THE CONVERGENCE OF GALERKIN-FOURIER METHOD FOR A SYSTEM OF EQUATIONS OF SCHRÖDINGER-BOUSSINESQ FIELD*

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I. Introduction

In [1, 2], Guo Bo-ling has investigated the global solutions for some systems of nonlinear Schrödinger equations and the problems of numerical computations. In [2], a continuous Galerkin definite element method has been presented, and the estimation of L_2 optimum error and the proof of convergence have been given. In [3], Makhankov has proposed the problem of the solutions for a system of equations of Schrödinger-Boussinesq field and has found the approximate solutions for the system

$$i\varepsilon_t+\varepsilon_{xx}-n\varepsilon=0,$$
$$\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}-\frac{\delta}{3}\frac{\partial^4}{\partial x^4}\right)n-\delta(n^2)_{xx}=|\varepsilon|^2_{xx}.$$

In [4, 5], a class of important equations of Boussinesq field

$$n_{tt}-n_{xx}-b(n^2)_{xx}+n_{xxxx}=0,$$

and

$$n_{tt}=n_{xx}+a(n^2)_{xx}+bn_{xxxx}\quad (a,\ b \text{ being constants})$$

have been proposed. In [6] the global solutions for some systems of equations of the complex Schrödinger field interacting with the real Boussinesq field are investigated, which satisfy the equations

$$i\varepsilon_t+\varepsilon_{xx}-n\varepsilon=0,$$
$$n_{tt}-n_{xx}-f(n)_{xx}+\alpha n_{xxxx}=|\varepsilon|^2_{xx}.$$

If $\alpha>0$ and certain conditions for the function $f(n)$ are satisfied, the existence and uniqueness of the global solution have been proved.

In this paper, by introducing the equation of the potential function $\varphi(x,t)$, we consider some systems of equations of complex Schrödinger field, interacting with the real Boussinesq field, as follows:

$$i\varepsilon_t+\varepsilon_{xx}-n\varepsilon=0, \tag{1.1}$$
$$n_t-\varphi_{xx}=0, \tag{1.2}$$
$$\varphi_t-n-f(n)+\alpha n_{xx}=|\varepsilon|^2 \tag{1.3}$$

with the periodic boundary conditions

$$\varepsilon(x,t)=\varepsilon(x+D,t),\quad n(x,t)=n(x+D,t),\quad \varphi(x,t)=\varphi(x+D,t) \tag{1.4}$$
$$-\infty<x<\infty,\ t\geqslant 0,$$

and initial conditions

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$$\varepsilon|_{t=0}=\varepsilon_0(x),\ n|_{t=0}=n_0(x),\ \varphi|_{t=0}=\varphi_0(x),\quad -\infty<x<\infty, \tag{1.5}$$

where D is a positive constant.

By using the Galerkin–Fourier method, we construct the approximate solutions of the problem (1.1)—(1.5) and obtain the estimation of L_2 optimum error. Finally, we prove that the approximate solutions converge to the exact solutions of the problem (1.1)—(1.5).

II. Galerkin–Fourier Method and the Estimation of the Approximate Solution

First we introduce some spaces and notations. Let Z be a complex function and $\bar{Z}$ a complex conjugate function of Z. Let $C^l_*(\Omega)=C^l([0, D])$ denote the space of complex functions, l times continuous differentiable over the interval $[0, D]$.

Let $L_p(\Omega)=L_p([0, D])$ denote the Lebesque space of complex measurable functions $u(x)$ with the p-th power of absolute value $|u|$ integrable over the interval $[0, D]$.

If we define the inner product

$$(u, v)=\int_0^D u(x)\bar{v}(x)dx,\quad \|u\|^2=(u, u),$$

then $L_2([0, D])$ is a Hilbert space.

Let $L_\infty(\Omega)=L_\infty([0, D])$ denote the Lebesque space of measurable functions $u(x)$ over the interval $[0, D]$, which are essentially bounded, with the norm

$$\|u\|_{L_\infty}=\text{ess.}\sup_{x\in\Omega}|u(x)|.$$

Let $H^l(\Omega)=H^l([0, D])$ denote the space of complex functions with generalized derivatives

$$D^k u(|k|\leqslant l)\in L_2([0, D]),$$

$$V^l=\{u\in H^l(\Omega)\,|\,u^j(0)=u^j(D),\quad 0\leqslant j\leqslant l-1\},\quad u^j=\frac{d^j u}{dx^j},$$

$$\|u\|_V^2=\|u\|^2+\left\|\frac{du}{dx}\right\|^2,\quad V=H^1,\ H=L_2.$$

Let F_k denote the projection from H to $H_k=\text{span}(v_{-k},\cdots, v_k)$,

$$F_k g=\sum_{j=-k}^{k}(g, v_j)v_j,$$

where $$v_j=\frac{1}{\sqrt{D}}\exp(iw_jx),\quad w_j=\frac{2\pi j}{\sqrt{D}},\quad v_j''(x)=-w_j^2v_j(x).$$

