion

$$I_{i_1,s}\times\cdots\times I_{i_r,s}\subset\Omega(A,\ b).$$

The inverse is also ture from (2.12).

3. Some Conclusions

As a direct consequence of Lemma 3 we have the following result.

Theorem 2. *An m-stage Runge-Kutta method $M(A,\ b)$ is AN-contractive if and only if it is algebraically stable, i. e.*

$$b_j\geqslant 0,\ j=1,\ 2,\ \cdots,\ m,\ Q\geqslant 0.$$

And it is AN-contractive if and only if it is contractive in an arbitrary small neighborhood of the origin, i. e. for some $\varepsilon>0$

$$\{\zeta\in C^m;\ |\zeta|<\varepsilon,\ \mathrm{Re}\, z_i\leqslant 0,\ i=1,\ 2,\ \cdots,\ m\}\subset\Omega(A,\ b).$$

Corollary 1. Given matrix $A=(a_{ij})_{m\times m}$, the system

$$(I_m-AZ)Y=F \tag{2.13}$$

has a unique solution for any F, $\zeta\in(C^-)^m$ if A is a lower triangle matrix or if there exist nonnegative weights $b_1,\ b_2,\ \cdots,\ b_m$ such that

$$Q=(q_{ij})_{m\times m}\geqslant 0 \quad \text{where } q_{ij}=b_ia_{ij}+b_ja_{ji}-b_ib_j.$$

Proof. The first part holds obviously. If there exist $b_1,\ b_2,\ \cdots,\ b_m$ such that $Q=(q_{ij})_{m\times m}\geqslant 0$, by Theorem 2 the method $M(A,\ b)$is AN-contractive, and by Theorem 1 $\det(I_m-AZ)\neq 0$ for any $\zeta\in(C^-)^m$. Hence system (2.13) has a unique solution for any $F\in C^m$.

Now we further examine the intersections of the contractivity region and the axis planes $C_i=R_i\times I_i$, $i=1,\ 2,\ \cdots,\ m$.

Set $\zeta_0\in R_i$ and $|\zeta_0|=x$. Then $\zeta_0\in S(A,\ b)$ if and only if x satisfies

$$2b_i(\mathrm{sign}(\zeta_0))x-q_{ii}x^2=0, \tag{2.14}$$

where $q_{ii}=2b_ia_{ii}-b_i^2$. When $b_i=0$, (2.14) is an identity for all $x\geqslant 0$, and if $q_{ii}\neq 0$, (2.14) has a non-zero root

$$x=2b_i(\mathrm{sign}(\zeta_0))/q_{ii}=2\mathrm{sign}(\zeta_0)/(2a_{ii}-b_i),$$

which is positive only if ζ_0 and $2a_{ii}-b_i$ have the same sign. Thus (2.14) shows that

$R_i\subset S(A,\ b)$ if $b_i=0$,

$S(A,\ b)$ has no intersection with R_i^- if $a_{ii}>b_i/2\neq 0$,

$S(A,\ b)$ has no intersection with R_i^+ if $a_{ii}<b_i/2\neq 0$,

the only intersection point of $S(A,\ b)$ and R_i is $\zeta=0$ if $a_{ii}=b_i/2\neq 0$.

Set $\zeta_0\in I_i$ and $|\zeta_0|=y$. Then by (2.7) $\zeta_0\in S(A,\ b)$ if and only if $q_{ii}y^2=0$, which shows that

$I_i\subset S(A,\ b)$ if $q_{ii}=0$,

$S(A,\ b)\cap I_i=\{\zeta=0\}$ if and only if $q_{ii}\neq 0$.

To find the intersection of $\Omega(A,\ b)$ and C_i we now consider $\zeta^*=(z_1^*,\ z_2^*,\ \cdots,\ z_m^*)$ where $z_i^*=x+\sqrt{-1}\,y$ and $z_j^*=0$ for $j\neq i$. Thus if $\zeta^*\in S(A,\ b)$ we have

$$2q_ix-q_{ii}(x^2+y^2)=0$$

or

$$\left(x-\frac{b_i}{q_{ii}}\right)^2+y^2=\left(\frac{b_i}{q_{ii}}\right)^2 \text{ if } q_{ii}\neq 0,$$

which represents a circle in the C_i complex plane as shown in Fig. 1.

