FINITE DIFFERENCE SOLUTION OF A NONLINEAR ELLIPTIC EQUATION*

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Abstract

The finite difference scheme is constructed for a nonlinear elliptic equation; its convergence is proved.

I. The Scheme

Let $x=(x_1, x_2, x_3)\in R^3$ and

$$\Omega=\{x|\ 0<x_m<1,\ 1\leqslant m\leqslant 3\}.$$

The boundary of Ω is Γ. We consider the following problem

$$\begin{cases} -\nabla\cdot(\nu(U)\nabla U)=f, & x\in\Omega, \\ U=0, & x\in\Gamma, \end{cases} \tag{1}$$

where $f\in C^\alpha(\Omega+\Gamma)$, $0<\alpha<1$, and $\nu(\varphi)$ is a twice differentiable function. Assume that there are positive constants ν_0, ν_1, C_1 and C_2 such that for all $\varphi(x)$,

$$0<\nu_0\leqslant\nu(\varphi(x))\leqslant\nu_1,$$

$$|\nu'(\varphi(x))|\leqslant C_1,\quad |\nu''(\varphi(x))|\leqslant C_2.$$

Let $$a(V,\ W,\ \nu(\varphi))=\sum_{m=1}^{3}\int_\Omega \nu(\varphi(x))\nabla V(x)\cdot\nabla W(x)dx.$$

The generalized solution of (1) means such a function $U(x)\in H_0^1(\Omega)$ that

$$a(U,\ W,\ \nu(U))=\int_\Omega f(x)W(x)dx,\quad \forall W\in C_0^\infty(\Omega). \tag{2}$$

Douglas, Dupont[1] proved that the problem (1) possesses a unique generalized solution $U(x)\in C^{2+\alpha}(\Omega+\Gamma)$, and so $U(x)$ is the classical solution of (1) too. They also proposed a finite element scheme for solving (1) with the proof of convergence.

In this paper we construct a finite difference scheme for solving (1) and prove the existence of the approximate solution and the convergence.

Let h be the mesh spacing of the variables $x_m (1\leqslant m\leqslant 3)$ and $Jh=1$, J being an integer. Let $j_m(x)$ be an integer and

$$\Omega_h=\{x|\ x_m=j_m(x)h,\ 1<j_m(x)<J-1,\ 1\leqslant m\leqslant 3\}.$$

If $x\notin\Omega_h$ and the distance from x to Ω_h equals h, then we say $x\in\Gamma_h$.

Let $$e_1=(1,\ 0,\ 0),\ e_2=(0,\ 1,\ 0),\ e_3=(0,\ 0,\ 1).$$

We define $$u_{x_m}(x)=\frac{u(x+he_m)-u(x)}{h}.$$

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$$u_{\bar{x}_m}(x)=\frac{u(x)-u(x-he_m)}{h},$$

$$\Delta_{h,m}^{\nu(\varphi)}u(x)=\frac{1}{2}[(\nu(\varphi(x))u_{x_m}(x))_{\bar{x}_m}+(\nu(\varphi(x))u_{\bar{x}_m}(x))_{x_m}],$$

$$\Delta_h^{\nu(\varphi)}u(x)=\sum_{m=1}^{3}\Delta_{h,m}^{\nu(\varphi)}u(x).$$

Let $u_h(x)$ be the approximation of $U(x)$. The finite difference scheme for solving (1) is the following

$$\begin{cases}-\Delta_h^{\nu(u_h)}u_h(x)=f(x), & x\in\Omega_h,\\ u_h(x)=0, & x\in\Gamma_h.\end{cases}\tag{3}$$

Let $R_h(x)$ be the truncation error. Then

$$\begin{cases}-\Delta_h^{\nu(U)}U(x)=f(x)+R_h(x), & x\in\Omega_h,\\ U(x)=0, & x\in\Gamma_h.\end{cases}\tag{4}$$

Because $U(x)\in C^{3+\alpha}(\Omega+\Gamma)$, so $|R_h(x)|\to 0$, as $h\to 0$.