Set $R_kg=g-F_kg$, $g\in H$. When $k\to\infty$, $R_kg\to 0$. From the Basswell inequality, we have

$$\|F_kg\|\leqslant\|g\|.$$

Here we construct the approximate solutions of the problem (1.1)—(1.5) by the Galerkin–Fourier method:

$$\begin{aligned}\varepsilon_k(\cdot, t)&=\varepsilon_k(t)=\sum_{j=-k}^{k}\alpha_j(t)v_j(x),\\ n_k(\cdot, t)&=n_k(t)=\sum_{j=-k}^{k}\beta_j(t)v_j(x),\\ \varphi_k(\cdot, t)&=\varphi_k(t)=\sum_{j=-k}^{k}\gamma_j(t)v_j(x).\end{aligned} \tag{2.1}$$

where $\varepsilon_k(t)$ denotes a complex function, $n_k(t)$ and $\varphi_k(t)$ denote real functions. They should satisfy the equations

$$(i\varepsilon_{kt}+\varepsilon_{kxx}-n_k\varepsilon_k,\ v_j(x))=0, \tag{2.2a}$$

$$(n_{kt}-\varphi_{kxx},\ v_j(x))=0, \tag{2.2b}$$

$$(\varphi_{kt}-n_k-f(n_k)+\alpha n_{kxx}-|\varepsilon_k|^2,\ v_j(x))=0, \quad j=-k, \cdots, k, \tag{2.2c}$$

where

$$\begin{aligned} \varepsilon_k|_{t=0}&=\varepsilon_{0k}(x)=F_k\varepsilon_0(x),\\ n_k|_{t=0}&=n_{0k}(x)=F_kn_0(x),\\ \varphi_k|_{t=0}&=\varphi_{0k}(x)=F_k\varphi_0(x). \end{aligned} \tag{2.3}$$

The problem (2.2a), (2.2b), (2.2c) can be consider as an initial value problem of the system of nonlinear ordinary differential equations of first order with unknown functions $\alpha_j(t)$, $\beta_j(t)$, $\gamma_j(t)$.

For the solutions of the problem (2.2), (2.3) there exist the following estimates:

Lemma 1. *Suppose $\varepsilon_0(x)\in L_2$, then we have*

$$\|\varepsilon_k(t)\|^2=\|\varepsilon_{0k}(x)\|^2\leqslant\|\varepsilon_0(x)\|^2. \tag{2.4}$$

Proof. Multiplying (2.2a) by $\bar{\alpha}_j(t)$ and summing for j, we obtain

$$(i\varepsilon_{kt}+\varepsilon_{kxx}-n_k\varepsilon_k,\ \varepsilon_k)=0.$$

Taking the imaginary part of the last equation, we obtain

$$\frac{d}{dt}\|\varepsilon_k\|^2=0$$

and derive

$$\|\varepsilon_k(t)\|^2=\|\varepsilon_k(0)\|^2=\|\varepsilon_{0k}\|^2\leqslant\|\varepsilon_0\|^2.$$

Lemma 2 (The Sobolev inequality). *If $\delta>0$ and $l\geqslant0$ are given, for the function $u(x)\in H^k$, there exists a constant C depending on δ and l such that*

$$\left\|\frac{\partial^l u}{\partial x^l}\right\|_{L_\infty}\leqslant\delta\left\|\frac{\partial^k u}{\partial x^k}\right\|+C\|u\|,\quad l<k \tag{2.5a}$$

and

$$\left\|\frac{\partial^l u}{\partial x^l}\right\|\leqslant\delta\left\|\frac{\partial^k u}{\partial x^k}\right\|+C\|u\|,\quad l<k, \tag{2.5b}$$

Lemma 3. *Suppose the following conditions*

(i) $\varepsilon_0(x),\ n_0(x),\ \varphi_0(x)\in H^1$,

(ii) $\alpha>0,\ \int_0^n f(z)dz\geqslant0$

are satisfied, then we have

$$\|\varepsilon_{kx}\|^2+\|n_k\|^2+\|\varphi_{kx}\|^2+\|n_{kx}\|^2\leqslant E_2, \tag{2.6}$$

where E_2 is a definite constant independent of k.

Proof. Multiplying equation (2.2a) by $\bar{\alpha}_{jt}(t)$ and summing up for j, we can obtain

$$(i\varepsilon_{kt}+\varepsilon_{kxx}-n_k\varepsilon_k,\ \varepsilon_{kt})=0.$$

Taking the real part in the above equality we get

$$\frac{d}{dt}\|\varepsilon_{kx}\|^2+\int_0^D n_k|\varepsilon_k|_t^2\,dx=0,$$

$$\int_0^D n_k|\varepsilon_k|_t^2\,dx=\frac{d}{dt}\int_0^D n_k|\varepsilon_k|^2\,dx-\int_0^D n_{kt}|\varepsilon_k|^2\,dx,$$

$$\frac{d}{dt}\|\varepsilon_{kx}\|^2+\frac{d}{dt}\int_0^D n_k|\varepsilon_k|^2\,dx-\int_0^D n_{kt}|\varepsilon_k|^2\,dx=0. \tag{2.7}$$

From (2.2c) we can obtain

$$(\varphi_{kt}-n_k-f(n_k)+\alpha n_{kxx}-|\varepsilon_k|^2,\ n_{kt})=0.$$

It follows that

$$-(|\varepsilon_k|^2,\ n_{kt})=-(\varphi_{kt},\ n_{kt})+(n_k,\ n_{kt})+(f(n_k),\ n_{kt})-\alpha(n_{kxx},\ n_{kt}). \tag{2.8}$$

