NUMERICAL ANALYSIS OF NAVIER-STOKES EQUATIONS BY A HYBRID FINITE ELEMENT METHOD*

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It is known that the advantages of the hybrid finite element method lie in many aspects. Firstly, the trouble of using finite elements of class $C^k(k\geqslant 1)$, which is sometimes required for conforming methods, is avoided. Secondly, some derivatives of the solutions can be obtained simultaneously, such as the stress tensor, which is of more importance sometimes. Thirdly, some special interpolation functions can be easily used for special goals, for instance, we may use the singular expansion as the interpolation functions for fracture mechanics.

The drawback of this method is also obvious, as more variables are involved in the equations to be solved, it is more complicated to construct the stiffness matrix for each element and the program would be more complicated. But the scale of algebraic systems is the same as that of the comforming methods. Therefore if the problem is of large scale and requires a high precision, the hybrid finite element method may be a good choice.

We have applied the hybrid finite element method to incompressible viscous flow[1-3], and discovered another advantage, namely, it improves convergence and stability.

If a primitive variables formulation is used for Navier-Stokes flow, then the Babuska-Brezzi condition is necessary for a conforming finite element approach, the degree of freedom of the velocity field should be much bigger than that of the pressure field, and it causes a loss of precision. For example, as the quadratic six nodes triangular elements are used for the Stokes problem, only a precision of $O(h)$ can be obtained[4], in contrast with the precision $O(h^2)$ for the same elements used in an elastic problem. Some authors have improved the results for Stokes flow, e.g. the work of Santos[5].

We discovered that for the hybrid finite element method, althrough some kinds of Babuska-Brezzi conditions have to be satisfied, they incur no loss of precision. For example, when the quadratic six-nodes triangular elements are used for the velocity field, the precision is $O(h^2)$. Therefore, we get the optimal degree of precision. Some numerical examples have shown that the approximate solution is in good agreement with the analytical solution by our method. In this paper we generalize our method to the nonlinear problem. In the first section we deduce some formulations of the variational problem formally, which means we do not use the terminology of Sobolev spaces, for this kind of statement can give the readers a more intuitional

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understanding of our method. In the second section, we discuss the Stokes problem, which is the foundation of the next section. Most of the material in Section 2 has been published, but we will give a new and simpler proof. In the third section we discuss the hybrid finite element method for Navier-Stokes equations.

§ 1. Some Variational Formulations

Let the fluid be incompressible, viscous, and Newtonian. The space is d-dimensional ($d=2$ or 3), and the governing equations of a stationary flow are

$$-\sigma_{ij,j}+u_j u_{i,j}=f_i, \quad 1\leqslant i\leqslant d, \tag{1.1}$$

$$\sigma_{ij}=\sigma_{ji}=2\nu\varepsilon_{ij}-p\delta_{ij}, \quad 1\leqslant i,\ j\leqslant d, \tag{1.2}$$

$$\varepsilon_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i}), \quad 1\leqslant i,\ j\leqslant d, \tag{1.3}$$

$$u_{i,i}=0, \tag{1.4}$$

where $x=(x_i)$ are the spatial cartesian coordinates, f_i are body forces, $\sigma=(\sigma_{ij})$ the stress tensor, which is symmetric, p the hydrostatic pressure, $u=(u_i)$ the velocity, $\varepsilon=(\varepsilon_{ij})$ the velocity strain tensor which is also symmetric, and ν the constant of viscosity. We assume the fluid density is $\rho=1$, and $(\ )_{,i}$ denotes partial differentiation with respect to x_i. For simplification, we will not indicate the range of indices $i, j, \cdots$.

Let us consider a domain $\Omega\subset R^d$, with boundary $\partial\Omega$. We consider the above equations with boundary value

$$u(x)=\bar{u}_0(x), \quad x\in\partial\Omega, \tag{1.5}$$

where $\bar{u}_0=(\bar{u}_{0i})$ is a known function satisfying

$$\int_{\partial\Omega}\bar{u}_0\cdot n\,dx=0,$$

where $n=(n_i)$ is the unit exterior normal vector along $\partial\Omega$.

First we consider the Stokes equation, that is, the convection term in equation (1.1) is ignored, it becomes

$$-\sigma_{ij,j}=f_i. \tag{1.6}$$

The boundary value problem (1.2)—(1.6) corresponds naturally to the following functional:

$$F_1(\varepsilon, u, p)=\int_\Omega\{\nu\varepsilon_{ij}(\varepsilon_{ij}-(u_{i,j}+u_{j,i}))+pu_{i,i}+f_iu_i\}dx$$

$$+\int_{\partial\Omega}(2\nu\varepsilon_{ij}-p\delta_{ij})n_j(u_i-\bar{u}_{0i})dx. \tag{1.7}$$

The critical point of (1.7) satisfies equation

$$F_1'(\varepsilon, u, p)=0, \tag{1.8}$$

that is,

$$\int_\Omega\nu\Big(\varepsilon_{ij}-\frac{1}{2}(u_{i,j}+u_{j,i})\Big)\mu_{ij}\,dx=0, \quad \forall\mu_{ij}(\mu_{ij}=\mu_{ji}), \tag{1.9}$$

$$\int_\Omega\{-\nu\varepsilon_{ij}(v_{i,j}+v_{j,i})+pv_{i,i}+f_iv_i\}\,dx+\int_{\partial\Omega}(2\nu\varepsilon_{ij}-p\delta_{ij})n_jv_i\,dx=0, \quad \forall v_i, \tag{1.10}$$

$$\int_\Omega u_{i,i}q\,dx=0, \quad \forall q, \tag{1.11}$$

$$\int_{\partial\Omega} (u_i - \bar{u}_{0i}) n_j (2\nu\mu_{ij} - q\delta_{ij}) dx = 0, \ \forall \mu_{ij}, \ \forall q. \tag{1.12}$$

It is clear that (1.9)—(1.12) are equivalent to (1.2)—(1.6). If the variable u in F_1 is regarded as a Lagrange multiplier with the constraint

$$2\nu\varepsilon_{ij,j} - p_{,i} + f_i = 0, \tag{1.13}$$

we can define another functional

$$F_2(\varepsilon, p) = \int_\Omega \nu\varepsilon_{ij}\varepsilon_{ij} dx - \int_{\partial\Omega} (2\nu\varepsilon_{ij} - p\delta_{ij}) n_j \bar{u}_{0i} dx. \tag{1.14}$$

The critical point of F_2 with constraint (1.13) should satisfy

$$\int_\Omega 2\nu\varepsilon_{ij}\mu_{ij} dx - \int_{\partial\Omega} (2\nu\mu_{ij} - q\delta_{ij}) n_j \bar{u}_{0i} dx = 0, \tag{1.15}$$

but μ_{ij} and q are no longer arbitrary, and the variables $\varepsilon_{ij} + \mu_{ij}$ and $p+q$ should satisfy equation (1.13).