II. Lemmas

We define

$$(u,\ v)=h^3\sum_{x\in\Omega_h}u(x)v(x),\quad \|u\|^2=(u,\ u),$$

$$\|u\|_{l^q}^q=h^3\sum_{x\in\Omega_h}|u(x)|^q,\quad \|u\|_\infty=\max_{x\in\Omega_h}|u(x)|,$$

$$|u|_1^2=\frac{1}{2}\sum_{m=1}^{3}(\|u_{x_m}\|^2+\|u_{\bar{x}_m}\|^2),$$

$$|u|_2^2=\frac{1}{2}\sum_{m=1}^{3}(|u_{x_m}|_1^2+|u_{\bar{x}_m}|_1^2),$$

....

$$\|u\|_p^2=\sum_{r=1}^{p}|u|_r^2+\|u\|^2,$$

and

$$S_m(u,\ v,\ \nu(\varphi))=\frac{h}{2}\sum_{\substack{x\in\Gamma_h\\ x_m=0}}\nu(\varphi(x))u(x+he_m)v(x+he_m)$$

$$+\frac{h}{2}\sum_{\substack{x\in\Gamma_h\\ x_m=1}}\nu(\varphi(x))u(x-he_m)v(x-he_m),$$

$$S(u,\ v,\ \nu(\varphi))=\sum_{m=1}^{3}S_m(u,\ v,\ \nu(\varphi)).$$

Let $\mathscr{H}$ be the space of the mesh function $u(x)$ such that

$$u(x)=0,\quad \text{for}\quad x\in\Gamma_h.$$

We define the scalar product

$$(u,\ v)_{\mathscr{H}}=\frac{1}{2}\sum_{m=1}^{3}[(u_{x_m},\ v_{x_m})+(u_{\bar{x}_m},\ v_{\bar{x}_m})]+S(u,\ v,\ 1)$$

and the norm

$$\|u\|_{\mathscr{H}}^2=(u,\ u)_{\mathscr{H}}.$$

Let

$$a_h(u,\ v,\ \nu(\varphi))=\frac{1}{2}\sum_{m=1}^{3}(\nu(\varphi),\ u_{x_m}v_{x_m}+u_{\bar{x}_m}v_{\bar{x}_m})+S(u,\ v,\ \nu(\varphi)).$$

Lemma 1. *For all $\varphi(x)$ and $u(x)$, we have*

$$\nu_0\|u\|^2_{\mathscr{H}}\leqslant a_h(u,\ u,\ \nu(\varphi))\leqslant\nu_1\|u\|^2_{\mathscr{H}}.$$

Lemma 2. *If* $u(x),\ v(x)\in\mathscr{H}$, then

$$(-\Delta_h^{\nu(\varphi)}u,\ v)=a_h(u,\ v,\ \nu(\varphi)).$$

Proof. Let $u(x)=u(\underset{\sim}{x},\ x_m),\quad 1\leqslant m\leqslant 3.$

We have from Abel's formula

$$h\sum_{j_m(x)=1}^{J-1}[(\nu(\varphi(x))u_{x_m}(x))_{\bar{x}_m}v(x)+\nu(\varphi(x))u_{x_m}(x)v_{x_m}(x)]$$
$$=\nu(\varphi(\underset{\sim}{x},\ 1-h))u_{x_m}(\underset{\sim}{x},\ 1-h)v(\underset{\sim}{x},\ 1)-\nu(\varphi(\underset{\sim}{x},\ 0))u_{x_m}(\underset{\sim}{x},\ 0)v(\underset{\sim}{x},\ h). \tag{5}$$

Similarly

$$h\sum_{j_m(x)=1}^{J-1}[(\nu(\varphi(x))u_{\bar{x}_m}(x))_{x_m}v(x)+\nu(\varphi(x))u_{\bar{x}_m}(x)v_{\bar{x}_m}(x)]$$
$$=\nu(\varphi(x,\ 1))u_{\bar{x}_m}(\underset{\sim}{x},\ 1)v(\underset{\sim}{x},\ 1-h)-\nu(\varphi(\underset{\sim}{x},\ h))u_{\bar{x}_m}(\underset{\sim}{x},\ h)v(\underset{\sim}{x},\ 0). \tag{6}$$

Since $u,\ v\in\mathscr{H}$, so

$$h\sum_{j_m(x)=1}^{J-1}\{2v(x)\Delta_{h,m}^{\nu(\varphi)}u(x)+\nu(\varphi(x))[u_{x_m}(x)v_{x_m}(x)+u_{\bar{x}_m}(x)v_{\bar{x}_m}(x)]\}$$
$$=\frac{-1}{h}[\nu(\varphi(\underset{\sim}{x},\ 1))u(\underset{\sim}{x},\ 1-h)v(\underset{\sim}{x},\ 1-h)+\nu(\varphi(\underset{\sim}{x},\ 0))u(\underset{\sim}{x},\ h)v(\underset{\sim}{x},\ h)]. \tag{7}$$