From (2.2b), we have

$$(n_{kt}-\varphi_{kxx},\ \varphi_{kt})=0.$$

Hence

$$-(n_{kt},\ \varphi_{kt})=-(\varphi_{kxx},\ \varphi_{kt})=\frac{1}{2}\frac{d}{dt}\|\varphi_{kx}\|^2 \tag{2.9}$$

and

$$(n_k,\ n_{kt})=\frac{1}{2}\frac{d}{dt}\|n_k\|^2,$$

$$(f(n_k),\ n_{kt})=\frac{d}{dt}\int_0^D\int_0^{n_k} f(z)\,dz\,dx,$$

$$-\alpha(n_{kxx},\ n_{kt})=\frac{\alpha}{2}\frac{d}{dt}\|n_{kx}\|^2.$$

Putting them into (2.8), we obtain

$$-(|\varepsilon_k|^2,\ n_{kt})=\frac{1}{2}\frac{d}{dt}\|\varphi_{kx}\|^2+\frac{1}{2}\frac{d}{dt}\|n_k\|^2+\frac{d}{dt}\int_0^D\int_0^{n_k} f(z)\,dz\,dx+\frac{\alpha}{2}\frac{d}{dt}\|n_{kx}\|^2.$$

Then substituting the resulting relations into (2.7), we have

$$\frac{d}{dt}\left[\|\varepsilon_{kx}\|^2+\int_0^D n_k|\varepsilon_k|^2dx+\frac{1}{2}\|\varphi_{kx}\|^2+\frac{1}{2}\|n_k\|^2+\int_0^D\int_0^{n_k} f(z)\,dz\,dx+\frac{\alpha}{2}\|n_{kx}\|^2\right]=0,$$

i. e.

$$E_k(t)\equiv\|\varepsilon_{kx}\|^2+\frac{1}{2}\|\varphi_{kx}\|^2+\frac{1}{2}\|n_k\|^2+\frac{\alpha}{2}\|n_{kx}\|^2$$
$$+\int_0^D n_k|\varepsilon_k|^2\,dx+\int_0^D\int_0^{n_k} f(z)\,dz\,dx=E_k(0).$$

Using the inequality $ab\leqslant\varepsilon a^2+\frac{1}{4\varepsilon}b^2$, Lemma 1 and Lemma 2 we have

$$\int_0^D n_k|\varepsilon_k|^2\,dx=(n_k,\ |\varepsilon_k|^2)\leqslant\frac{1}{4}\|n_k\|^2+\|\varepsilon_k\|^4\leqslant\frac{1}{4}\|n_k\|^2+\|\varepsilon_k\|^2_{L_\infty}\|\varepsilon_k\|^2$$

$$\leqslant\frac{1}{4}\|n_k\|^2+2(\delta^2\|\varepsilon_{kx}\|^2+C^2\|\varepsilon_{0k}\|^2)\|\varepsilon_{0k}\|^2.$$

By the hypothesis $\int_0^{n_k} f(z)dz\geqslant 0$ of this lemma, we have

$$\|\varepsilon_{kx}(t)\|^2+\frac{1}{2}\|n_k(t)\|^2+\frac{1}{2}\|\varphi_{kx}(t)\|^2+\frac{\alpha}{2}\|n_{kx}(t)\|^2-\frac{1}{4}\|n_k(t)\|^2$$
$$-2\delta^2\|\varepsilon_{kx}(t)\|^2\|\varepsilon_{0k}\|^2-2C^2\|\varepsilon_{0k}\|^4\leqslant E_k(0)\leqslant E_0,$$

where E_0 is a definite constant. Hence

$$(1-2\delta^2\|\varepsilon_{0k}\|^2)\|\varepsilon_{kx}(t)\|^2+\frac{1}{4}\|n_k(t)\|^2+\frac{1}{2}\|\varphi_{kx}(t)\|^2+\frac{\alpha}{2}\|n_{kx}(t)\|^2\leqslant E_0+2C^2\|\varepsilon_0\|^4. \tag{2.10}$$

Choose a suitable small number δ, such that

$$1-2\delta^2\|\varepsilon_{0k}\|^2\geqslant 1-2\delta^2\|\varepsilon_0\|^2>\frac{1}{4}.$$

Let $\delta_0=\min\left(\frac{1}{4},\ \frac{\alpha}{2}\right)$. Then

$$\|\varepsilon_{kx}(t)\|^2+\|n_{kx}(t)\|^2+\|\varphi_{kx}(t)\|^2+\|n_k(t)\|^2\leqslant\frac{1}{\delta_0}(|E_0|+2C^2\|\varepsilon_0\|^4)=E_2,$$

that is, (2.6) is satisfied. This completes the proof of Lemma 3.

Corollary 1. If the conditions of Lemma 3 are satisfied, then we have

$$\|\varepsilon_k\|_{L_\infty}\leqslant E_2',\quad \|n_k\|_{L_\infty}\leqslant E_2',$$

where the constant E_2' is independent of k.

Proof. By Lemmas 2 and 3, we can immediately prove it.

Lemma 4. *Suppose that the conditions of Lemma 3 are satisfied, and that* $\varepsilon_0(x)\in H^2$, $n_0(x)\in H^2$, $\varphi_0\in H^2$, and $f(n)\in C^3$. Then the following estimation is true

$$\|n_{kx}\|^2+\|n_{kt}\|^2+\|n_{kxx}\|^2+\|\varepsilon_{kt}\|^2\leqslant E_3, \tag{2.10}$$

where E_3 is a definite constant independent of k and t.

