BOUNDS ON CONDITION NUMBER OF A MATRIX*

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Abstract

For each vector norm $\|x\|$, a matrix A has its operator norm $\|A\|=\max_{x\neq0}\frac{\|Ax\|}{\|x\|}$ and a condition number $P(A)=\|A\|\|A^{-1}\|$. Let U be the set of the whole of norms defined on C^n. It is shown that for a nonsingular matrix $A\in C^{n\times n}$, there is no finite upper bound of $P(A)$ while $\|\cdot\|$ varies on U if $A\neq\alpha I$; on the other hand, it is shown that $\inf_{\|\cdot\|\in U}\|A\|\|A^{-1}\|=\rho(A)\rho(A^{-1})$ and in which case this infimum can or cannot be attained, where $\rho(A)$ denotes the spectral radius of A.

Let $\|x\|$ be a norm defined on the linear space C^n. Then a matrix $A\in C^{n\times n}$, treated as a linear operator on C^n, has a norm $\|A\|=\max_{x\neq0}\frac{\|Ax\|}{\|x\|}$ correspondingly. We denote by $P(A)=\|A\|\|A^{-1}\|$ the condition number of a nonsingular matrix A. This is a basic concept in numerical algebra and is important in some other fields of numerical analysis. Under certain circumstances, one takes the product of spectral radius $\rho(A)\rho(A^{-1})$, namely the ratio $|\lambda_1|/|\lambda_n|$ where λ_1 and λ_n are the largest and smallest eigenvalues of A by norm, to characterize the condition of A. $P(A)$ depends on the selected norm $\|\cdot\|$ while $\rho(A)\rho(A^{-1})$ is determined only by the matrix itself. Now we reveal their relationship.

Denote by U the set of the whole of norms defined on C^n. We begin with the upper bound of $P(A)$ while $\|\cdot\|$ varies on U. Obviously, when $A=\alpha I$, where I is the identity matrix and α is a nonzero scalar, $P(A)=1$ for any norm. Otherwise, we have

Theorem 1. *Let $A\in C^{n\times n}$, be nonsingular and $A\neq\alpha I$. Then there is no finite upper bound of $P(A)$ while $\|\cdot\|$ varies on U.*

Proof. Let $\|x\|_\infty=\max_i|x_i|$, it is known that the corresponding norm of the matrix is

$$\|A\|_\infty=\max_i\sum_j|a_{ij}|. \tag{1}$$

Now we divide the matrices involved in the condition of the theorem into two cases: (1) at least one nonzero element on the off-diagonal, (2) a diagonal form with $a_{ii}\neq a_{jj}$ for some $i\neq j$ due to $A\neq\alpha I$.

In case (1), supposing $a_{ij}\neq0$, we take

$$Q_s=\begin{bmatrix}1&&&&\\&\ddots&&&\\&&s&&\\&&&\ddots&\\&&&&1\end{bmatrix}(\text{row } i),$$

a matrix different from the identity matrix only by the element $[Q_s]_{ii}=s\neq0$. With

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notice of the nonsingularity of Q_s, we can define a norm $\|\cdot\|_{q(s)}$ with a parameter s such as

$$\|x\|_{q(s)}=\|Q_s x\|_\infty$$

and then correspondingly,

$$\|A\|_{q(s)}=\max_{x\neq 0}\frac{\|Ax\|_{q(s)}}{\|x\|_{q(s)}}=\max_{x\neq 0}\frac{\|Q_sAx\|_\infty}{\|Q_sx\|_\infty}=\max_{y\neq 0}\frac{\|Q_sAQ_s^{-1}y\|_\infty}{\|y\|_\infty}=\|Q_sAQ_s^{-1}\|_\infty.$$

Through calculation and from (1) we can obtain

$$\|A\|_{q(s)}\geqslant|sa_{ij}|. \tag{2}$$

From (2), it can be seen that no finite upper bound of $\|A\|_{q(s)}$ exists while $|s|$ tends to infinity.