According to the basic idea of the hybrid finite element method we make a discretization for domain Ω, which means we construct subdomains Ω_k ($k=1, \cdots, N$) of Ω such that

$$\Omega_k \cap \Omega_l = \phi, \quad k \neq l, \quad \bar{\Omega} = \bigcup_{k=1}^{N} \bar{\Omega}_k.$$

Then we define the following functional:

$$F_3(\varepsilon, u, p, \bar{u}) = \int_\Omega \{\nu\varepsilon_{ij}(\varepsilon_{ij} - (u_{i,j} + u_{j,i})) + pu_{i,i} + f_i u_i\} dx$$
$$+ \sum_k \int_{\partial\Omega_k} (2\nu\varepsilon_{ij} - p\delta_{ij}) n_j (u_i - \bar{u}_i) dx, \tag{1.16}$$

where n is the unit exterior normal vector along $\partial\Omega_k$, and $\bar{u}$ is defined on $\bigcup_k \partial\Omega_k$ and satisfies the boundary value (1.5) on $\partial\Omega$. The critical point of functional F_3 satisfies

$$\int_\Omega \nu\left(\varepsilon_{ij} - \frac{1}{2}(u_{i,j} + u_{j,i})\right)\mu_{ij} dx = 0, \quad \forall \mu_{ij}, \tag{1.17}$$

$$\int_\Omega \{-\nu\varepsilon_{i,j}(v_{i,j} + v_{j,i}) + pv_{i,i} + f_i v_i\} dx$$
$$+ \sum_k \int_{\partial\Omega_k} (2\nu\varepsilon_{ij} - p\delta_{ij}) n_j v_i dx = 0, \quad \forall v_i, \tag{1.18}$$

$$\int_\Omega u_{i,i} q \, dx = 0, \quad \forall q, \tag{1.19}$$

$$\int_{\partial\Omega_k} (u_i - \bar{u}_i) n_j (2\nu\mu_{ij} - q\delta_{ij}) dx = 0, \quad \forall \mu_{ij}, \forall q, \forall k, \tag{1.20}$$

$$\sum_k \int_{\partial\Omega_k} (2\nu\varepsilon_{ij} - p\delta_{ij}) n_j \bar{v}_i dx = 0, \quad \forall \bar{v}_i. \tag{1.21}$$

Equation (1.21) corresponds to the balance of forces at the boundary of elements. It is easy to see that equations (1.17)—(1.21) with constraint (1.5) are equivalent to the original problem. Similarly, we may regard u as a Lagrange multiplier with the constraint (1.13) in each element. Then we define a functional

$$F_4(\varepsilon, p, \bar{u}) = \int_\Omega \nu\varepsilon_{ij}\varepsilon_{ij} dx - \sum_k \int_{\partial\Omega_k} (2\nu\varepsilon_{ij} - p\delta_{ij}) n_j \bar{u}_i dx. \tag{1.22}$$

It should be noticed that the derivatives in (1.13) are distributions in general and are defined in each Ω_k, not Ω. The original problem (1.2)—(1.6) is equivalent to a variational problem with constraints (1.13) and (1.5):

$$\int_\Omega 2\nu \varepsilon_{ij}\mu_{ij}\,dx-\sum_k\int_{\partial\Omega_k}(2\nu\mu_{ij}-q\delta_{ij})n_j\bar{u}_i\,dx=0,\quad \forall\mu_{ij},\forall q, \tag{1.23}$$

$$\sum_k\int_{\partial\Omega_k}(2\nu\varepsilon_{ij}-p\delta_{ij})n_j\bar{v}_i\,dx=0,\quad \forall v_i, \tag{1.24}$$

where again μ_{ij} and q are not arbitrary, and the variables $\varepsilon_{ij}+\mu_{ij}$ and $p+q$ should satisfy equation (1.13).

Now we return to the problem (1.1)—(1.5). It is clear that (1.17)—(1.21) with constraint (1.5) are equivalent to it if we replace f_i in equation (1.18) with $f_i-u_ju_{i,j}$. By the same consideration, we may use (1.23), (1.24) to solve the problem (1.1)—(1.5) if we replace the constraint (1.13) with

$$2\nu\varepsilon_{ij,j}-p_{,i}+f_i-u_ju_{i,j}=0. \tag{1.25}$$

But the unknown u_i does not appear in (1.23), (1.24), and so (1.25) is not a constraint. Here we use an approximate approach. By some definite interpolation formula we extend function $\bar{u}$ from $\bigcup_k\partial\Omega_k$ to the whole domain $\bar{\Omega}$, which is denoted by $\mathscr{I}\bar{v}$; then we replace u in (1.25) with $\mathscr{I}\bar{u}$. Since $\mathscr{I}\bar{u}$ is different from u, the new problem is no longer equivalent to (1.1)—(1.5). However, since replacement of the true solutions by some interpolation functions is the basic idea of the finite element method, the approach used here is quite reasonable.

From now on we use the simpler notation u_i for $\bar{v}_i$ and consider problem (1.23)—(1.25), (1.5). We will consider the linear case, that is Stokes problem, in Section 2, and nonlinear case in Section 3.

For the finite element scheme, ε_{ij}, p, u_i are expressed in terms of polynomials. (1.23)—(1.25), (1.5) is a nonlinear algebraic system, which can be solved by some effective methods. For instance, if we use the Newton–Rapson algorithm and let $\Delta\varepsilon_{ij}=\varepsilon_{ij}^{(m+1)}-\varepsilon_{ij}^{(m)}$, $\Delta p=p^{(m+1)}-p^{(m)}$, $\Delta u_i=u_i^{(m+1)}-u_i^{(m)}$, then we get a linear algebraic system for $\Delta\varepsilon_{ij}$, Δp, Δu_i as follows:

$$\int_\Omega 2\nu\Delta\varepsilon_{ij}\mu_{ij}\,dx-\sum_k\int_{\partial\Omega_k}(2\nu\mu_{ij}-q\delta_{ij})n_j\Delta u_i\,dx=0,\quad\forall\mu_{ij},\ \forall q,$$

$$\sum_k\int_{\partial\Omega_k}(2\nu\Delta\varepsilon_{ij}-\Delta p\delta_{ij})n_jv_i\,dx=0,\quad\forall v_i,$$

$$2\nu\Delta\varepsilon_{ij,j}-\Delta p_{,i}-\Delta u_ju_{i,j}^{(m)}-u_j^{(m)}\Delta u_{i,j}=-r,$$

$$\Delta u(x)=0,\quad x\in\partial\Omega,$$

where

$$r=2\nu\varepsilon_{ij,j}^{(m)}-p_{,i}^{(m)}+f_i-u_j^{(m)}u_{i,j}^{(m)}.$$

§ 2. The Stokes Problem

For linear equations which are not elliptic, the following two lemmas are basic, which are the generalization of the well known Lax-Milgram theorem, Brezzi's theory[6] and Cea's lemma[7].