Proof. Set $\varepsilon_{kt}=E_k$, $n_{kt}=N_k$. Then multiplying equation (2.2c) by $(-w_j^2)$, we have

$$(\varphi_{kt}-n_k-f(n_k)+\alpha n_{kxx}-|\varepsilon_k|^2,\ v_j''(x))=0.$$

Hence

$$(\varphi_{ktxx}-n_{kxx}-f(n_k)_{xx}+\alpha n_{kxxxx}-|\varepsilon_k|^2_{xx},\ v_j(x))=0.$$

On the other hand, differentiating (2.2b) with respect to t, we have

$$(n_{ktt}-\varphi_{kxxt},\ v_j(x))=0.$$

Putting it into the above equation we obtain

$$(n_{ktt}-n_{kxx}-f(n_k)_{xx}+\alpha n_{kxxxx}-|\varepsilon_k|^2_{xx},\ v_j(x))=0.$$

Multiplying this equation by $\bar\beta_{jt}$ and summing for j, we have

$$(n_{ktt}-n_{kxx}-f(n_k)_{xx}+\alpha n_{kxxxx}-|\varepsilon_k|^2_{xx},\ n_{kt})=0. \tag{2.11}$$

Estimate these inner products respectively as follows

$$(n_{ktt},\ n_{kt})=\frac{1}{2}\frac{d}{dt}\|n_{kt}\|^2=\frac{1}{2}\frac{d}{dt}\|N_k\|^2,$$

$$-(n_{kxx},\ n_{kt})=\frac{1}{2}\frac{d}{dt}\|n_{kx}\|^2,$$

$$\alpha(n_{kxxxx},\ n_{kt})=\alpha(n_{kxx},\ n_{kxxt})=\frac{\alpha}{2}\frac{d}{dt}\|n_{kxx}\|^2.$$

Because

$$(f(n_k)_{xx},\ n_{kt})=\left(\frac{\partial^2 f}{\partial n_k^2}\left(\frac{\partial n_k}{\partial x}\right)^2+\frac{\partial f}{\partial n_k}\frac{\partial^2 n_k}{\partial x^2},\ n_{kt}\right),$$

hence

$$\begin{aligned}(f''(n_k)(n_{kx})^2,\ n_{kt})&\leqslant\|f''(n_k)\|_{L_\infty}\frac{1}{2}(\|(n_{kx})^2\|^2+\|n_{kt}\|^2)\\&\leqslant k[\|n_{kt}\|^2+\|n_{kx}\|^2_{L_\infty}\|n_{kx}\|^2]\\&\leqslant k[\|N_k\|^2+E_2 2(\delta^2\|n_{kxx}\|^2+C^2\|n_k\|^2)]\\&\leqslant k_1[\|N_k\|^2+\|n_{kxx}\|^2+1]\end{aligned}$$

and

$$(f'(n_k)n_{kxx},\ n_{kt})\leqslant\|f'(n_k)\|_{L_\infty}\frac{1}{2}(\|n_{kt}\|^2+\|n_{kxx}\|^2)\leqslant K_2(\|N_k\|^2+\|n_{kxx}\|^2).$$

So

$$(f(n_k)_{xx},\ n_{kt})\leqslant K_1(\|N_k\|^2+\|n_{kxx}\|^2+1)+K_2(\|N_k\|^2+\|n_{kxx}\|^2).$$

Then we estimate

$$(-|\varepsilon_k|^2_{xx},\ n_{kt})\leqslant|(|\varepsilon_k|^2_{xx},\ n_{kt})|$$

and

$$|\varepsilon_k|^2_{xx}=(\varepsilon_k\bar{\varepsilon}_k)_{xx}=\varepsilon_{kxx}\bar{\varepsilon}_k+2\varepsilon_{kx}\bar{\varepsilon}_{kx}+\varepsilon_k\bar{\varepsilon}_{kxx}.$$

Multiplying (2.2a) by $(-w_j^2)$, we obtain

$$(i\varepsilon_{kt}+\varepsilon_{kxx}-n_k\varepsilon_k,\ v_j'')=0.$$

Multiplying the above equality by $\bar{a}_j$ and summing up for j, we have

$$(i\varepsilon_{kt}+\varepsilon_{kxx}-n_k\varepsilon_k,\ \varepsilon_{kxx})=0,$$

$$(\varepsilon_{kxx},\ \varepsilon_{kxx})=-(i\varepsilon_{kt},\ \varepsilon_{kxx})+(n_k\varepsilon_k,\ \varepsilon_{kxx}),$$

i. e.

$$\|\varepsilon_{kxx}\|^2\leqslant\|\varepsilon_{kt}\|\cdot\|\varepsilon_{kxx}\|+\|n_k\varepsilon_k\|\cdot\|\varepsilon_{kxx}\|=(\|\varepsilon_{kt}\|+\|n_k\varepsilon_k\|)\|\varepsilon_{kxx}\|.$$

Therefore

$$\|\varepsilon_{kxx}\|\leqslant\|\varepsilon_{kt}\|+\|n_k\varepsilon_k\|.$$

It follows that

$$\begin{aligned}|(|\varepsilon_k|^2_{xx},\ n_{kt})|&\leqslant|(\varepsilon_{kxx}\bar{\varepsilon}_k+\varepsilon_k\bar{\varepsilon}_{kxx},\ n_{kt})|+2|(\varepsilon_{kx}\bar{\varepsilon}_{kx},\ n_{kt})|\\&\leqslant2\|\varepsilon_k\|_{L_\infty}\cdot\|\varepsilon_{kxx}\|\cdot\|n_{kt}\|+2|(\varepsilon_{kx}\bar{\varepsilon}_{kx},\ n_{kt})|\\&\leqslant2\|\varepsilon_k\|_{L_\infty}(\|\varepsilon_{kt}\|+\|n_k\varepsilon_k\|)\|n_{kt}\|+2\|\varepsilon_{kx}\|_{L_\infty}\|\varepsilon_{kx}\|\|n_{kt}\|\\&\leqslant\|\varepsilon_k\|_{L_\infty}[\|\varepsilon_{kt}\|^2+\|n_{kt}\|^2+\|\varepsilon_k\|_{L_\infty}(\|n_k\|^2+\|n_{kt}\|^2)]\\&\quad+2\|\varepsilon_{kx}\|_{L_\infty}\cdot\|\varepsilon_{kx}\|\cdot\|n_{kt}\|\\&\leqslant K_3(\|E_k\|^2+\|N_k\|^2+1)+K_4(C^2\|\varepsilon_k\|^2+\delta^2\|\varepsilon_{kxx}\|^2+\|n_{kt}\|^2)\\&\leqslant K_5(\|E_k\|^2+\|N_k\|^2+1).\end{aligned}$$