Then the conclusion follows

Lemma 3.[4]. *For all* $u(x),\ v(x)\in\mathscr{H}$, we have

$$\|uv\|^2\leqslant 8\|u\|^{\frac{1}{2}}\|v\|^{\frac{1}{2}}|u|_1^{\frac{3}{2}}|v|_1^{\frac{3}{2}};$$

then

$$\|u\|^4_{l^4}\leqslant 8\|u\|\,|u|_1^3.$$

Lemma 4. *For all* $u(x),\ v(x)$ *and* $w(x)$,

$$|(u,\ v,\ w)|\leqslant\|u\|_{l^m}\|v\|_{l^n}\|w\|_{l^k},$$

where $m,\ n,\ k>1,\ \frac{1}{m}+\frac{1}{n}+\frac{1}{k}=1.$

Proof. For all non-negative numbers a_j, b_j and c_j, we have

$$\sum_{j=1}^{J-1}a_jb_jc_j\leqslant\Big(\sum_{j=1}^{J-1}a_j^m\Big)^{\frac{1}{m}}\Big(\sum_{j=1}^{J-1}b_j^n\Big)^{\frac{1}{n}}\Big(\sum_{j=1}^{J-1}c_j^k\Big)^{\frac{1}{k}}. \tag{8}$$

From which the conclusion follows.

III. The Existence of the Approximate Solution

Theorem 1. *For all* $h>0$, *the scheme* (3) *possesses solution* u_h *such that*

$$\|u_h\|_{\mathscr{H}}\leqslant\frac{m_0^{\frac{1}{2}}\|f\|}{\nu_0},$$

where

$$m_0=\sup_{u\in\mathscr{H}}\frac{\|u\|^2}{\|u\|^2_{\mathscr{H}}}.$$

Proof. Define the operator σ, where $u=\sigma v$ is determined by

$$\begin{cases}-\Delta_h^{\nu(v)}u=f, & x\in\Omega_h,\\ u=0, & x\in\Gamma_h.\end{cases} \tag{9}$$

Taking the scalar product of (9) with $u(x)$, we get from Lemma 2

$$a_h(u, u, \nu(v)) \leq \|f\| \|u\| \leq m_0^{\frac{1}{2}} \|f\| \|u\|_{\mathscr{H}}.$$

Then from Lemma 1,

$$\|u\|_{\mathscr{H}} \leq \frac{m_0^{\frac{1}{2}} \|f\|}{\nu_0}.$$

So the range of σ is contained in a closed ball in $\mathscr{H}$.

Assume $u^{(l)} = \sigma v^{(l)}$, $l=1, 2$. Then

$$\begin{cases} -\Delta_h^{\nu(v^{(1)})} u^{(1)}(x) + \Delta_h^{\nu(v^{(2)})} u^{(2)}(x) = 0, & x \in \Omega_h, \\ u^{(1)}(x) - u^{(2)}(x) = 0, & x \in \Gamma_h. \end{cases}$$

Let $\tilde{u}(x) = u^{(1)}(x) - u^{(2)}(x)$, $\tilde{\nu}(x) = \nu(v^{(1)}(x)) - \nu(v^{(2)}(x))$. Then

$$\begin{cases} \Delta_h^{\nu(v^{(1)})} \tilde{u}(x) = \Delta_h^{\tilde{\nu}} u^{(2)}(x), & x \in \Omega_h, \\ \tilde{u}(x) = 0, & x \in \Gamma_h, \end{cases} \tag{10}$$

whence
$$a_h(\tilde{u}, \tilde{u}, \nu(v^{(1)})) = a_h(u^{(2)}, \tilde{u}, \tilde{\nu}).$$

Since $\tilde{\nu}(x) = 0$ on Γ_h, we obtain

$$\nu_0 \|\tilde{u}\|_{\mathscr{H}}^2 \leq C_1 \|v^{(1)} - v^{(2)}\|_\infty \|u^{(2)}\|_{\mathscr{H}} \|\tilde{u}\|_{\mathscr{H}}.$$

