NONNEGATIVE INTERPOLATION*

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Abstract

The problem discussed in this paper is to determine a nonnegative interpolating polynomial which takes the prescribed nonnegative values $y_0, y_1, \cdots, y_n$ at given distinct points $x_0, x_1, \cdots, x_n$:

$$p(x_i)=y_i, \quad i=0, 1, \cdots, n.$$

This paper shows: (1) $2n$ is the least number of m such that there exists a polynomial $p\in P_m^+$, the set of all nonnegative polynomials of degree $\leqslant m$, satisfying the above equations for any choice of $y_i\geqslant 0$. (2) The above equations have a unique solution in P_{2n}^+ if and only if at most one of the y_i's is nonzero.

1. Introduction

Let $x_0, x_1, \cdots, x_n$ be $n+1$ distinct points in the interval $[a, b]$ satisfying

$$a\leqslant x_0<x_1<\cdots<x_n\leqslant b.$$

Let $y_0, y_1, \cdots, y_n$ be any prescribed values satisfying $y_i\geqslant 0$, $i=0, 1, \cdots, n$. Let P_m denote the set of all polynomials of degree equal to m or less. In this paper we consider the following problem: to find a nonnegative polynomial p such that

$$p(x_i)=y_i, \quad i=0, 1, \cdots, n. \tag{1}$$

That is to say, if we denote by P_m^+ the set of all nonnegative polynomials of degree equal to m or less, then our problem is to determine a polynomial $p\in P_m^+$ satisfying the above interpolating conditions.

It is not hard to see that the problem is not necessarily solvable for $m=n$. For example, taking $n=1$, $x_0=a$, $x_1=\frac{1}{2}(a+b)$, $y_0=1$ and $y_1=0$, there does not exist a $p\in P_1^+$ satisfying (1). Therefore we are concerned with the existence of a nonnegative polynomial satisfying (1), in particular, the least number of m for which the problem is always solvable for any choice of $y_i\geqslant 0$, $i=0, 1, \cdots, n$. That is the content of Section 2. Section 3 will discuss uniqueness of such a polynomial.

2. Existence

The main result in this section is as follows.

Theorem 1. *$2n$ is the least number of m such that there exists a polynomial $p\in P_m^+$ satisfying the equations (1) for any choice of $y_i\geqslant 0$, $i=0, 1, \cdots, n$.*

Proof. First, we are going to show that there exists a polynomial $p\in P_{2n}^+$ satisfying (1) for any choice of $y_i\geqslant 0$. To do this, put

$$w(x)=(x-x_0)(x-x_1)\cdots(x-x_n),$$

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$$l_i(x)=\frac{w(x)}{(x-x_i)w'(x_i)},\quad i=0,\ 1,\ \cdots,\ n,$$

here
$$w'(x_i)=\prod_{\substack{j=1\\ j\neq i}}^{n}(x_i-x_j),\quad i=0,\ 1,\ \cdots,\ n.$$

Then
$$p(x)=\sum_{i=0}^{n}y_i l_i^2(x) \tag{2}$$
is such a polynomial.

In fact, it is clear that $p\in P_{2n}^{+}$. On the other hand, by an observation one can see that

$$l_i(x_j)=\begin{cases}1, & j=i,\\ 0, & j\neq i.\end{cases}$$

Hence
$$p(x_i)=y_i,\quad i=0,\ 1,\ \cdots,\ n.$$

We turn now to show the minimality of $m=2n$. To the end let us consider some choice of y_i:

$$y_0=1,\quad y_1=\cdots=y_n=0.$$

Suppose that $p\in P_m^{+}$ satisfies

$$p(x_i)=y_i,\quad i=0,\ 1,\ \cdots,\ n,$$

i. e.,
$$p(x_0)=1,\quad p(x_1)=\cdots=p(x_n)=0.$$

Since $p\geqslant 0$, each of $x_1,\ \cdots,\ x_n$ is a zero of at least multiplicity 2 of p. Meanwhile $p\neq 0$, then p should contain a factor $(x-x_1)^2\cdots(x-x_n)^2$. This means that p has degree at least $2n$ and therefore $m\geqslant 2n$. This proves the minimality of $m=2n$.

3. Uniqueness

A simple example indicates that a solution of the interpolatory problem in P_{2n}^{+} is not unique for some choice of $y_i\geqslant 0$, $i=0,\ 1,\ \cdots,\ n$.

Example. Let $n=2$, $[a,\ b]\equiv[-1,\ 1]$, $x_0=-1$, $x_1=0$, $x_2=1$, $y_0=1$, $y_1=0$, and $y_2=1$. Both x^2 and x^4 are the polynomials in P_4^{+} satisfying (1). Generally, $p_t=tx^4+(1-t)x^2$ with $0\leqslant t\leqslant 1$ is also such a polynomial.