Substituting these estimations into (2.11), we get

$$\frac{d}{dt}\Big[\|n_{kt}\|^2+\|n_{kx}\|^2+\frac{\alpha}{2}\|n_{kxx}\|^2\Big]\leqslant K_6[\|n_{kt}\|^2+\|n_{kxx}\|^2+\|\varepsilon_{kt}\|^2]+K_7. \tag{2.12}$$

On the other hand, differentiating (2.2a) with respect to t and multiplying the resulting relation by $\bar{a}_j(t)$, and finally summing up for j, we obtain

$$(i\varepsilon_{ktt}+\varepsilon_{ktxx}-n_{kt}\varepsilon_k-n_k\varepsilon_{kt},\ \varepsilon_{kt})=0.$$

Taking the imaginary part of the above equality, we have

$$\frac{1}{2}\frac{d}{dt}\|\varepsilon_{kt}\|^2-\mathrm{Im}(n_{kt}\varepsilon_k,\ \varepsilon_{kt})=0,$$

$$\frac{d}{dt}\|\varepsilon_{kt}\|^2\leqslant\|\varepsilon_k\|_{L_\infty}(\|\varepsilon_{kt}\|^2+\|n_{kt}\|^2).$$

Combining it with (2.12), we get

$$\frac{d}{dt}[\|\varepsilon_{kt}\|^2+\|n_{kt}\|^2+\|n_{kx}\|^2+\|n_{kxx}\|^2]\leqslant K_9[\|\varepsilon_{kt}\|^2+\|n_{kt}\|^2+\|n_{kx}\|^2+\|n_{kxx}\|^2]+K_{10}.$$

By using the Gronwall inequality, we can obtain

$$\|n_{kt}\|^2+\|n_{kx}\|^2+\|n_{kxx}\|^2+\|\varepsilon_{kt}\|^2\leqslant E_3,$$

that is (2.10) is true. The proof of Lemma 4 is thus completed.

Corollary 2. If the conditions of Lemma 4 are satisfied, then the following estimations remain valid:

$$\|\varphi_{kx}\|_{L_\infty}\leqslant E_3',\quad\|n_{kx}\|_{L_\infty}\leqslant E_3',\quad\|\varphi_{kxx}\|\leqslant E_3',\quad\|\varepsilon_{kxx}\|\leqslant E_3', \tag{2.13}$$

where E'_3 is a constant independent of k.

Differentiating (2.2a), (2.2b), (2.2c) with respect to t several times respectively, and using the property of the basic functions $v''_j(x) = -w_j^2 v_j(x)$, we can derive analogous estimates of the previous lemmas.

Lemma 5. *If the conditions of Lemma 4 are satisfied and*

(i) $s_0(x) \in H^4$, $n_0(x) \in H^4$,

(ii) $f(n) \in C^3$,

then we have

$$\|s_{kxxt}\|^2 + \|s_{ktt}\|^2 + \|n_{ktt}\|^2 + \|n_{kxt}\|^2 + \|n_{kxxt}\|^2 \leqslant E_4, \tag{2.14}$$
$$\|s_{kxx}\|^2_{L_\infty} + \|n_{kxx}\|^2_{L_\infty} \leqslant E_4,$$

where E_4 is a definite constant independent of k, t.

Theorem 1. *Assume that the conditions of Lemma 5 are satisfied. Then there exists the global classical solution $\{s(x, t), n(x, t), \varphi(x, t)\}$ of the periodic initial value problem* (1.1)—(1.5):

$$s_t \in L^\infty(0, T; H^2), \quad n_t \in L^\infty(0, T; H^2), \quad \varphi_t \in L^\infty(0, T; H_2),$$
$$s_{tt} \in L^\infty(0, T; L_2), \quad n_{tt} \in L^\infty(0, T; L_2), \quad \varphi_{tt} \in L^\infty(0, T; L_2).$$

Proof. From Lemmas 1—5, by means of the uniform estimations of k, the compact principle, and the Sobolev imbedding theorem, the proof is obtained immediately.

III. The Convergence of the Galerkin-Fourier Method

Let
$$s - s_k = \Sigma, \; n - n_k = N, \; \varphi - \varphi_k = \phi.$$

Then from equations (2.2a) and (1.1), we get

$$(i(s - s_k)_t + (s - s_k)_{xx} - (ns - n_k s_k), \; v_j(x)) = 0.$$

Let
$$v = F_k s - s_k = s - R_k s - s_k = \Sigma - R_k s = \sum_j a_j v_j(x).$$

Then from the last equation we have

$$(i\Sigma_t + \Sigma_{xx} - Ns - n_k \Sigma, \; \Sigma - R_k s) = 0. \tag{3.1}$$

From equations (2.2b) and (1.2), the following equation

$$((n - n_k)_t - (\varphi - \varphi_k)_{xx}, \; v_j(x)) = 0$$

is got. If we set

$$v = F_k n - n_k = n - R_k n - n_k = N - R_k n = \sum_j b_j v_j(x),$$

then from the above equation it follows that

$$(N_t - \phi_{xx}, \; N - R_k n) = 0. \tag{3.2}$$

Similarly, from (2.2c) and (1.3), we have

$$((\varphi - \varphi_k)_t - (n - n_k) - (f(n) - f(n_k)) + \alpha(n - n_k)_{xx} - (|s|^2 - |s_k|^2), \; v_j(x)) = 0.$$