Let X and Y be two real Hilbert spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, X' and Y' be their dual spaces with norms $\|\cdot\|_{X'}$ and $\|\cdot\|_{Y'}$ respectively, and $a(\cdot,\cdot)$: $X\times X\to R$, $b(\cdot,\cdot)$: $X\times Y\to R$ be two continuous bilinear forms with norms

$$\|a\| = \sup_{\substack{u,v\in X\\ u,v\neq 0}} \frac{a(u,v)}{\|u\|_X\|v\|_X}, \quad \|b\| = \sup_{\substack{v\in X\\ \mu\in Y\\ v\neq 0,\mu\neq 0}} \frac{b(v,\mu)}{\|v\|_X\|\mu\|_Y}.$$

For $\psi\in Y'$, we set

$$V(\psi)=\{v\in X;\ b(v,\mu)=\langle\psi,\mu\rangle,\ \forall\mu\in Y\}.$$

Then we have

Lemma 1. *If*

1° *for any* $\chi\in X'$, $\psi\in Y'$, *the problem: to find* $u\in V(\psi)$ *such that*

$$a(u,v)=\langle\chi,v\rangle,\quad \forall v\in V(0),$$

has a unique solution u, *and there exists a constant* α, *such that*

$$\|u\|_X\leqslant\alpha(\|\chi\|_{X'}+\|\psi\|_{Y'}),$$

2° *there exists a constant* $\beta>0$, *such that*

$$\sup_{\substack{v\in X\\ v\neq 0}}\frac{b(v,\mu)}{\|v\|_X}\geqslant\beta\|\mu\|_Y,\quad \forall\mu\in Y,$$

then the problem: to find $(u,\lambda)\in X\times Y$ *such that*

$$a(u,v)+b(v,\lambda)=\langle\chi,v\rangle,\quad \forall v\in X,$$
$$b(u,\mu)=\langle\psi,\mu\rangle,\ \forall\mu\in Y,$$

has a unique solution for any $\chi\in X'$ *and* $\psi\in Y'$. *Moreover, there exists a constant* C, *which depends on* α, β, $\|a\|$ *only, such that the solution satisfies*

$$\|u\|_X+\|\lambda\|_Y\leqslant C(\|\chi\|_{X'}+\|\psi\|_{Y'}).$$

The proof of this lemma is easy and so is omitted. We will always use C as constants which are not necessarily the same.

We consider the approximate solution of the problem

$$b(u,\mu)=\langle\psi,\mu\rangle,\quad \forall\mu\in Y. \tag{2.1}$$

Let X_h, Y_h denote closed subspaces of X and Y respectively.

Lemma 2. *If the problem: to find* $u_h\in X_h$, *such that*

$$b(u_h,\mu)=\langle\psi,\mu\rangle,\quad \forall\mu\in Y_h, \tag{2.2}$$

has a unique solution u_h *for any* $\psi\in Y'$, *and there exists a constant* γ *such that*

$$\|u_h\|_X\leqslant\gamma\|\psi\|_{Y'},$$

and if u *is a solution of problem* (2.1), *then*

$$\|u-u_h\|_X\leqslant(\gamma\|b\|+1)\inf_{v\in X_h}\|u-v\|_X.$$

Proof. Because Y_h is a subspace of Y, u is a solution of (2.1),

$$b(u,\mu)=\langle\chi,\mu\rangle,\quad \forall\mu\in Y_h;$$

in virtue of (2.2),

$$b(u-u_h,\mu)=0,\quad \forall\mu\in Y_h.$$

Taking an arbitrary $v\in X_h$, we obtain

$$b(v-u_h,\mu)=b(v-u,\mu),\quad \forall\mu\in Y_h. \tag{2.3}$$

Set $b(v-u,\mu)=\langle\phi,\mu\rangle$, where $\phi\in Y'$. Then

$$\|\phi\|_{Y'}\leqslant\|b\|\cdot\|v-u\|_X.$$

By the assumption of this lemma and (2.3),

$$\|v-u_h\|_X \leqslant \gamma \|\phi\|_{Y'} \leqslant \gamma \|b\| \cdot \|v-u\|_X.$$

Hence
$$\|u-u_h\|_X \leqslant \|u-v\|_X + \|v-u_h\|_X \leqslant (\gamma\|b\|+1)\|u-v\|_X.$$

Then the conclusion follows.

In order to apply the above results to the Stokes problem, we introduce the following spaces. Let $\Omega \subset R^d$ be a bounded domain with Lipschitz continuous boundary. We define a space as

$$E(\Omega)=\{(\varepsilon, p);\ \varepsilon_{ij}=\varepsilon_{ji},\ \varepsilon_{ij}\in L^2(\Omega),\ p\in L^2(\Omega),$$
$$2\nu\varepsilon_{ij,j}-p_{,i}\in L^{4/3}(\Omega)\},$$

the norm of which is defined by

$$\|(\varepsilon, p)\|_{E(\Omega)}=\Big(\sum_{i,j}\|\varepsilon_{ij}\|^2_{L^2(\Omega)}+\|p\|^2_{L^2(\Omega)}\Big)^{1/2}+\|2\nu\varepsilon_{ij,j}-p_{,i}\|_{L^{4/3}(\Omega)}.$$

Then $E(\Omega)$ is a Banach space.

Define a subspace of $E(\Omega)$ as

$$E_0(\Omega)=\{(\varepsilon, p)\in E(\Omega);\ 2\nu\varepsilon_{ij,j}-p_{,i}=0\}.$$

It is easy to see that $E_0(\Omega)$ is a Hilbert space under the above norm.

There are some basic properties about the spaces $E(\Omega)$ and $E_0(\Omega)$:

1° $(C^\infty(\overline{\Omega}))^{\frac{1}{2}d(d+1)+1}$ is dense in $E(\Omega)$;

2° the mapping $(\varepsilon, p)\mapsto(2\nu\varepsilon_{ij}-p\delta_{ij})n_j|_{\partial\Omega}$ defined on $(C^\infty(\overline{\Omega}))^{\frac{1}{2}d(d+1)+1}$ can be extended continuously to a linear continuous mapping from $E(\Omega)$ onto $(H^{-1/2}(\partial\Omega))^d$;

3° $\Big(\sum_{i,j}\|\varepsilon_{ij}\|^2_{L^2(\Omega)}+\Big(\int_\Omega p\,dx\Big)^2\Big)^{1/2}$ is an equivalent norm for elements in $E_0(\Omega)$.

The proof of the first two properties is similar to that of the space $H(\mathrm{div};\ \Omega)$ in [4], the key step of which is $H^1(\Omega)\hookrightarrow L^4(\Omega)$ by the Sobolev imbedding theorem. For the last one, readers may consult [2].