In case (2), supposing $a_{ii}\neq a_{jj}$, we take

$$T_s=\begin{bmatrix}1 & & & & \\ & \ddots & & & \\ & & 1 & 1 & \\ & & & \ddots & \\ & & 1 & 1+\varepsilon & \\ & & & & \ddots \\ & & & & & 1\end{bmatrix}\begin{matrix}\\ \\ (\text{row } i)\\ \\ (\text{row } j),\\ \\ \\ \end{matrix}$$

where ε is a positive real number. We define

$$\|x\|_{t(\varepsilon)}=\|T_\varepsilon x\|_\infty$$

with a parameter ε. Through calculation we have

$$\|A\|_{t(\varepsilon)}=\|T_\varepsilon AT_\varepsilon^{-1}\|_\infty\geqslant\frac{1}{\varepsilon}|a_{ii}-a_{jj}|. \tag{3}$$

When $\varepsilon\to 0$, there is no finite upper bound of $\|A\|_{t(\varepsilon)}$. With notice of $\|A^{-1}\|\geqslant\rho(A^{-1})>0$ we conclude that $P(A)$ has no finite upper bound in both cases.

For any norm it is known that $\|A\|\,\|A^{-1}\|\geqslant\rho(A)\rho(A^{-1})$. Now we go further to prove the following theorem.

Theorem 2. *Let $A\in C^{n\times n}$, and be nonsingular. Then*

$$\inf_{\|\cdot\|\in U}\|A\|\,\|A^{-1}\|=\rho(A)\rho(A^{-1}). \tag{4}$$

Proof. Let Q be the matrix transforming A into Jordan canonical form, namely,

$$Q^{-1}AQ=J=\begin{bmatrix}J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_m\end{bmatrix}, \tag{5}$$

where

$$J_i=\begin{bmatrix}\lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1\\ & & & \lambda_i\end{bmatrix}, \tag{5a}$$

and is of order n_i, $\sum_{i=1}^{m} n_i=n$,

At the same time we have,

$$Q^{-1}A^{-1}Q=J^{-1}=\begin{bmatrix} J_1^{-1} & & & \\ & J_2^{-1} & & \\ & & \ddots & \\ & & & J_m^{-1} \end{bmatrix}, \tag{6}$$

where J_i^{-1}, due to the form (5a) of J_i, is also an upper triangular form with diagonal elements λ_i^{-1},

$$J_i^{-1}=\begin{bmatrix} \lambda_i^{-1} & \times & \times & \cdots & \times \\ & \lambda_i^{-1} & \times & \cdots & \times \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \lambda_i^{-1} \end{bmatrix}, \tag{6a}$$

Let D_s be a diagonal matrix as follows:

$$D_s=\begin{bmatrix} 1 & & & & \\ & s & & & \\ & & s^2 & & \\ & & & \ddots & \\ & & & & s^{n-1} \end{bmatrix}, \tag{7}$$

where s is a positive real number. Then we define

$$\|x\|_{d(s)}=\|D_s^{-1}Q^{-1}x\|_\infty.$$

Correspondingly we have

$$\|A\|_{d(s)}=\|D_s^{-1}Q^{-1}AQD_s\|=\|D_s^{-1}JD_s\|_\infty \tag{8}$$

and

$$\|A^{-1}\|_{d(s)}=\|D_s^{-1}J^{-1}D_s\|_\infty. \tag{9}$$

From the Jordan canonical form (5), (5a) and the diagonal form (7) of D_s, we obtain

$$D_s^{-1}JD_s=\begin{bmatrix} J_1(s) & & & \\ & J_2(s) & & \\ & & \ddots & \\ & & & J_m(s) \end{bmatrix}, \tag{8a}$$

where

$$J_i(s)=\begin{bmatrix} \lambda_i & s & & \\ & \lambda_i & \ddots & \\ & & \ddots & s \\ & & & \lambda_i \end{bmatrix}. \tag{8b}$$

On the other hand, from (6) and (6a) we obtain

$$D_s^{-1}J^{-1}D_s=\begin{bmatrix} J_1^{-1}(s) & & & \\ & J_2^{-1}(s) & & \\ & & \ddots & \\ & & & J_m^{-1}(s) \end{bmatrix}, \tag{9a}$$

where

$$J_i^{-1}(\varepsilon)=\begin{bmatrix}\lambda_i^{-1} & O(\varepsilon) & O(\varepsilon^2) & \cdots & O(\varepsilon^{n_i-1}) \\ & \lambda_i^{-1} & O(\varepsilon) & \cdots & O(\varepsilon^{n_i-2}) \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \lambda_i^{-1}\end{bmatrix}. \tag{9b}$$