If we denote $v = F_k\varphi - \varphi_k = \varphi - R_k\varphi - R_k = \phi - R_k\varphi = \sum_j c_j v_j(x)$, then from the last equation it follows that

$$(\phi_t - N - f'(n^*)N + \alpha N_{xx} - (|s|^2 - |s_k|^2), \; \phi - R_k\varphi) = 0, \tag{3.3}$$

where $\min(n, n_k) \leqslant n^* \leqslant \max(n, n_k)$. Taking the imaginary part of equation (3.1), we have

$$(\Sigma_t, \Sigma) = \mathrm{Im}(Ns, \Sigma) - \mathrm{Im}(i\Sigma_t + \Sigma_{xx} - Ns - n_k\Sigma, \; R_k s).$$

Then

$$\frac{1}{2}\frac{d}{dt}\|\Sigma\|^2\leqslant\frac{1}{2}\|\varepsilon\|_{L_\infty}(\|N\|^2+\|\Sigma\|^2)+\mathrm{Re}\left\{\frac{d}{dt}(\Sigma,\ R_k\varepsilon)-(\Sigma,\ R_k\varepsilon_t)\right\}$$
$$+\mathrm{Im}(\Sigma,\ R_k\varepsilon_{xx})+\mathrm{Im}(-N\varepsilon-n_k\Sigma,\ R_k\varepsilon).$$

Integrating the above inequality with respect to t, we have

$$\|\Sigma(t)\|^2\leqslant\|\Sigma(0)\|^2+C\int_0^t(\|N(\tau)\|^2+\|\Sigma(\tau)\|^2)d\tau$$
$$+\frac{1}{2}\|\Sigma(t)\|^2+\frac{1}{2}\|\Sigma(0)\|^2+2\|R_k\varepsilon(t)\|^2+2\|R_k\varepsilon(0)\|^2$$
$$+\int_0^t(\|\Sigma(\tau)\|^2+\|R_k\varepsilon_{xx}\|^2)d\tau+\int_0^t(\|\Sigma(\tau)\|^2+\|R_k\varepsilon_t\|^2)d\tau$$
$$+\int_0^t\|\varepsilon\|_{L_\infty}(\|N\|^2+\|R_k\varepsilon\|^2)d\tau+\int_0^t\|n_k\|_{L_\infty}(\|\Sigma\|^2+\|R_k\varepsilon\|^2)d\tau.$$

That is, we have

$$\|\Sigma(t)\|^2\leqslant\delta_k^{(1)}+C_1\int_0^t(\|N(\tau)\|^2+\|\Sigma(\tau)\|^2)d\tau, \tag{3.4}$$

where $\delta_k^{(1)}$, as well as below $\delta_k^{(j)}(j=2,\ 3,\ \cdots)$, denotes a definite constant, which tends to zero when $k\to\infty$. From (3.2), we obtain

$$(N_t,\ N)=(\phi_{xx},\ N)+(N_t,\ R_kn)-(\phi_{xx},\ R_kn)$$
$$=-(\phi_x,\ N_x)+\frac{d}{dt}(N,\ R_kn)-(N,\ R_kn_t)-(\phi,\ R_kn_{xx}),$$

i. e.

$$\frac{1}{2}\frac{d}{dt}\|N\|^2\leqslant\frac{1}{2}(\|\phi_x\|^2+\|N_x\|^2)+\frac{d}{dt}(N,\ R_kn)$$
$$+\frac{1}{2}(\|N\|^2+\|R_kn_t\|^2)+\frac{1}{2}(\|\phi\|^2+\|R_kn_{xx}\|^2).$$

Integrating the above inequality with respect to t, we obtain

$$\|N(t)\|^2\leqslant\|N(0)\|^2+\int_0^t(\|\phi_x\|^2+\|N_x\|^2)d\tau$$
$$+\frac{1}{2}\|N(t)\|^2+2\|R_kn\|^2+\frac{1}{2}\|N(0)\|^2+2\|R_kn(0)\|^2$$
$$+\int_0^t(\|N\|^2+\|R_kn_t\|^2+\|\phi\|^2+\|R_kn_{xx}\|^2)d\tau,$$

i. e.

$$\|N(t)\|^2\leqslant\delta_k^{(2)}+C_2\int_0^t(\|N\|^2+\|N_x\|^2+\|\phi\|^2+\|\phi_x\|^2)d\tau. \tag{3.5}$$

Expanding (3.3), we estimate the inner product with ϕ respectively

$$(\phi_t,\ \phi)=\frac{1}{2}\frac{d}{dt}\|\phi\|^2,$$
$$(N,\ \phi)\leqslant\frac{1}{2}(\|N\|^2+\|\phi\|^2),$$
$$(f'(n^*)N,\ \phi)\leqslant\|f'(n^*)\|_{L_\infty}\frac{1}{2}(\|N\|^2+\|\phi\|^2),$$
$$(\alpha N_{xx},\ \phi)=-\alpha(N_x,\ \phi_x)\leqslant|\alpha|\frac{1}{2}(\|N_x\|^2+\|\phi_x\|^2),$$

$$(|\varepsilon|^2-|\varepsilon_k|^2,\ \phi)=((|\varepsilon|+|\varepsilon_k|)(|\varepsilon|-|\varepsilon_k|),\ \phi)$$
$$\leqslant(\|\varepsilon\|_{L_\infty}+\|\varepsilon_k\|_{L_\infty})\frac{1}{2}(\|\Sigma\|^2+\|\phi\|^2).$$