Space $E_0(\Omega)$ is decomposed as the direct sum of subspaces:

$$E_0(\Omega)=F_0(\Omega)\oplus F_1(\Omega),$$

where
$$F_0(\Omega)=\Big\{(\varepsilon, p)\in E_0(\Omega);\ \int_\Omega p\,dx=0\Big\}$$
$$F_1(\Omega)=\{(0, p)\in E_0(\Omega);\ p=\text{constant}\}.$$

It is natural to apply $\big(\sum_{i,j}\|\varepsilon_{ij}\|^2_{L^2(\Omega)}\big)^{1/2}$ as the norm of element (ε, p) in $F_0(\Omega)$ by property 3°. We assume that the discretization of Ω in Section 1 is so defined that the boundary $\partial\Omega_k$ of each element satisfies the Lipschitz condition. Then we define the following cartesian product

$$Y=\prod_k F_0(\Omega_k),$$

which is one of the basic spaces for our problem. Another basic space is

$$Z=\{u=(u_k);\ u_k\in H^{\frac{1}{2}}(\partial\Omega_k),\ \forall k\};$$

the norm of an element in Z is defined by

$$\|u\|_Z=\inf\{\|v\|_{(H^1(\Omega))^d};\ v\in(H^1(\Omega))^d,\ v=u,\ \text{as } x\in\bigcup_k\partial\Omega_k\}.$$

We introduce a subspace of Z:

$$Z_0=\{u\in Z;\ u=0,\ \text{as } x\in\partial\Omega\}.$$

Since the solution of our problem with boundary value (1.5) may change a constant hydrostatic pressure, for definiteness, we define the third basic space

$$P=\left\{p_o\in L^2(\Omega);\ p_o \text{ equals to a constant } p_k \text{ on } \Omega_k,\ \forall k,\ \int_\Omega p_o\,dx=0\right\}.$$

We define three bilinear forms:

$$a((\varepsilon,p),(\mu,q))=2\nu\int_\Omega \varepsilon_{ij}\mu_{ij}\,dx \tag{2.4}$$

on $Y\times Y$,

$$b((\varepsilon,p),u)=\sum_k\int_{\partial\Omega_k}(2\nu\varepsilon_{ij}-p\delta_{ij})n_ju_i\,dx \tag{2.5}$$

on $Y\times Z$, and

$$c(p_o,u)=\sum_k p_k\int_{\partial\Omega_k} n_iu_i\,dx \tag{2.6}$$

on $P\times Z$.

We assume that $f\in(L^{4/3}(\Omega))^d$, $\bar{u}_0\in(H^{1/2}(\partial\Omega))^d$. Take an arbitrary $(\tilde{\varepsilon},\tilde{p})\in\prod_k E(\Omega_k)$ such that (1.13) is satisfied on each element Ω_k and $\int_\Omega\tilde{p}\,dx=0$. This is easy because there are $\frac{1}{2}d(d+1)+1$ unknowns in d equations (the technique will be shown in the next section). Then we take an arbitrary $\tilde{u}\in H^1(\Omega)$ such that it satisfies equation (1.5). If (ε,p,u) is a solution of problem (1.23), (1.24), let

$$\varepsilon_0=\varepsilon-\tilde{\varepsilon},\quad p_o|_{\Omega_k}=\frac{1}{\operatorname{meas}\Omega_k}\int_{\Omega_k}(p-\tilde{p})\,dx,\quad p_0=p-\tilde{p}-p_o,$$
$$u_0=u-\tilde{u},$$

then $(\varepsilon_0,p_0)\in Y$, $u_0\in Z_0$, and $p_o\in P$. Substituting $\varepsilon=\varepsilon_0+\tilde{\varepsilon}$, $p=p_0+p_o+\tilde{p}$, $u=u_0+\tilde{u}$ into (1.23), (1.24) and removing each term involving $\tilde{\varepsilon}$, $\tilde{p}$, $\tilde{u}$ to the right hand side, we obtain the following equations for unknowns $(\varepsilon_0,p_0)\in Y$, $u_0\in Z_0$ and $p_o\in P$:

$$a((\varepsilon_0,p_0),(\mu,q))-b((\mu,q),u_0)=\langle\chi,(\mu,q)\rangle,\quad\forall(\mu,q)\in Y, \tag{2.7}$$
$$b((\varepsilon_0,p_0),v)-c(p_o,v)=\langle\psi,v\rangle,\ \forall v\in Z_0, \tag{2.8}$$
$$c(q_o,u_0)=\langle\phi,q_o\rangle,\quad\forall q_o\in P, \tag{2.9}$$

where $\chi\in Y'$, $\psi=Z_0'$, $\phi\in P'$.

We are going to prove the following:

Theorem 1. *Let $V(\phi)=\{u\in Z_0;\ c(q_o,u)=\langle\phi,q_o\rangle,\ \forall q_o\in P\}$. If there exist constants $\alpha>0$, $\beta>0$ such that*

$$\sup_{\substack{(\mu,q)\in Y\\(\mu,q)\neq0}}\frac{b((\mu,q),v)}{\|(\mu,q)\|_Y}\geq\alpha\|v\|_Z,\quad\forall v\in V(0), \tag{2.10}$$

$$\sup_{\substack{v\in Z_0\\v\neq0}}\frac{c(q_o,v)}{\|v\|_Z}\geq\beta\|q_o\|_P,\quad\forall q_o\in P, \tag{2.11}$$

then problem (2.7)—(2.9) has a unique solution, and there is a constant C which depends on ν, α, β, and $\|a\|$, $\|b\|$, $\|c\|$ only, such that

$$\|(\varepsilon_0,p_0)\|_Y+\|u_0\|_Z+\|p_o\|_P\leq C(\|\chi\|_{Y'}+\|\psi\|_{Z_0'}+\|\phi\|_{P'}).$$

Proof. By (2.4) we have

$$a((\varepsilon,p),(\varepsilon,p))=2\nu\|(\varepsilon,p)\|_Y^2,\quad\forall(\varepsilon,p)\in Y.$$