Let $v^{(1)} \to v^{(2)}$. Then $\|\tilde{u}\|_{\mathscr{H}} \to 0$. Hence σ is continuous. From Brouwer's theorem, the scheme (3) has at least a solution $u_h(x)$ such that

$$\|u_h\|_{\mathscr{H}} \leq \frac{m_0^{\frac{1}{2}} \|f\|}{\nu_0}.$$

IV. The Convergence

Theorem 2. *There exists a positive constant C_3 such that*

$$\|u_h - U\|_{\mathscr{H}} \leq C_3 \|R_h\|.$$

Proof. Let $\tilde{u}_h(x) = u_h(x) - U(x)$, $\tilde{\nu}(x) = \nu(u_h(x)) - \nu(U(x))$. Then

$$\begin{cases} -\Delta_h^{\nu(u_h)} \tilde{u}_h(x) = \Delta_h^{\tilde{\nu}} U(x) - R_h(x), & x \in \Omega_h, \\ \tilde{u}_h(x) = 0, & x \in \Gamma_h. \end{cases}$$

By Lemma 2, we have

$$a_h(\tilde{u}_h, \tilde{u}_h, \nu(u_h)) = -a(U, \tilde{u}_h, \tilde{\nu}) - (\tilde{u}_h, R_h).$$

We have from Lemma 1

$$\nu_0 \|\tilde{u}_h\|_{\mathscr{H}}^2 \leq |a_h(U, \tilde{u}_h, \tilde{\nu})| + |(\tilde{u}_h, R_h)|. \tag{11}$$

Clearly

$$|(\tilde{u}_h, R_h)| \leq m_0^{\frac{1}{2}} \|R_h\| \|\tilde{u}_h\|_{\mathscr{H}}. \tag{12}$$

We now estimate $|a_h(U, \tilde{u}_h, \tilde{\nu})|$. Since $U(x)$, $\tilde{u}_h(x) \in \mathscr{H}$, so

$$S(u, \tilde{u}_h, \tilde{\nu}) = 0.$$

Therefore it follows from Lemma 4 that

$$|(\tilde{\nu}, U_{x_m} \tilde{u}_{h, x_m})| \leq C_4 |\tilde{u}_h|_{1'} |U_{x_m}|_{\nu} |\tilde{u}_{h, x_m}|.$$

We have from Lemma 3

$$\|\tilde{u}_h\|_{1'}^3 \leq \|\tilde{u}_h\| \|\tilde{u}_h\|_{1'}^2 \leq C_5 |\tilde{u}_h|^{\frac{3}{2}} |\tilde{u}_h|_1^{\frac{3}{2}}.$$

So

$$|a_h(U,\ \tilde{u}_h,\ \tilde{\nu})|\leqslant C_6\|\tilde{u}_h\|^{\frac{1}{2}}\|\tilde{u}_h\|_1^{\frac{3}{2}}. \tag{13}$$

By substituting (12) and (13) into (11), we obtain

$$\nu_0\|\tilde{u}_h\|_{\mathscr{H}}\leqslant C_7(\|\tilde{u}_h\|^{\frac{1}{2}}\|\tilde{u}_h\|_{\mathscr{H}}^{\frac{1}{2}}+\|R_h\|).$$

Therefore

$$\|\tilde{u}_h\|_{\mathscr{H}}\leqslant C_8(\|\tilde{u}_h\|+\|R_h\|). \tag{14}$$

To estimate $\|\tilde{u}_h\|$, we consider the following conjugate problem

$$\begin{cases} -\Delta_h^{\nu(U)}\varphi(x)+\dfrac{1}{2}\displaystyle\sum_{m=1}^{3}\nu'(U(x))(U_{x_m}(x)\varphi_{x_m}(x)+U_{\bar{x}_m}(x)\varphi_{\bar{x}_m}(x))=\tilde{u}(x),\ x\in\Omega_h, \\ \varphi(x)=0,\quad x\in\Gamma_h. \end{cases} \tag{15}$$

By the theorems of Thomée[3], we have

$$\|\varphi\|_2\leqslant C_9\|\tilde{u}_h\|.$$

Taking the scalar product of (15) with $\tilde{u}_h(x)$, we have

$$\|\tilde{u}_h\|^2=a_h(\varphi,\ \tilde{u}_h,\ \nu(U))+\frac{1}{2}\sum_{m=1}^{3}(\tilde{u}_h,\ \nu'(U)(U_{x_m}\varphi_{x_m}+U_{\bar{x}_m}\varphi_{\bar{x}_m})). \tag{16}$$