Then we estimate the inner product with $R_k\varphi$ similarly. Hence, from (3.3) we obtain the following inequality

$$\frac{1}{2}\frac{d}{dt}\|\phi\|^2\leqslant\frac{1}{2}(\|N\|^2+\|\phi\|^2)+\frac{1}{2}\|f'(n^*)\|_{L_\infty}(\|N\|^2+\|\phi\|^2)$$
$$+\frac{|\alpha|}{2}(\|N_x\|^2+\|\phi_x\|^2)+C(\|\Sigma\|^2+\|\phi\|^2)$$
$$+\frac{d}{dt}(\phi,\ R_k\varphi)-(\phi,\ R_k\varphi_t)+\frac{1}{2}(\|N\|^2+\|R_k\varphi\|^2)$$
$$+\frac{1}{2}\|f'(n^*)\|_{L_\infty}(\|N\|^2+\|R_k\varphi\|^2)+\frac{|\alpha|}{2}(\|N\|^2+\|R_k\varphi_{xx}\|^2)$$
$$+(\|\varepsilon\|_{L_\infty}+\|\varepsilon_k\|_{L_\infty})\frac{1}{2}(\|\Sigma\|^2+\|R_k\varphi\|^2).$$

Rearranging the above inequality and integrating it with respect to t, we have

$$\|\phi(t)\|^2\leqslant\|\phi(0)\|^2+C\int_0^t(\|\Sigma\|^2+\|N\|^2+\|N_x\|^2+\|\phi\|^2+\|\phi_x\|^2)d\tau$$
$$+C\int_0^t(\|R_k\varphi\|^2+\|R_k\varphi_{xx}\|^2)d\tau+\frac{1}{2}\|\phi\|^2+2\|R_k\varphi\|^2$$
$$+\int_0^t(\|\phi\|^2+\|R_k\varphi_t\|^2)d\tau+\frac{1}{2}\|\phi(0)\|^2+2\|R_k\varphi(0)\|^2.$$

Hence
$$\|\phi(t)\|^2\leqslant\delta_k^{(3)}+C_3\int_0^t(\|\Sigma\|^2+\|N\|^2+\|N_x\|^2+\|\phi\|^2+\|\phi_x\|^2)d\tau. \tag{3.6}$$

In the following, we must estimate the derivatives of Σ, N and ϕ. Differentiating (2.2b), (1.2) with respect to t respectively and then taking the difference of the two results obtained, we obtain the following formula

$$(N_{tt}-\phi_{xxt},\ v_j)=0. \tag{3.7}$$

On the other hand, subtracting (1.3) from (2.2c) and using $v_j''=-w_j^2v_j(x)$ we have

$$(\phi_t-N-(f(n)-f(n_k))+\alpha N_{xx}-(|\varepsilon|^2-|\varepsilon_k|^2,\ v_j'')=0.$$

Integrating the above equality by parts gives

$$(\phi_{txx}-N_{xx}-(f(n)-f(n_k))_{xx}+\alpha N_{xxxx}-(|\varepsilon|^2-|\varepsilon_k|^2)_{xx},\ v_j)=0. \tag{3.8}$$

Canceling ϕ_{txx} from (3.7) and (3.8), and letting

$$v_j=F_kn_t-n_{kt}=n_t-R_kn_t-n_{kt}=N_t+R_kn_t,$$

we obtain

$$(N_{tt}-N_{xx}-(f(n)-f(n_k))_{xx}+\alpha N_{xxxx}-(|\varepsilon|^2-|\varepsilon_k|^2)_{xx},\ N_t)$$
$$=(N_{tt}-N_{xx}-(f(n)-f(n_k))_{xx}+\alpha N_{xxxx}-(|\varepsilon|^2-|\varepsilon_k|^2)_{xx},\ R_kn_t). \tag{3.9}$$

We estimate these inner products as follows:

$$(N_{tt},\ N_t)=\frac{1}{2}\frac{d}{dt}\|N_t\|^2,$$
$$-(N_{xx},\ N_t)=\frac{1}{2}\frac{d}{dt}\|N_x\|^2,$$
$$(f(n)-f(n_k))_x=f'(n)n_x-f'(n_k)n_{kx},$$

$$(f(n)-f(n_k))_{xx}=(f''(n)-f''(n_k))(n_k)^2+f''(n_k)(n_x^2-n_{kx}^2)$$
$$+(f'(n)-f'(n_k))n_{xx}+f'(n_k)(n_{xx}-n_{kxx})$$
$$=f''(n^*)N(n_x)^2+f''(n_k)(n_x+n_{kx})N_x+f''(n^*)Nn_{xx}+f'(n_k)\cdot N_{xx}$$

and $$|((f(n)-f(n_k))_{xx},\ N_t)|\leqslant|(f'''(n^*)N(n_x)^2,\ N_t)|$$
$$+|f''(n_k)(n_x+n_{kx})N_x,\ N_t)|+|(f''(n^*)Nn_{xx},\ N_t)|+|(f'(n_k)N_{xx},\ N_t)|$$
$$\leqslant\|f'''(n^*)\|_{L_\infty}\|n_x\|^2_{L_\infty}\frac{1}{2}(\|N\|^2+\|N_t\|^2)+\|f''(n_k)\|_{L_\infty}(\|n_x\|_{L_\infty}+\|n_{kx}\|_{L_\infty})$$
$$\times\frac{1}{2}(\|N_x\|^2+\|N_t\|^2)+\|f''(n^*)\|_{L_\infty}\|n_{xx}\|_{L_\infty}\frac{1}{2}(\|N\|^2+\|N_t\|^2)$$
$$+\frac{1}{2}\|f'(n_k)\|_{L_\infty}\frac{1}{2}(\|N_{xx}\|^2+\|N_t\|^2)\leqslant O(\|N\|^2+\|N_t\|^2+\|N_x\|^2+\|N_{xx}\|^2).$$