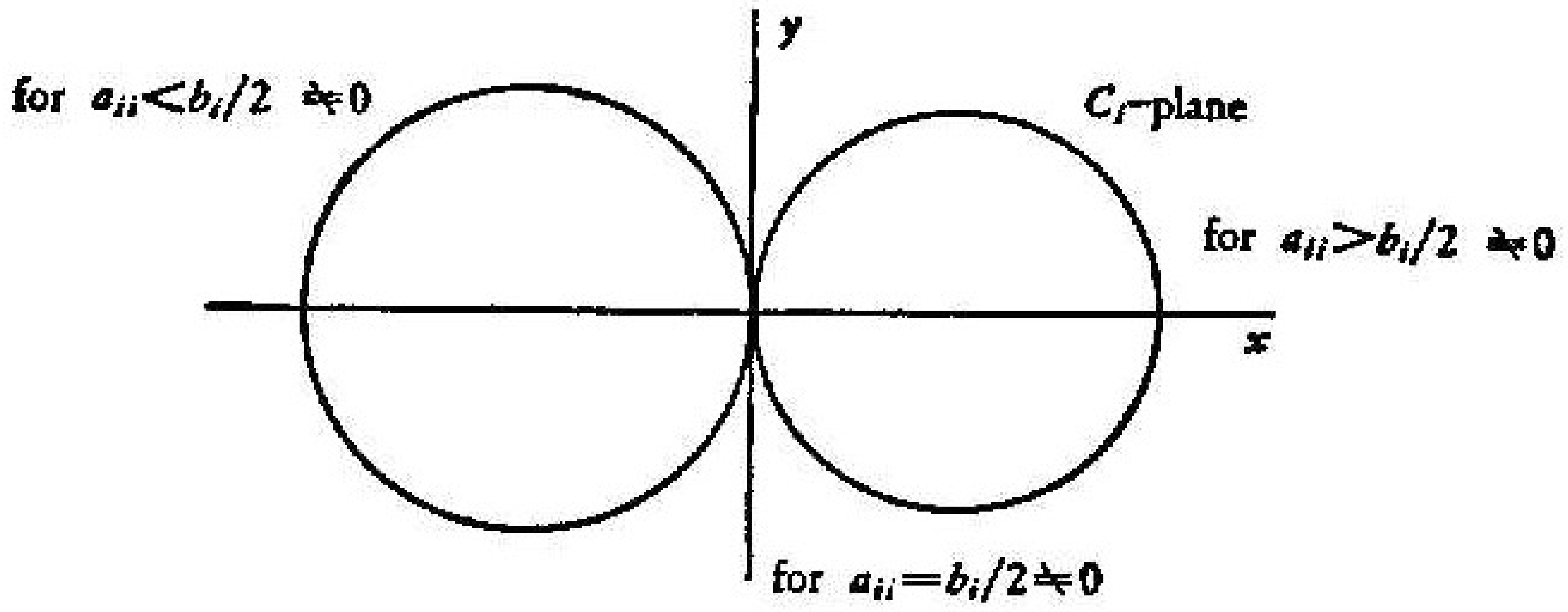

Fig. 1

When $b_i=0$, we have $q_{ii}=2b_ia_{ii}-b_i^2=0$; hence

$$C_i=I_i\times R_i\subset S(A,\ b).$$

From these we have the following result.

Theorem 3. *For any given Runge-Kutta method $M(A,\ b)$, the intersection of the contractivity region $\Omega\ (A,\ b)$ and the C_i-plane is always either the whole C_i-plane or a generalized disk. More precisely, with notations $\Omega_i=\Omega\ (A,\ b)\cap C_i$ and $r=|b_i/q_{ii}|$ we have*

$$\begin{aligned}
&\Omega_i=C_i && \text{if } b_i=0,\\
&\Omega_i=\{\zeta\in C_i;\ \operatorname{Re}\zeta\leqslant 0\} && \text{if } b_i>0,\ a_{ii}=b_i/2,\\
&\Omega_i=\{\zeta\in C_i;\ |\zeta+r|\leqslant r\} && \text{if } b_i>0,\ a_{ii}<b_i/2,\\
&\Omega_i=\{\zeta\in C_i;\ |\zeta-r|\geqslant r\} && \text{if } b_i>0,\ a_{ii}>b_i/2,\\
&\Omega_i=\{\zeta\in C_i;\ \operatorname{Re}\zeta\geqslant 0\} && \text{if } b_i<0,\ a_{ii}=b_i/2,\\
&\Omega_i=\{\zeta\in C_i;\ |\zeta+r|\geqslant r\} && \text{if } b_i<0,\ a_{ii}<b_i/2,\\
&\Omega_i=\{\zeta\in C_i;\ |\zeta-r|\leqslant r\} && \text{if } b_i<0,\ a_{ii}>b_i/2.
\end{aligned}$$