Combining (8)—(8b) and (9)—(9b), we have

$$\lim_{\varepsilon\to 0}\|A\|_{d(\varepsilon)}\|A^{-1}\|_{d(\varepsilon)}=\lim_{\varepsilon\to 0}\frac{\max_i|\lambda_i|+\varepsilon}{\min_i\|\lambda_i\|+O(\varepsilon)}=\rho(A)\rho(A^{-1}).$$

With notice of $\|A\|\|A^{-1}\|\geqslant\rho(A)\rho(A^{-1})$ we obtain (4).

For the nonsingular Hermitian matrix, it is known that $\|A\|\|A^{-1}\|=\rho(A)\rho(A^{-1})$ with Euclidean norm. It means in this case that the infimum of $P(A)$ can be attained. Now we show whether or not the infimum of $P(A)$ can be attained in general case.

Theorem 3a. *Let $A\in C^{n\times n}$, be nonsingular and has no Jordan block corresponding to the largest and smallest eigenvalues by norm, namely, there exists a matrix Q such that*

$$Q^{-1}AQ=\begin{bmatrix}\lambda_1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & \lambda_1 & & & & & & \\ & & & J_2 & & & & & \\ & & & & \ddots & & & & \\ & & & & & J_{m-1} & & & \\ & & & & & & \lambda_m & & \\ & & & & & & & \ddots & \\ & & & & & & & & \lambda_m\end{bmatrix}, \tag{10}$$

where $|\lambda_1|>|\lambda_2|\geqslant\cdots\geqslant|\lambda_{m-1}|>|\lambda_m|>0$. Then we have

$$\inf_{\|\cdot\|\in U}\|A\|\|A^{-1}\|=\rho(A)\rho(A^{-1}). \tag{11}$$

Proof. In the proof of Theorem 2, it can be seen that in the case of Jordan form (10) there exists a real number $\varepsilon_0>0$ for the norm $\|x\|_{d(\varepsilon)}=\|D_\varepsilon^{-1}Q^{-1}x\|_\infty$, as $\varepsilon\leqslant\varepsilon_0$

$$\|A\|_{d(\varepsilon)}=|\lambda_1|=\rho(A)$$

and

$$\|A^{-1}\|_{d(\varepsilon)}=1/|\lambda_m|=\rho(A^{-1})$$

hold. So we complete the proof.

Except the case of Theorem 3a, there is no any norm $\|\cdot\|$ which can make $P(A)$ attain its infimum.

Theorem 3b. *Let $A\in C^{n\times n}$, be nonsingular and has Jordan block corresponding to its largest or smallest eigenvalue by norm. Then there is no any norm $\|\cdot\|\in U$ which can make $\|A\|\|A^{-1}\|=\rho(A)\rho(A^{-1})$.*

Proof. It suffices to prove that no norm can make $\|A\|=|\lambda_1|$ if there is a Jordan block corresponding to λ_1. In this case, there exist nonzero vectors q_1, q_2 such that

$$Aq_1=\lambda_1q_1,$$
$$Aq_2=q_1+\lambda_1q_2.$$

Making use of these two equalities iteratively, we get

$$A^k q_2 = k\lambda_1^{k-1} q_1 + \lambda^k q_2.$$

Assume that a norm $\|\cdot\| \in U$ makes $\|A\| = |\lambda_1|$. It follows that for any positive integer k,

$$k|\lambda_1|^{k-1}\|q_1\| - |\lambda_1|^k\|q_2\| \leqslant \|A^k q_2\| \leqslant \|A^k\|\,\|q_2\| \leqslant |\lambda_1|^k\|q_2\|$$

and then

$$k \leqslant 2|\lambda_1|\,\|q_2\|/\|q_1\|.$$

The inequality means that the constant $2|\lambda_1|\|q_2\|/\|q_1\|$ can be bigger than any positive integer. This falsity shows the impossibility of $\|A\| = |\lambda_1|$.

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