Because $\tilde{\nu}=0$ on Γ_h, so

$$\begin{aligned} a_h(\varphi,\ \tilde{u}_h,\ \nu(U)) &= -a_h(\varphi,\ U,\ \nu(U))+a_h(\varphi,\ u_h,\ \nu(U)) \\ &= a_h(\varphi,\ u_h,\ \nu(u_h))-a_h(\varphi,\ U,\ \nu(U)) \\ &\quad -a_h(\varphi,\ U,\ \tilde{\nu})-a_h(\varphi,\ \tilde{u}_h,\ \tilde{\nu}). \end{aligned}$$

Then we get from (16)

$$\begin{aligned} \|\tilde{u}_h\|^2 &= a_h(\varphi,\ u_h,\ \nu(u_h))-a_h(\varphi,\ U,\ \nu(U))-\frac{1}{2}\sum_{m=1}^{3}(\tilde{\nu},\ \varphi_{x_m}\tilde{u}_{h,x_m}+\varphi_{\bar{x}_m}\tilde{u}_{h,\bar{x}_m}) \\ &\quad +\frac{1}{2}\sum_{m=1}^{3}(\tilde{u}_h\nu'(U)-\tilde{\nu},\ \varphi_{x_m}U_{x_m}+\varphi_{\bar{x}_m}U_{\bar{x}_m}). \end{aligned} \tag{17}$$

Now we are going to estimate the terms of the right hand side of (17). Firstly by taking the scalar product of (3) and (4) with $\varphi(x)$, we have

$$a_h(\varphi,\ u_h,\ \nu(u_h))=(f,\ \varphi)$$

and

$$a_h(\varphi,\ U,\ \nu(U))=(f,\ \varphi)+(R_h,\ \varphi).$$

So

$$|a_h(\varphi,\ u_h,\ \nu(u_h))-a_h(\varphi,\ U,\ \nu(U))|\leqslant\|R_h\|\|\varphi\|\leqslant C_{10}\|R_h\|\|\tilde{u}_h\|. \tag{18}$$

From the discrete imbedding theorem we have[5] for all $v_h\in\mathscr{H}$

$$\|v_h\|_{l^4}\leqslant C_{11}\|v_h\|_1$$

and so

$$\begin{aligned} \left|\frac{1}{2}\sum_{m=1}^{3}(\tilde{\nu},\ \varphi_{x_m}\tilde{u}_{h,x_m}+\varphi_{\bar{x}_m}\tilde{u}_{h,\bar{x}_m})\right| &\leqslant\frac{C_1}{2}\|\tilde{u}_h\|_{l^4}|\tilde{u}_h|_1\cdot\sum_{m=1}^{3}(\|\varphi_{x_m}\|_{l^4}+\|\varphi_{\bar{x}_m}\|_{l^4}) \\ &\leqslant C_{12}\|\varphi\|_2\|\tilde{u}_h\|^{\frac{1}{2}}\|\tilde{u}_h\|_{\mathscr{H}}^{\frac{3}{2}}\leqslant C_{13}\|\tilde{u}_h\|^{\frac{3}{2}}\|\tilde{u}_h\|_{\mathscr{H}}^{\frac{3}{2}}. \end{aligned} \tag{19}$$

Finally because

$$\begin{aligned} |\tilde{\nu}-\tilde{u}_h\nu'(U)| &= |\nu'(U+\xi_1\tilde{u}_h)\tilde{u}_h-\nu'(U)\tilde{u}_h| \\ &= |\xi_1\nu''(U+\xi_1\xi_2\tilde{u}_h)\tilde{u}_h^2|\leqslant C_2|\tilde{u}_h|^2, \end{aligned}$$

we obtain

$$\left|\frac{1}{2}\sum_{m=1}^{3}(\tilde{\nu}-\tilde{u}_h\nu'(U),\ \varphi_{x_m}U_{x_m}+\varphi_{\bar{x}_m}U_{\bar{x}_m})\right|\leqslant C_{14}\|\tilde{u}_h^2\||\varphi|_1\leqslant C_{15}\|\tilde{u}_h\|^{\frac{1}{2}}|\tilde{u}|_1^{\frac{3}{2}}|\varphi|_1$$
$$\leqslant C_{16}\|\tilde{u}_h\|^{\frac{3}{2}}\|\tilde{u}_h\|_{\mathscr{F}}^{\frac{3}{2}}. \tag{20}$$