Hence owing to the Lax-Milgram theorem, the problem

$$a((\varepsilon,p),(\mu,q))=\langle\chi,(\mu,q)\rangle,\quad\forall(\mu,q)\in Y,$$

for any $\chi\in Y'$ has a unique solution (ε,p), and satisfies

$$\|(\varepsilon, p)\|_Y \leqslant \frac{1}{2\nu}\|\chi\|_{Y'}.$$

This is also true if we replace Y with any closed subspace of Y. By Brezzi's theory and (2.11), there is a $\tilde{u}_0 \in Z_0$, such that

$$c(q_c, \tilde{u}_0) = \langle \phi, q_c \rangle, \quad \forall q_c \in P$$

and

$$\|\tilde{u}_0\|_Z \leqslant \frac{1}{\beta}\|\phi\|_{P'}.$$

Set $u_0 = v_0 + \tilde{u}_0$. According to (2.10) and Lemma 1, the problem

$$a((\varepsilon_0, p_0), (\mu, q)) - b((\mu, q), v_0) = \langle \chi, (\mu, q)\rangle + b((\mu, q), \tilde{u}_0), \quad \forall (\mu, q) \in Y,$$

$$b((\varepsilon_0, p_0), v) = \langle \psi, v \rangle, \quad \forall v \in V(0)$$

has a unique solution $(\varepsilon_0, p_0) \in Y$, $v_0 \in V(0)$, and

$$\|(\varepsilon_0, p_0)\|_Y + \|v_0\|_Z \leqslant C\{\|\chi\|_{Y'} + \|\psi\|_{Z_0'} + \|\phi\|_{P'}\}.$$

That is, there is a unique solution $(\varepsilon_0, p_0) \in Y$, $u_0 \in V(\phi)$, which satisfies

$$a((\varepsilon_0, p_0), (\mu, q)) - b((\mu, q), u_0) = \langle \chi, (\mu, q)\rangle, \quad \forall (\mu, q) \in Y, \tag{2.12}$$

$$b((\varepsilon_0, p_0), v) = \langle \psi, v \rangle, \quad \forall v \in V(0),$$

and

$$\|(\varepsilon_0, p_0)\|_Y + \|u_0\|_Z \leqslant C\{\|\chi\|_{Y'} + \|\psi\|_{Z_0'} + \|\phi\|_{P'}\}.$$

We define a bilinear form on $(Y \times Z_0) \times (Y \times Z_0)$ as

$$A((\varepsilon_0, p_0, u_0), (\mu, q, v)) = a((\varepsilon_0, p_0), (\mu, q)) - b((\mu, q), u_0) - b((\varepsilon_0, p_0), v).$$

Then (2.12) can be rewritten as

$$A((\varepsilon_0, p_0, u_0), (\mu, q, v)) = \langle \chi, (\mu, q)\rangle - \langle \psi, v\rangle, \quad \forall (\mu, q, v) \in Y \times V(0),$$

and problem (2.7)—(2.9) can be rewritten as

$$A((\varepsilon_0, p_0, u_0), (\mu, q, v)) + c(p_c, v) = \langle \chi, (\mu, q)\rangle - \langle \psi, v\rangle, \ \forall (\mu, q, v) \in Y \times Z_0,$$

$$c(q_c, u_0) = \langle \phi, q_c \rangle, \quad \forall q_c \in P.$$

Once again we apply Lemma 1 and the conclusion follows.

Problem (2.7)—(2.9) is equivalent to the original boundary value problem of Stokes flow, so there is a unique solution[3]. On the other hand, it is not hard to verify conditions (2.10) and (2.11) directly[3].

We discuss the finite element approximation of (2.7)—(2.9). Let $Y_h \subset Y$, $Z_h \subset Z_0$, $P_h \subset P$ be closed subspaces, where h is the parameter of discretization. (As a matter of fact, $P_h = P$ and it is already of finite dimensions.) The approximate problem of (2.7)—(2.9) will be: to find $(\varepsilon_h, p_h, u_h, p_{ch}) \in Y_h \times Z_h \times P_h$, such that

$$a((\varepsilon_h, p_h), (\mu, q)) - b((\mu, q), u_h) = \langle \chi, (\mu, q)\rangle, \quad \forall (\mu, q) \in Y_h, \tag{2.13}$$

$$b((\varepsilon_h, p_h), v) - c(p_{ch}, v) = \langle \psi, v \rangle, \quad \forall v \in Z_h, \tag{2.14}$$

$$c(q_c, u_h) = \langle \phi, q_c \rangle, \forall q_c \in P_h. \tag{2.15}$$

We define a bilinear form

$$B((\varepsilon_0, p_0, u_0, p_c), (\mu, q, v, q_c)) = A((\varepsilon_0, p_0, u_0), (\mu, q, v)) + c(p_c, v) + c(q_c, u_0)$$

on $(Y \times Z_0 \times P) \times (Y \times Z_0 \times P)$. By Theorem 1 and Lemma 2, it is easy to prove (the bilinear form $b(\cdot, \cdot)$ in Lemma 2 is $B(\cdot, \cdot)$ now):

Theorem 2. *Let* $\tilde{V}(0)=\{u\in Z_h;\ c(q_o, u)=0,\ \forall q_o\in P_h\}$. *If there exist constants* $\tilde{\alpha}>0$, $\tilde{\beta}>0$ *independent of* h, *such that*

$$\sup_{\substack{(\mu,q)\in Y_h\\(\mu,q)\neq 0}}\frac{b((\mu, q), v)}{\|(\mu, q)\|_Y}\geqslant\tilde{\alpha}\|v\|_Z,\quad \forall v\in\tilde{V}(0), \tag{2.16}$$

$$\sup_{\substack{v\in Z_h\\ v\neq 0}}\frac{c(q_o, v)}{\|v\|_Z}\geqslant\tilde{\beta}\|q_o\|_P,\quad \forall q_o\in P_h, \tag{2.17}$$

then problem (2.13)—(2.15) *has a unique solution, and there is a constant* C *independent of* h, *such that*

$$\|(\varepsilon_h, p_h)\|_Y+\|u_h\|_Z+\|p_{oh}\|_P\leqslant C(\|\chi\|_{Y'}+\|\psi\|_{Z_0'}+\|\phi\|_{P'}), \tag{2.18}$$

and the following error estimation holds:

$$\|(\varepsilon_0-\varepsilon_h, p_0-p_h)\|_Y+\|u_0-u_h\|_Z+\|p_o-p_{oh}\|_P$$
$$\leqslant C\{\inf_{(\mu,q)\in Y_h}\|(\varepsilon_0-\mu, p_0-q)\|_Y+\inf_{v\in Z_h}\|u_0-v\|_Z+\inf_{q_o\in P_h}\|p_o-q_o\|_P\}. \tag{2.19}$$

We have proved in [2] that condition (2.16) is nothing but a local condition, it holds if and only if the so-called rank condition holds in each element. And condition (2.17) is the weakest form of the Babuska–Brezzi condition for Navier–Stokes equations, because $p_o\in P$ is a piecewise constant function. It is known that quite a few kinds of elements satisfy (2.17), for instance the quadratic six nodes triangular elements. Therefore, if we use quadratic six-nodes interpolation formulae for velocity, then the second term on the right hand side of (2.19) would be $O(h^2)$, and if we take the space Y_h "sufficiently large", then any degree of precision can be achieved for the first term, and the scale of space Y_h does not affect the order of the algebraic system to be solved, because when we solve (2.13)—(2.15), the variables (ε, p) are eliminated in advance. The third term is zero in our case.

We have given some kinds of elements which satisfy conditions (2.16), (2.17)[1,3].