Corollary 2. A Runge-Kutta method $M(A,\ b)$ is C_i^--contractive if and only if it is R_i^--contractive, and this happens if and only if

$$b_i\geqslant 0 \text{ and } a_{ii}\geqslant b_i/2.$$

By Theorem 2 we see that $M(A,\ b)$ is R_i^--contractive only when $b_i=0$ or $b_i>0$ and $a_{ii}>b_i/2$, but then $M(A,\ b)$ is C_i^--contractive also. So this corollary holds.

Dahlquist and Jeltsch[4] introduced the generalized disks

$$D(r)=\begin{cases}\{\lambda\in C;\ |\lambda+r|\leqslant r\} & \text{if } r>0,\\ \{\lambda\in C;\ \operatorname{Re}\lambda\leqslant 0\} & \text{if } r=\infty,\\ \{\lambda\in C;\ |\lambda+r|\geqslant -r\} & \text{if } r<0\end{cases}$$

and the following circle contractivity concept.

Definition 4. *A Runge-Kutta method $M(A,\ b)$ is called r-circle contractive if $D(r)$ is the largest generalized disk with $r\neq 0$ such that*

$$|K(\zeta)|\leqslant 1 \text{ for all } \zeta\in D^m(r).$$

Theorem 3 shows that the intersection of the contractivity region with each individual complex plane C_i has the disk shape (the interior or exterior region of a circle). So this theorem gives evidence to the concept of circle contractivity in some extent.

A Runge-Kutta method $M(A, b)$ is proved in [4] to be r-circle contractive if and only if $b_j\geqslant 0$ for $j=1, 2, \cdots, m$ and $\rho=-\frac{1}{r}$ is the largest number such that

$$(w, Qw)\geqslant\rho(w, Bw), \text{ for all } w\in R^m, \tag{2.15}$$

where $B=\text{diag}(b_1, b_2, \cdots, b_m)$.

By Theorem 3 we can provide a simple bound for the radius r of an r-circle contractive method.

Corollary 3. Assume that $M(A, b)$ is r-circle contractive. Then

$$r\leqslant\{\max_{\substack{1\leqslant i\leqslant m\\ b_i\neq 0}}(2a_{ii}-b_i)\}^{-1};$$

especially,

$$r=\{\max_{\substack{1\leqslant i\leqslant m\\ b_i\neq 0}}(2a_{ii}-b_i)\}^{-1}$$

if $Q=(q_{ij})$ is a diagonal matrix.

The first part holds directly from Theorem 3 and the second part follows from (2.15).

4. Examples

In this section we show the feature of the contractivity regions of some implicit Runge-Kutta methods.

Example 1. The 2-stage Runge-Kutta method $M(A, b)$:

$$A=\begin{bmatrix}\frac{1}{8} & \frac{1}{8}\\ \frac{3}{8} & \frac{3}{8}\end{bmatrix},\quad b=\left(\frac{1}{2}, \frac{1}{2}\right)^T.$$

This method is A-stable, since the R-function satisfies

$$|R(z)|=\left|\frac{2+z}{2-z}\right|\leqslant 1 \quad \text{for } \operatorname{Re} z\leqslant 0.$$

Here we have $b_1, b_2>0$ and

$$a_{11}-b_1/2=-\frac{1}{8}<0,\quad a_{22}-b_2/2=\frac{1}{8}>0.$$

By Theorem 3 we find easily that the intersections of the contractivity region with C_1-and C_2-plane are

$$\Omega_1=\{\zeta\in C_1;\ |\zeta+4|\leqslant 4\},\quad \Omega_2=\{\zeta\in C_2;\ |\zeta-4|\geqslant 4\}$$

(see Fig. 2) where Ω_1 is a bounded domain. Hence this method is not AN-contractive. However, it is r-circle contractive with $r=4$, since $\{\max_i(2a_{ii}-b_i)\}^{-1}=4$ and Q is a diagonal matrix.

