ASYMPTOTIC EXPANSION FOR THE DERIVATIVE OF FINITE ELEMENTS*

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Abstract

It is proved in this paper that there exists an expansion for the derivative of the linear finite element approximation to a model Dirichlet problem in a polygonal domain with a piecewise uniform triangulation.

Superconvergence of the derivative of finite elements has been studied in many works, see [1—24]. To the authors' knowledge, no expansion theorem has been proved for the finite element derivative. The aim of the paper is to show that such an expansion theorem can be derived by the expansion method in [11—14] for the finite element solution and some estimates in [6, 16, 17, 22—24] for the Green function.

Consider the model problem: Find $u\in H_0^1(\Omega)$ such that

$$-\Delta u=f \text{ in } \Omega, \tag{1}$$

where Ω is a convex polygonal domain. We approximate (1) by the linear elements. First we divide Ω into several triangular subdomains $\Omega_i(i=1, 2, \cdots, M)$ meeting at a point A (see Fig. 1) and then divide each Ω_i into a uniform triangulation:

$$\Omega_i=\bigcup_j \Omega_{ij}$$

with triangles Ω_{ij} ($j=1, 2, \cdots, 4^N$). So we have a piecewise uniform partition

$$\Omega=\bigcup_i \Omega_i=\bigcup_i\bigcup_j \Omega_{ij}. \tag{2}$$

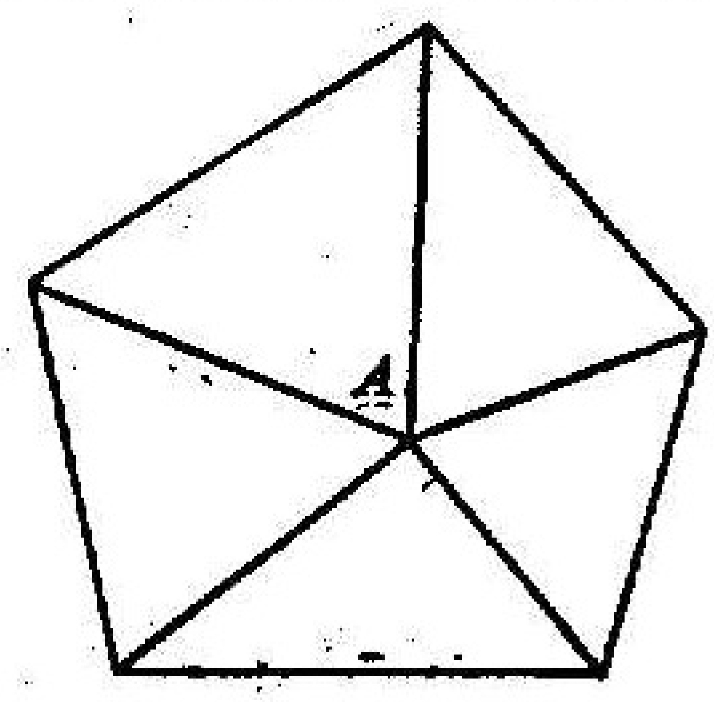

Fig. 1 (Ω)

Let $h=1/2^N$, and let u^h and u^I be the linear finite element approximation and the linear interpolation of u respect to the partition (2) respectively.

Consider an interior domain

$$\Omega_0=\{z\in\Omega: \text{dist}\,(z, \partial\Omega_i)\geqslant\delta>0\},$$

where δ is an arbitrary constant and $\partial\Omega_i$ the edges of $\Omega_i(1\leqslant i\leqslant M)$.

Theorem. *If $u\in H^{4,q}(\Omega)$, then there exists a function $w\in H^{2,q}(\Omega_0)$ independent of h such that*

$$\nabla(u^h-u^I)(z_0)=h^2\nabla w(z_0)+O(h^{3-2/q})\|u\|_{4,q} \tag{3}$$

for $z_0\in\Omega_0$, where $q\in(2, q_0)$ with $q_0\in(2, \infty)$ dependent on Ω (see (5)).

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Proof. Let $G_{z_0}(z)$ be the Green function with singularity at z_0 and $G^h_{z_0}(z)$ the linear finite element approximation to $G_{z_0}(z)$, Then, from [11, 13, 14], we have

$$(u^h-u^I)(z_0)=\int_\Omega \nabla G^h_{z_0}\nabla(u-u^I)dz$$
$$=h^2P_h(z_0)+h^2Q_h(z_0)+O(h^3)\|u\|_{4,q}\|G^h_{z_0}\|_{1,p}$$

with the piecewise linear functions P_h and Q_h defined by

$$P_h(z_0)=\sum_{i=1}^M\int_{\Omega_i} G^h_{z_0}D^4_iu\,dz,$$
$$Q_h(z_0)=\sum_{i=1}^M\Big(\int_{\partial\Omega_i} G^h_{z_0}D^3_iu\,dz+G^h_{z_0}(A)D^2_iu(A)\Big), \tag{4}$$

where D^j_i denotes some linear combinations of some order ($\leqslant j$) derivatives of u, $\frac{1}{p}+\frac{1}{q}=1$.