§ 3. The Navier–Stokes Problem

According to the notations in Section 2, the problem is: to find $(\varepsilon_0, p_0, u_0, p_o)\in Y\times Z_0\times P$, such that

$$B((\varepsilon_0, p_0, u_0, p_o), (\mu, q, v, q_o))=\langle\chi, (\mu, q)\rangle-\langle\psi, v\rangle+\langle\phi, q_o\rangle,$$
$$\forall(\mu, q, v, q_o)\in Y\times Z_0\times P, \tag{3.1}$$

or we replace $Y\times Z_0\times P$ by $Y_h\times Z_h\times P_h$ and consider the corresponding problem $(3.1)_h$, but now χ and ψ take quite complicated nonlinear form, with the expressions

$$\langle\chi, (\mu, q)\rangle=-\int_\Omega 2\nu\tilde{\varepsilon}_{ij}\mu_{ij}\,dx+\sum_k\int_{\partial\Omega_k}(2\nu\mu_{ij}-q\delta_{ij})n_j\tilde{u}_i\,dx, \tag{3.2}$$

$$\langle\psi, v\rangle=-\sum_k\int_{\partial\Omega_k}(2\nu\tilde{\varepsilon}_{ij}-\tilde{p}\delta_{ij})n_jv_i\,dx, \tag{3.3}$$

where $(\tilde{\varepsilon}, \tilde{p})$ satisfies

$$2\nu\tilde{\varepsilon}_{ij,j}-\tilde{p}_{,i}-u_ju_{i,j}+f_i=0 \tag{3.4}$$

on each element Ω_k. Here $\tilde{u}\in(H^1(\Omega))^d$ satisfies equation (1.5). We can take $\tilde{u}$ such that

$$\|\tilde{u}\|_{(H^1(\Omega))^d}\leqslant C\|\bar{v}_0\|_{(H^{1/2}(\partial\Omega))^d}. \tag{3.5}$$

The function u in (3.4) is an extension of u from $\bigcup_k\partial\Omega_k$ to the whole domain $\bar{\Omega}$. We can also define a suitable interpolation operator $\mathscr{I}$: $Z\to(H^1(\Omega))^d$ such that

$$\|\mathscr{I}u\|_{(H^1(\Omega))^d}\leqslant C\|u\|_Z. \tag{3.6}$$

We take $\mathscr{I}u$ as the function u in (3.4). Now we show a technique to solve $(\tilde{\varepsilon},\tilde{p})$ from (3.4), such that the estimation

$$\|(\tilde{\varepsilon},\tilde{p})\|_{E(\Omega_k)}\leqslant C\sum_i\|u_ju_{i,j}-f_i\|_{L^{4/3}(\Omega_k)} \tag{3.7}$$

holds. When $d=2$, we may set $\tilde{p}=0$, $\tilde{\varepsilon}_{11}=-\tilde{\varepsilon}_{22}$. Then (3.4) becomes an elliptic system with the unknowns $\tilde{\varepsilon}_{11}$ and $\tilde{\varepsilon}_{12}$. We can solve it on the whole plane by Fourier transform and get $(\tilde{\varepsilon},\tilde{p})$. When $d=3$, we may set $\tilde{\varepsilon}_{13}=0$, $\tilde{p}=0$ and solve the first two equations by the above technique; then we set $f_2=0$ and solve a similar elliptic system on (x_2,x_3) plane. Finally we sum them up and get $(\tilde{\varepsilon},\tilde{p})$. In practice, we may assume that f and u are polynomials and let $(\tilde{u},\tilde{p})$ be polynomials too. Then we determine some coefficients and obtain $(\tilde{\varepsilon},\tilde{p})$.

By (3.2),

$$\begin{aligned}|\langle\chi,(\mu,q)\rangle|&\leqslant 2\nu\|\tilde{\varepsilon}_{ij}\|_{L^2(\Omega)}\|\mu_{ij}\|_{L^2(\Omega)}+\sum_k\|(\mu,q)\|_{F_k(\Omega_k)}\|\tilde{u}\|_{(H^{1/2}(\partial\Omega))^d}\\&\leqslant 2\nu\Big(\sum_{i,j}\|\tilde{\varepsilon}_{ij}\|^2_{L^2(\Omega)}\Big)^{1/2}\Big(\sum\|\mu_{ij}\|^2_{L^2(\Omega)}\Big)^{1/2}\\&\quad+\|(\mu,q)\|_Y\|\tilde{u}\|_{(H^1(\Omega))^d}.\end{aligned}$$

Then by (3.5) and the definition of Y we obtain

$$\|\chi\|_{Y'}\leqslant 2\nu\Big(\sum_{i,j}\|\tilde{\varepsilon}_{ij}\|^2_{L^2(\Omega)}\Big)^{1/2}+C\|v_0\|_{(H^{1/2}(\partial\Omega))^d}. \tag{3.8}$$

By (3.3) and property 2° of spaces $E(\Omega_k)$ in Section 2,

$$|\langle\psi,v\rangle|\leqslant C\sum_k\|(\tilde{\varepsilon},\tilde{p})\|_{E(\Omega_k)}\|v\|_{(H^1(\Omega_k))^d}.$$

By (3.7) we get

$$|\langle\psi,v\rangle|\leqslant C\sum_k\sum_i\|u_ju_{i,j}-f_i\|_{L^{4/3}(\Omega_k)}\cdot\|v\|_{(H^1(\Omega_k))^d}.$$

Hence

$$\begin{aligned}\|\psi\|_{Z'}&\leqslant C\Big\{\sum_k\sum_i\|u_ju_{i,j}-f_i\|^2_{L^{4/3}(\Omega_k)}\Big\}^{1/2}\\&\leqslant C\Big\{\max_k\sum_i\|u_ju_{i,j}-f_i\|^{2/3}_{L^{4/3}(\Omega_k)}\cdot\sum_k\sum_i\|u_ju_{i,j}-f_i\|^{4/3}_{L^{4/3}(\Omega_k)}\Big\}^{1/2}\\&\leqslant C\sum_i\|u_ju_{i,j}-f_i\|^{1/3}_{L^{4/3}(\Omega)}\cdot\Big\{\sum_k\sum_i\|u_ju_{i,j}-f_i\|^{4/3}_{L^{4/3}(\Omega_k)}\Big\}^{1/2}\\&\leqslant C\sum_i\|u_ju_{i,j}-f_i\|_{L^{4/3}(\Omega)};\end{aligned}$$

but $H^1(\Omega)\subsetneq L^4(\Omega)$; so

$$\|\psi\|_{Z'}\leqslant C\{\|u\|^2_{(H^1(\Omega))^d}+\|f\|_{(L^{4/3}(\Omega))^d}\}. \tag{3.9}$$

Similarly, we can estimate the right hand side of (3.8) and get

$$\|\chi\|_{Y'}\leqslant C\{\|u\|^2_{(H^1(\Omega))^d}+\|f\|_{(L^{4/3}(\Omega))^d}+\|u_0\|_{(H^{1/2}(\partial\Omega))^d}\}. \tag{3.10}$$

We consider two closed neighborhoods of the origin in spaces $(L^{4/3}(\Omega))^d\times(H^{1/2}(\partial\Omega))^d$ and $(H^1(\Omega))^d$:

$$N(\delta)=\{(f, \bar{u}_0)\in (L^{4/3}(\Omega))^d\times (H^{1/2}(\partial\Omega))^d;\ \|f\|_{(L^{4/3}(\Omega))^d}+\|\bar{u}_0\|_{(H^{1/2}(\partial\Omega))^d}\leqslant\delta\},$$

$$B(\eta)=\{u\in (H^1(\Omega))^d;\ \|u\|_{(H^1(\Omega))^d}\leqslant\eta\}.$$

Then we prove the existence and uniqueness theorem.