Example 2. The Runge-Kutta methods of types III_A and III_B based on Lobatto formulas (see [5] or [6]). These methods have positive weights and their nodes

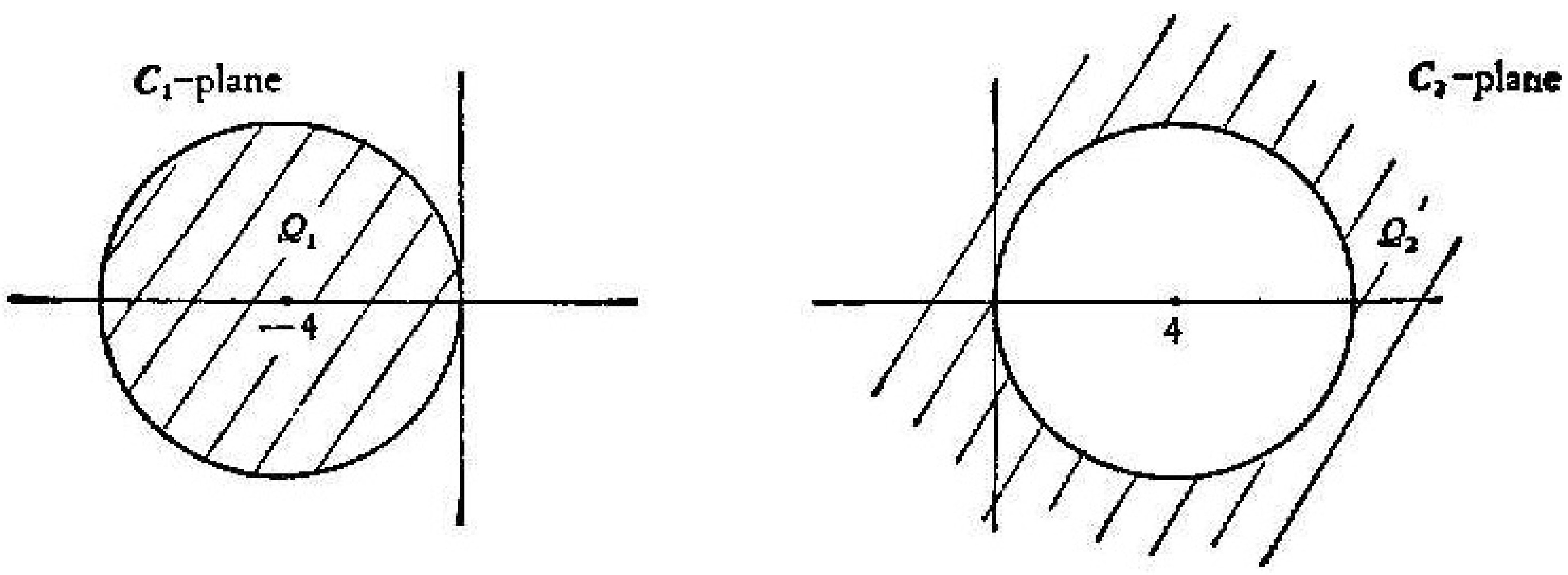

Fig. 2

satisfy $c_i = c_j$ for $i \neq j$. The family of $\mathrm{III_A}$ methods is characterized by

$$a_{1j}=0 \text{ and } a_{mj}=b_j \text{ for } j=1, 2, \cdots, m$$

and the family of $\mathrm{III_B}$ by

$$a_{jm}=0 \text{ and } a_{j1}=b_1 \text{ for } j=1, 2, \cdots, m.$$

It is known that the $\mathrm{III_A}$ and $\mathrm{III_B}$ methods are A-stable. For $\mathrm{III_A}$ methods we note that

$$2a_{11}-b_1=-b_1<0,\ 2a_{mm}-b_m=b_m>0.$$

Hence by Theorem 3

$$\Omega_1=\left\{\zeta\in C_1;\ \left|\zeta+\frac{1}{b_1}\right|\leqslant\frac{1}{b_1}\right\},\ \ \Omega_m=\left\{\zeta\in C_m;\ \left|\zeta-\frac{1}{b_m}\right|\geqslant\frac{1}{b_m}\right\}.$$