Also, $$(\alpha N_{xxxx},\ N_t)=\alpha\frac{1}{2}\frac{d}{dt}\|N_{xx}\|^2,$$

$$(|\varepsilon|^2-|\varepsilon_k|^2)_{xx}=(\varepsilon\bar\varepsilon)_{xx}-(\varepsilon_k\bar\varepsilon_k)_{xx}$$
$$=\varepsilon_{xx}\bar\varepsilon+2\varepsilon_x\bar\varepsilon_x+\varepsilon\bar\varepsilon_{xx}-\varepsilon_{kxx}\bar\varepsilon_k-2\varepsilon_{kx}\bar\varepsilon_{kx}-\varepsilon_k\bar\varepsilon_{kxx}$$
$$=(\varepsilon_{xx}-\varepsilon_{kxx})\bar\varepsilon+\varepsilon_{kxx}(\bar\varepsilon-\bar\varepsilon_k)+2(|\varepsilon_x|^2-|\varepsilon_{kx}|^2)$$
$$+\varepsilon(\bar\varepsilon_{xx}-\bar\varepsilon_{kxx})+\bar\varepsilon_{kxx}(\varepsilon-\varepsilon_k)$$
$$=\Sigma_{xx}\bar\varepsilon+\varepsilon_{kxx}\bar\Sigma+2(|\varepsilon_x|+|\varepsilon_{kx}|)(|\varepsilon_x|-|\varepsilon_{kx}|)+\varepsilon\bar\Sigma_{xx}+\bar\varepsilon_{kxx}\Sigma.$$

Hence $$|(|\varepsilon|^2_{xx}-|\varepsilon_k|^2_{xx},\ N_t)|\leqslant|(\Sigma_{xx}\bar\varepsilon,\ N_t)|+|(\varepsilon_{kxx}\bar\Sigma,\ N_t)|$$
$$+2((|\varepsilon_x|+|\varepsilon_{kx}|)|\Sigma_x|,\ |N_t|)+|(\varepsilon\bar\Sigma_{xx},\ N_t)|+|(\bar\varepsilon_{kxx}\Sigma,\ N_t)|$$
$$\leqslant\|\varepsilon\|_{L_\infty}\frac{1}{2}(\|\Sigma_{xx}\|^2+\|N_t\|^2)+\|\varepsilon_{kxx}\|_{L_\infty}\frac{1}{2}(\|\Sigma\|^2+\|N_t\|^2)$$
$$+\|\varepsilon\|_{L_\infty}\frac{1}{2}(\|\Sigma_{xx}\|^2+\|N_t\|^2)+(\|\varepsilon_x\|_{L_\infty}+\|\varepsilon_{kx}\|_{L_\infty})(\|\Sigma_x\|^2+\|N_t\|^2)$$
$$+\|\varepsilon_{kxx}\|_{L_\infty}\frac{1}{2}(\|\Sigma\|^2+\|N_t\|^2)\leqslant O(\|\Sigma\|^2+\|\Sigma_x\|^2+\|\Sigma_{xx}\|^2+\|N_t\|^2).$$

Similarly, $$(N_{tt},\ R_kn_t)=\frac{d}{dt}(N_t,\ R_kn_t)-(N_t,\ R_kn_{tt}),$$

$$(N_{xx},\ R_kn_t)\leqslant\frac{1}{2}(\|N_{xx}\|^2+\|R_kn_t\|^2),$$
$$|((f(n)-f(n_k))_{xx},\ R_kn_t)|\leqslant O(\|N\|^2+\|N_x\|^2+\|N_{xx}\|^2+\|R_kn_t\|^2),$$
$$(\alpha N_{xxxx},\ R_kn_t)=\alpha(N_{xx},\ R_kn_{txx})\leqslant O(\|N_{xx}\|^2+\|R_kn_{txx}\|^2),$$
$$(|\varepsilon|^2_{xx}-|\varepsilon_k|^2_{xx},\ R_kn_t)\leqslant O(\|\Sigma\|^2+\|\Sigma_x\|^2+\|\Sigma_{xx}\|^2+\|R_kn_t\|^2).$$

Substituting these inequalities into (3.9), we get

$$\frac{1}{2}\frac{d}{dt}(\|N_t\|^2+\|N_x\|^2+\alpha\|N_{xx}\|^2)$$
$$\leqslant\frac{d}{dt}(N_t,\ R_kn_t)+O(\|R_kn_t\|^2+\|R_kn_{tt}\|^2+\|R_kn_{txx}\|^2)$$
$$+O(\|N\|^2+\|N_t\|^2+\|N_x\|^2+\|N_{xx}\|^2+\|\Sigma\|^2+\|\Sigma_x\|^2+\|\Sigma_{xx}\|^2).$$

Integrating the above inequality with respect to t, where

$$\int_0^t\frac{d}{dt}(N_t,\ R_kn_t)dt\leqslant\frac{1}{4}\|N_t\|^2+\|R_kn_t\|^2+\frac{1}{4}\|N_t(0)\|^2+\|R_kn_t(0)\|^2$$