Theorem 3. *There exist constants $\eta>0$ and $\delta>0$, such that if $(f, \bar{u}_0)\in N(\delta)$, then the solution of problem (3.1) exists and is unique with the constraint $\mathscr{I}(u_0+\tilde{u})\in B(\eta)$. And if the hypotheses of Theorem 2 hold, then the same is true for problem $(3.1)_h$, where η and δ are independent of h.*

Proof. We prove for problem (3.1); the proof for problem $(3.1)_h$ is the same.

Taking an arbitrary $u^*\in (H^1(\Omega))^d$ and replacing u in (3.4) with u^*, we get a linear Stokes problem. The solution of (3.1) in this case is denoted by $(\varepsilon_0, p_0, u_0, p_o)$. Let $u=\mathscr{I}(u_0+\tilde{u})$. Then the mapping $u^*\mapsto u$ defines a nonlinear operator $F:(H^1(\Omega))^d\to (H^1(\Omega))^d$.

By Theorem 1 and (3.5), (3.6), (3.9), (3.10) we obtain

$$\|u\|_{(H^1(\Omega))^d}\leqslant C\{\|u^*\|^2_{(H^1(\Omega))^d}+\|f\|_{(L^{4/3}(\Omega))^d}+\|\bar{u}_0\|_{(H^{1/2}(\partial\Omega))^d}\}. \tag{3.11}$$

We take $\eta\leqslant\frac{1}{2C}, \delta=\frac{\eta}{2C}$. Then F maps $B(\eta)$ into $B(\eta)$. Suffice it to prove F is a contraction.

Suppose v^* and $w^*\in B(\eta)$ and $v=F(v^*)$, $w=F(w^*)$. Let $u^*=v^*-w^*$, $u=v-w$. Then by Theorem 1 and (3.6), (3.9), (3.10) we get

$$\|u\|_{(H^1(\Omega))^d}\leqslant C\sum_i\|v_j^*v_{i,j}^*-w_j^*w_{i,j}^*\|_{L^{4/3}(\Omega)}=C\sum_i\|v_j^*u_{i,j}^*+u_j^*w_{i,j}^*\|_{L^{4/3}(\Omega)}$$

$$\leqslant C\{\|v^*\|_{(H^1(\Omega))^d}\cdot\|u^*\|_{(H^1(\Omega))^d}+\|u^*\|_{(H^1(\Omega))^d}\cdot\|w^*\|_{(H^1(\Omega))^d}\},$$

that is

$$\|u\|_{(H^1(\Omega))^d}\leqslant C\eta\|u^*\|_{(H^1(\Omega))^d}.$$

We take η such that $C\eta<1$. Then the equation

$$u=F(u)$$

has a unique solution in the ball $B(\eta)$.

Finally we consider the error estimation. First we discuss the difference between the solutions of (3.1) and $(3.1)_h$.

Lemma 3. *Under the hypotheses of Theorem 2, there exists a constant δ_1, $0<\delta_1\leqslant\delta$, such that if $(f, \bar{u}_0)\in N(\delta_1)$, then (2.19) holds for the solutions $(\varepsilon_0, p_0, u_0, p_o)$ of (3.1) and $(\varepsilon_h, p_h, u_h, p_{oh})$ of $(3.1)_h$.*

Proof. Let $u=\mathscr{I}(u_0+\tilde{u})$. We substitute it in (3.4), and get $\tilde{\varepsilon}_{ij}$ and $\tilde{p}$. Then we get χ, ψ by (3.2), (3.3). With the same χ, ψ and ϕ, we solve (3.1) to obtain $(\varepsilon_0, p_0, u_0, p_o)$, and solve $(3.1)_h$ to obtain $(\varepsilon_h^*, p_h^*, u_h^*, p_{oh}^*)$. Each solution exists and is unique because they are linear problems.

By (2.19),

$$\|(\varepsilon_0-\varepsilon_h^*, p_0-p_h^*)\|_Y+\|u_0-u_h^*\|_Z+\|p_o-p_{oh}^*\|_P$$

$$\leqslant C\{\inf_{(\mu,q)\in Y_h}\|(\varepsilon_0-\mu, p_0-q)\|_Y+\inf_{v\in Z_h}\|u_0-v\|_Z+\inf_{q\in P_h}\|p_o-q_o\|_P\}. \tag{3.12}$$

$(\varepsilon_h^*-\varepsilon_h, p_h^*-p_h, u_h^*-u_h, p_{oh}^*-p_{oh})$ is a solution of the linear problem (2.13)—(2.15) with some χ, ψ and $\phi=0$. Applying (3.2), (3.3) and with some computation we get χ and ψ in this case. Then we apply (2.18) and Theorem 3, and get

$$\begin{aligned}&\|(\varepsilon_h^*-\varepsilon_h,\ p_h^*-p_h)\|_Y+\|u_h^*-u_h\|_Z+\|p_{ch}^*-p_{ch}\|_P\\&\leqslant C\{\|\mathscr{I}(u_0+\tilde{u})\|_{(H^1(\Omega))^d}\cdot\|\mathscr{I}(u_0-u_h)\|_{(H^1(\Omega))^d}\\&\quad+\|\mathscr{I}(u_h+\tilde{u})\|_{(H^1(\Omega))^d}\cdot\|\mathscr{I}(u_0-u_h)\|_{(H^1(\Omega))^d}\}\\&\leqslant C\eta\|\mathscr{I}(u_0-u_h)\|_{(H^1(\Omega))^d}\leqslant C\eta\|u_0-u_h\|_Z.\end{aligned}\tag{3.13}$$

We take η small enough such that $C\eta<1$, and $\delta_1=\min\left(\delta,\frac{\eta}{2C}\right)$. Then the conclusion follows from (3.12) and (3.13).