We see that $\mathrm{III_A}$ methods are not R_1^--contractive, and so can not be AN-contractive. However, they may be r-circle contractive with $r\leqslant 1/b_1$. For $\mathrm{III_B}$ methods, since

$$2a_{11}-b_1=b_1>0,\ \ 2a_{mm}-b_m=-b_m<0,$$

we find $\quad \Omega_1=\left\{\zeta\in C_1;\ \left|\zeta-\frac{1}{b_1}\right|\geqslant\frac{1}{b_1}\right\},\ \ \Omega_m=\left\{\zeta\in C_m;\ \left|\zeta+\frac{1}{b_m}\right|\leqslant\frac{1}{b_m}\right\}.$

Therefore, $\mathrm{III_B}$ methods are not R_m^--contractive, but may be r-circle contractive with $r\leqslant 1/b_m$.

Example 3. The family of m-stage diagonal implicit methods given by

$$A=\begin{bmatrix}\lambda & & & & \\ b_1 & \lambda & & & \\ b_1 & b_2 & \lambda & & \\ \vdots & \vdots & \ddots\ \vdots & & \\ b_1 & b_2 & \cdots & b_{m-1} & \lambda\end{bmatrix},\ \ b=(b_1,\ b_2,\ \cdots,\ b_m)^T,$$

where $\lambda,\ b_1,\ b_2,\ \cdots,\ b_{m-1}\neq 0$.

The K-function of these methods has the form

$$K(\zeta)=\prod_{j=1}^{m}\{(1+(b_j-\lambda)\zeta_j)/(1-\lambda\zeta_j)\}.$$

Noting that $|K(\zeta)|\leqslant 1$ for $\zeta\in(C^-)^m$ if and only if

$$|(1+(b_j-\lambda)\zeta_j)/(1-\lambda\zeta_j)|\leqslant 1 \quad \text{for } \zeta_j\in C_j^-,\ j=1, 2, \cdots, m,$$

we see that these methods are AN-contractive if and only if they are R_j^--contractive for $j=1, 2, \cdots, m$, i. e.

$$2\lambda\geqslant b_j\geqslant 0, \quad \text{for } j=1, 2, \cdots, m.$$

One may ask whether R_j^--contractivity ($j=1, 2, \cdots, m$) or $(R^-)^m$-contractivity implies AN-contractivity. This is not ture. To show this, we have the following example.

Example 4. A 2-stage Runge-Kutta method given by

$$A=\begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{bmatrix}, \quad b=(0, 1)^T.$$

We first show that it is $(R^-)^2$-contractive and A-stable. The K-function is

$$K(\zeta)=\frac{1-z_1-\frac{1}{2}z_1z_2}{(z_1-1)(z_2-1)}, \quad \zeta=(z_1, z_2).$$

It is easy to check that when z_1 and z_2 take nonnegative real values, $|K(\zeta)|\leqslant 1$. Hence this method is $(R^-)^2$-contractive. We now examine the R-function

$$R(\zeta)=K(zI)=\frac{1-z-\frac{1}{2}z^2}{(z-1)^2}.$$

Noting that $R(z)$ is analytic in C^- and that on the imaginary axis ($z=\sqrt{-1}\,y$) we have

$$|R(\sqrt{-1}y)|=\frac{\left(1+\frac{1}{2}y^2\right)^2+y^2}{(1-y^2)^2+4y^2}\leqslant 1,$$

by the maximum modulus theorem of complex function we have

$$|R(z)|\leqslant 1, \quad \text{for } z\in C^-.$$

Hence this method is A-stable.

Since the matrix Q of this method is

$$Q=\begin{bmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix}$$

which has the negative eigenvalue $(1-\sqrt{2})/2$, the condition $Q\geqslant 0$ does not hold and by Theorem 2 it is not AN-contractive. This method is not r-circle contractive either since for all $\rho\in R$ the matrix

$$Q-\rho B=\begin{bmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1-\rho \end{bmatrix}$$

has the negative eigenvalue $\{(1-\rho)-\sqrt{(1-\rho)^2+1}\}/2$, i. e. condition (2.15) is not valid.

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