Let z' and z'' be any two vertices at a typical triangle element $\Omega_{ij}\subset\Omega_0$. Then

$$(u^h-u^I)(z')-(u^h-u^I)(z'')=\int_\Omega \nabla(G^h_{z'}-G^h_{z''})\nabla(u-u^I)dz$$
$$=h^2(P_h(z')-P_h(z''))+h^2(Q_h(z')-Q_h(z''))$$
$$+O(h^3)\|u\|_{4,q}\|G^h_{z'}-G^h_{z''}\|_{1,p}.$$

Setting

$$Q(z_0)=\sum_{i=1}^M\Big(\int_{\partial\Omega_i} G_{z_0}D^3_iu\,dz+G_{z_0}(A)D^2_iu(A)\Big)\in H^{2,q}(\Omega_0)$$

and using the inequalities due to Frehse, Rannacher and Scott[5,6,16] and Schatz and Wahlbin[17]

$$\|G^h_{z'}-G^h_{z''}\|_{1,p}\leqslant Kh^{1-2/q},$$
$$|G^h_{z'}(z)-G_{z'}(z)|\leqslant Kh^{2-2/q}\quad \text{for } z\in\partial\Omega_i$$

we obtain

$$(u^h-u^I)(z')-(u^h-u^I)(z'')=h^2(P_h(z')-P_h(z''))$$
$$+h^2(Q(z')-Q(z''))+O(h^{4-2/q})\|u\|_{4,q},$$

hence, for $z_0\in\Omega_{ij}$,

$$\nabla(u^h-u^I)(z_0)=h^2\nabla P_h(z_0)+h^2\nabla Q^I(z_0)+O(h^{3-2/q})\|u\|_{4,q}.$$

Let

$$F(z)=D^4_iu(z)\quad \text{if } z\in\Omega_i$$

and find $P\in H^1_0(\Omega)\cap H^{2,q}(\Omega)$ such that

$$-\Delta P=F\quad \text{in } \Omega.$$

Then

$$P(z_0)=\sum_{i=1}^M\int_{\Omega_i} G_{z_0}D^4_iu\,dz\in H^{2,q}(\Omega). \tag{5}$$

Note from (4) that P_h is just the linear finite element approximation to P, we have by [16]

$$|\nabla(P_h-P)|\leqslant Kh^{1-2/q}\|P\|_{2,q,\Omega},$$
$$|\nabla(Q^I-Q)|\leqslant Kh^{1-2/q}\|Q\|_{2,q,\Omega}.$$

and (3) follows with $w=P+Q$.

Let $\overline{\nabla}$ be the nodal point averaged gradient defined by Křížek and Neittaanmäki[18]. Then it is easy to prove that

$$\frac{1}{3}\overline{\nabla}(4u^{h/2}-u^{h})(z_0)=\nabla u(z_0)+O(h^{3-2/q})$$

for z_0 being nodal points of the original h–mesh. Therefore, we have

Corollary. If $u\in H^{4,q}(\Omega)$, then

$$\frac{1}{3}\overline{\nabla}(4u^{h/2}-u^{h})(z_0)-\nabla u(z_0)=O(h^{3-2/q})\|u\|_{4,q},$$

$$\overline{\nabla}u^{h/2}(z_0)-\nabla u(z_0)=\frac{1}{3}\overline{\nabla}(u^{h/2}-u^{h})(z_0)+O(h^{3-2/q})\|u\|_{4,q}$$

for z_0 being nodal points of the original h–mesh.

We remark that J. Frehse has proposed a useful estimate

$$|\nabla(u^h-u)|\leqslant k_1h+k_2h^2$$

with a good constant k_1 and a bad constant k_2 for the usual regularity triangulation.

We also remark that the splitting extrapolation method presented in [15] for the 5–point difference scheme can be generalized to the case for the bilinear elements in a domain with uniform rectangular partition. Let $u(h_1, h_2)$ be the bilinear finite element approximation with mesh sizes h_1 and h_2 along the directions x_1 and x_2. Then we have

$$\frac{1}{3}\overline{\nabla}\left(4u\left(\frac{h_1}{2},h_2\right)+4u\left(h_1,\frac{h_2}{2}\right)-5u(h_1,h_2)\right)(z_0)-\nabla u(z_0)=O(h^{3-\varepsilon}).$$

The proof is based on an inequality due to Zhu Qi–ding and will be published elsewhere.

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