Precisely, we have the following criterion of uniqueness of a solution of the interpolatory problem in P_{2n}^{+}.

Theorem 2. *The polynomial* (2) *is the unique one satisfying the equations* (1) *from P_{2n}^{+} if and only if at most one of the y_i's is nonzero.*

Proof. Sufficiency. Suppose that, say, $y_i=0$, $\forall i\neq k$ for some index k. Thus

$$p(x)=y_k l_k^2(x).$$

Let $q\in P_{2n}^{+}$ satisfy

$$q(x_i)=y_i,\quad i=0,\ 1,\ \cdots,\ n$$

for such a choice of the y_i's. Then we obtain

$$p(x_i)=q(x_i),\quad i=0,\ 1,\ \cdots,\ n$$

and
$$p'(x_i)=q'(x_i)=0,\quad i=0,\ \cdots,\ k-1,\ k+1,\ \cdots,\ n.$$

Noting that $p,\ q\in P_{2n}$, this yields that $p=q$.

Necessity. Denote $I=\{i: y_i>0\}$ and $J=\{j: y_j=0\}$. Suppose, on the contrary, that card $I>1$, where card I denotes the cardinality of I. Write $s=\text{card } J$. Now let $q\in P_{s+t}$ satisfy

$$q(x_j) = q'(x_j) = 0 \quad \text{for all } j \in J$$

and

$$q(x_i) = y_i \quad \text{for all } i \in I.$$

Such a polynomial, as a Hermite interpolation polynomial, must exist.

Our proof would be completed if we could show that for some $t>0$, $r_t = (1-t)p + tq \geq 0$, here $p(x) = \sum_{i=0}^{n} y_i l_i^2(x) = \sum_{i\in I} y_i l_i^2(x)$. Indeed, since $s = \text{card } J = n+1-\text{card } I < n$, $P_{n+s} \subset P_{2n}$ and hence $q \in P_{2n}$. Meanwhile $q \neq p$ because q is of degree less than $2n$ and p is of degree $2n$. Then $r_t \in P_{2n}^+$ but $r_t \neq p$. On the other hand,

$$r_t(x_i) = (1-t)p(x_i) + tq(x_i) = y_i, \quad i = 0, 1, \cdots, n.$$

Thus r_t is a different solution from p of the equations (1), a contradiction.

The remainder of the proof is devoted to showing how to select t so that $r_t \geq 0$.

First, take $z = x_j$ with $j \in J$. It is easy to verify that

$$(l_i^2(x))'' = 2l_i(x) l_i''(x) + 2(l_i'(x))^2$$

and

$$l_i'(x) = \frac{w'(x)(x-x_i) - w(x)}{w'(x_i)(x-x_i)^2}.$$

For $i \in I$, since $l_i(z) = l_i(x_j) = 0$, we have

$$(l_i^2(x))''_{x=z} = 2\left(\frac{w'(z)}{w'(x_i)(z-x_i)}\right)^2,$$

which is positive. On the other hand, since the second derivative of $r_t(x)$ with respect to x

$$r_t''(x) = (1-t)\sum_{i\in I} y_i (l_i^2(x))'' + tq''(x),$$

its value at $t=0$ and $x=z$

$$r_0''(z) = 2\sum_{i\in I} y_i \left(\frac{w'(z)}{w'(x_i)(z-x_i)}\right)^2$$

is also positive. By continuity for such a point z there exists a number $t_z > 0$ and a neighbourhood N_z of z such that

$$r_t''(x) > 0, \quad \forall x \in N_z, \ \forall t \in [0, t_z].$$

Now from $r_t(z) = r_t'(z) = 0$ it follows that

$$r_t(x) \geq 0, \quad \forall x \in N_z, \ \forall t \in [0, t_z]. \tag{3}$$

Next, take $z \in [a, b] \backslash \{x_j : j \in J\}$. In this case $r_0(z) = p(z) > 0$. So by continuity for such a point z there also exists a number $t_z > 0$ and a neighbourhood N_z of z such that (3) is valid.

Now the collection of sets $\{N_z : z \in [a, b]\}$ is a cover of the compact interval $[a, b]$, from which we may select a finite subcover, say $N_{z_1}, \cdots, N_{z_M}$. Let $t_{z_1}, \cdots, t_{z_M}$ be the corresponding numbers and take $t = \min\{t_{z_1}, \cdots, t_{z_M}\}$. Obviously, $t > 0$. For such a t we can claim $r_t \geq 0$. Indeed, for any $x \in [a, b]$, there is an index $K (1 \leq K \leq M)$, such that $x \in N_{z_K}$, whence by (3) $r_t(x) \geq 0$ because $t \leq t_{z_K}$.

The proof of the theorem is completed.

To conclude this section, we mention the following obvious fact, the proof for which is omitted.

Theorem 3. *The set of solutions satisfying the equations* (1) *in* P_{2n}^+ *is convex.*