Substituting (18)—(20) into (17), we get

$$\|\tilde{u}_h\|\leqslant C_{17}(\|\tilde{u}_h\|^{\frac{1}{2}}\|\tilde{u}_h\|_{\mathscr{F}}^{\frac{3}{2}}+\|R_h\|),$$

whence

$$\|\tilde{u}_h\|\leqslant C_{18}(\|\tilde{u}_h\|_{\mathscr{F}}^{3}+\|R_h\|). \tag{21}$$

Since $\|R_h\|\to 0$ as $h\to 0$, we have from (14) and (21)

$$\|\tilde{u}_h\|\leqslant C_{19}(\|\tilde{u}_h\|^3+\|R_h\|).$$

If we can prove

$$\|\tilde{u}_h\|\to 0,\quad \text{as } h\to 0,$$

then the conclusion follows.

Indeed $U(x)$ is both a classical solution and a generalized solution of (1). On the other hand

$$a_h(u_h,\ W,\ \nu(u_h))=(f,\ W),\quad \forall W\in C_0^\infty(\Omega). \tag{22}$$

We interpolate $u_h(x)$ to get the piecewise linear function $U_h(x)$. Then for all h,

$$\|U_h\|_{H_0^1}\leqslant C_{20}\|u_h\|_{\mathscr{F}}\leqslant\frac{C_{20}m_0^{\frac{1}{2}}\|f\|}{\nu_0}.$$

So we can extract a subsequence $\{U_{h_i}(x)\}$ such that $U_{h_i}(x)$ tends to $\overline{U}(x)$ weakly in $H_0^1(\Omega)$ while strongly in $L^2(\Omega)$. Let $w_h(x)$ be a mesh function for which

$$\begin{cases} w_h(x)=W(x), & x\in\Omega_h,\\ w_h(x)=0, & x\in\Gamma_h.\end{cases}$$

Then interpolating $w_h(x)$ to get the piecewise linear function $W_h(x)$, we have

$$\left|a(\overline{U},\ W,\ \nu(\overline{U}))-\int_\Omega f(x)W(x)\,dx\right|$$
$$\leqslant|a(\overline{U},\ W-W_{h_i},\ \nu(\overline{U}))|+\left|\int_\Omega(\nu(\overline{U})\nabla\overline{U}-\nu(U_{h_i})\nabla U_{h_i})\nabla W_{h_i}dx\right|$$
$$+|a(U_{h_i},\ W_{h_i},\ \nu(U_{h_i}))-a_h(U_{h_i},\ W_{h_i},\ \nu(u_{h_i}))|$$
$$+\left|(f,\ W_{h_i})-\int_\Omega fW\,dx\right|. \tag{23}$$

It can be shown that all the terms on the right hand side of (23) tend to zero as $h\to 0$. Therfore

$$a(\overline{U},\ W,\ \nu(\overline{U}))=\int_\Omega fW\,dx.$$

So $\overline{U}(x)$ is the generalized solution of (1), i. e. $\overline{U}(x)=U(x)$.

It is clear that from each sequence we can extract a subsequence whose limit function equals $U(x)$. So $\{U_h\}$ tends to U strongly in $L^2(\Omega)$. On the other hand for sufficiently small h, we have

$$\|\tilde{u}_h\|^2=\|u_h-U\|^2\leqslant C_{21}\|U_h-U\|^2=C_{21}\int(U_h(x)-U(x))^2dx.$$

So $\|\tilde{u}_h\|^2\to 0$ as $h\to 0$. This completes the proof.

The results of this paper can be generalized into a more general case

$$\begin{cases} -\nabla\cdot(\nu(x, U)\nabla U(x)) = f(x, U), & x\in\Omega, \\ U(x) = g(x), & x\in\Gamma, \end{cases}$$

where $\nu'_U(x, U)$ and $\nu'_x(x, U)$ are Lipschitz continuous in x and U, $0<\nu_0\leqslant\nu(x, \varphi)\leqslant\nu_1$, for all φ and $x\in\Omega$, and $g(x)$ is sufficiently smooth.

The technique used in this paper can be applied to quasi-linear equations (see [3, 4]) such as

$$\begin{cases} \boldsymbol{U}(x)\cdot\nabla\boldsymbol{U}(x) - \nu\nabla^2\boldsymbol{U}(x) = 0, & x\in\Omega, \\ \nabla\cdot\boldsymbol{U}(x) = 0, & x\in\Omega, \\ \boldsymbol{U}(x) = 0, & x\in\Gamma. \end{cases}$$

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