Suppose σ^t, ε^t, u^t, p^t are the exact solutions of problem (1.1)—(1.5). We estimate the difference between it and the approximate solution by $(3.1)_h$. If we replace $u_iu_{i,j}$ in (3.4) with $u_i^tu_{i,j}^t$, then the solution of problem (3.1) is exact, which is denoted by $(\varepsilon_0^t,\ p_0^t,\ u_0^t,\ p_c^t)$.

Theorem 4. *Under the hypotheses of Theorem 2, there exists a constant* δ_2, $0<\delta_2\leqslant\delta_1$, *such that if* $(f,\ \bar{u}_0)\in N(\delta_2)$, *the solution* $(\varepsilon_h,\ p_h,\ u_h,\ p_{ch})$ *of* $(3.1)_h$ *satisfies the following error estimation:*

$$\begin{aligned}&\|(\varepsilon_0^t-\varepsilon_h,\ p_0^t-p_h)\|_Y+\|u_0^t-u_h\|_Z+\|p_c^t-p_{ch}\|_P\\&\leqslant C\{\inf_{(\mu,q)\in Y_h}\|(\varepsilon_0^t-\mu,\ p_0^t-q)\|_Y+\inf_{v\in Z_h}\|u_0^t-v\|_Z\\&\quad+\inf_{q_c\in P_h}\|p_c^t-q_c\|_P+\|u^t-\mathscr{I}u^t\|_{(H^1(\Omega))^d}\},\end{aligned}\tag{3.14}$$

where $\mathscr{I}u^t$ *denotes the interpolation function of the restriction of* u^t *on* $\bigcup_k\partial\Omega_k$.

Proof. By Theorem 1 and (3.6) we have

$$\begin{aligned}&\|(\varepsilon_0^t-\varepsilon_0,\ p_0^t-p_0)\|_Y+\|u_0^t-u_0\|_Z+\|p_c^t-p_c\|_P\\&\leqslant C\{\|u^t\|_{(H^1(\Omega))^d}\|u^t-\mathscr{I}(u_0+\tilde{u})\|_{(H^1(\Omega))^d}+\|\mathscr{I}(u_0+\tilde{u})\|_{(H^1(\Omega))^d}\\&\quad\times\|u^t-\mathscr{I}(u_0+\tilde{u})\|_{(H^1(\Omega))^d}\}\leqslant C\eta\|u^t-\mathscr{I}(u_0+\tilde{u})\|_{(H^1(\Omega))^d}\\&\leqslant C\eta\{\|u^t-\mathscr{I}u^t\|_{(H^1(\Omega))^d}+\|\mathscr{I}(u^t-u_0-\tilde{u})\|_{(H^1(\Omega))^d}\}\\&\leqslant C\eta\{\|u^t-\mathscr{I}u^t\|_{(H^1(\Omega))^d}+\|u_0^t-u_0\|_Z\},\end{aligned}$$

provided $(f,\ \bar{u}_0)\in N(\delta)$. Hence if η is so small that $C\eta<1$, then we obtain

$$\|(\varepsilon_0^t-\varepsilon_0,\ p_0^t-p_0)\|_Y+\|u_0^t-u_0\|_Z+\|p_c^t-p_c\|_P\leqslant C\eta\|u^t-\mathscr{I}u^t\|_{(H^1(\Omega))^d}.\tag{3.15}$$

Let $\delta_2=\min\left(\delta_1,\ \frac{\eta}{2C}\right)$. By Lemma 3 and (3.15) we get

$$\begin{aligned}&\|(\varepsilon_0^t-\varepsilon_h,\ p_0^t-p_h)\|_Y+\|u_0^t-u_h\|_Z+\|p_c^t-p_{ch}\|_P\\&\leqslant C\{\inf_{(\mu,q)\in Y_h}\|(\varepsilon_0-\mu,\ p_0-q)\|_Y+\inf_{v\in Z_h}\|u_0-v\|_Z\\&\quad+\inf_{q_c\in P_h}\|p_c-q_c\|_P+\eta\|u^t-\mathscr{I}u^t\|_{(H^1(\Omega))^d}\},\end{aligned}\tag{3.16}$$

but

$$\inf_{(\mu,q)\in Y_h}\|(\varepsilon_0-\mu,\ p_0-q)\|_Y\leqslant\inf_{(\mu,q)\in Y_h}\|(\varepsilon_0^t-\mu,\ p_0^t-q)\|_Y+\|(\varepsilon_0^t-\varepsilon_0,\ p_0^t-p_0)\|_Y,$$

$$\inf_{v\in Z_h}\|u_0-v\|_Z\leqslant\inf_{v\in Z_h}\|u_0^t-v\|_Z+\|u_0^t-u_0\|_Z,$$

$$\inf_{q_c\in P_h}\|p_c-q_c\|_P\leqslant\inf_{q_c\in P_h}\|p_c^t-q_c\|_P+\|p_c^t-p_c\|_P.$$

Substituting them into (3.16) and applying (3.15), we obtain (3.14).

Remark. To consider the influence of viscosity ν, we replace the variables u_i, ε_{ij} in equations (1.1)—(1.4) with new variables νu_i, $\nu\varepsilon_{ij}$, and accordingly $\bar{u}_0(x)$ in equation (1.5) with $\nu\bar{u}_0(x)$. For simplification these are still denoted by u_i, ε_{ij}, and

$\bar{u}_0(x)$ respectively. Then, equation (1.1) becomes

$$-\sigma_{ij,j}+\frac{1}{\nu^2}u_j u_{i,j}=f_i \tag{1.1'}$$

and ν becomes 1 in equation (1.2). After that the linearized Stokes problem is independent of ν, and so is the constant C in Theorem 2. The conclusions for Navier-Stokes equations depend on ν because equation (1.1)′ does. Now, inequality (3.11) becomes

$$\|u\|_{(H^1(\Omega))^d}\leqslant C\left\{\frac{1}{\nu^2}\|u^*\|^2_{(H^1(\Omega))^d}+\|f\|_{(L^{4/3}(\Omega))^d}+\|\bar{u}_0\|_{(H^{1/2}(\partial\Omega))^d}\right\},$$

where it should be noticed that u, u^*, and u_0 are all new variables. If we return to the primitive variables, it is easy to see that the inequalities included in Theorem 3 would be

$$\|f\|_{(L^{4/3}(\Omega))^d}+\nu\|\bar{u}_0\|_{(H^{1/2}(\partial\Omega))^d}\leqslant\nu^2\delta,$$

$$\|\mathscr{I}(u_0+\tilde{u})\|_{(H^1(\Omega))^d}\leqslant\nu\eta,$$

and the inequality included in Theorem 4 would be

$$\|f\|_{(L^{4/3}(\Omega))^d}+\nu\|\bar{u}_0\|_{(H^{1/2}(\partial\Omega))^d}\leqslant\nu^2\delta_2,$$

where δ, η, and δ_2 are independent of ν.

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