FINITE DIFFERENCE SOLUTIONS OF THE NONLINEAR MUTUAL BOUNDARY PROBLEMS FOR THE SYSTEMS OF FERRO-MAGNETIC CHAIN*

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§1

The Landau-Lifschitz equation for one-dimensional isotropic Heisenberg ferro-magnetic chain

$$\boldsymbol{s}_t=\boldsymbol{s}\times\boldsymbol{s}_{xx}+\boldsymbol{s}\times\boldsymbol{h} \tag{1}$$

is a strongly degenerate parabolic system, where $\boldsymbol{s}=(s_1, s_2, s_3)$ is a three-dimensional unknown vector function, $\boldsymbol{h}=(0, 0, h(t))$, $h(t)$ is a constant or a function of t, "$\times$" denotes the cross-product operator of two three-dimensional vectors[1-4]. In [5] the weak solutions of the periodic boundary problems and the initial problems for more general systems of ferro-magnetic chain

$$z_t=z\times z_{xx}+f(x, t, z) \tag{2}$$

are constructed, where $z=(u, v, w)$ and $f(x, t, z)$ are three-dimensional vector functions. In [6] some simple boundary problems for the system (2) are considered and their finite difference solutions are obtained in [7]. For the systems of ferro-magnetic chain with several variables

$$z_t=z\times\Delta z+f(x, t, z), \tag{3}$$

the homogeneous boundary problem is studied in [8], where $x=(x_1, x_2, \cdots, x_n)$.

In the present work for the system (2) of ferro-magnetic chain the nonlinear mutual boundary problem

$$\begin{aligned} z_x(0, t)&=\mathrm{grad}_0\,\psi(z(0, t), z(l, t)),\\ -z_x(l, t)&=\mathrm{grad}_1\,\psi(z(0, t), z(l, t)) \end{aligned} \tag{4}$$

with the initial condition

$$z(x, 0)=\varphi(x) \tag{5}$$

is considered in the rectangular domain $Q_T=\{0\leqslant x\leqslant l, 0\leqslant t\leqslant T\}$, by means of finite difference method, where $\psi(z_0, z_1)$ is a scalar function of two three-dimensional vector variables $z_0, z_1\in\mathbf{R}^3$, $\varphi(x)$ is a three-dimensional vector function and "grad_0" and "grad_1" denote the gradient operators with respect to z_0 and z_1 respectively.

Suppose that the following assumptions for the systems (2) of ferro-magnetic chain, the nonlinear mutual boundary conditions (4) and the initial vector function $\varphi(x)$ are valid.

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(I) $f(x, t, z)$ is a three-dimensional continuous vector function for $(x, t, z) \in Q_T \times \mathbb{R}^3$, $f_z(x, t, z)$ is also continuous and $f(x, t, z)$ satisfies the condition of semiboundedness

$$(z-y)\cdot(f(x, t, z)-f(x, t, y)) \leqslant b|z-y|^2, \tag{6}$$

where $(x, t) \in Q_T$, $z, y \in \mathbb{R}^3$ and b is a constant.

(II) $\psi(z_0, z_1)$ is a continuously differentiable with respect to vector variables $z_0, z_1 \in \mathbb{R}^3$.

(III) $\varphi(x) \in H^1(0, l)$.

Let us divided the rectangular domain Q_T into small grids by the parallel lines $x = x_j$ $(j=0, 1, \cdots, J)$ and $t = t_n$ $(n=0, 1, \cdots, N)$, where $x_j = jh$, $t_n = n\Delta t$ and $Jh = l$, $N\Delta t = T$. Denote the three-dimensional discrete vector function on the grid point (x_j, t_n) by z_j^n $(j=0, 1, \cdots, J;\ n=0, 1, \cdots, N)$.

Corresponding to the system (2) of ferro-magnetic chain we construct the finite difference system

$$\frac{z_j^n - z_j^{n-1}}{\Delta t} = z_j^n \times \frac{\Delta_+ \Delta_- z_j^n}{h^2} + f_j^n, \quad j=1, 2, \cdots, J-1;\ n=1, 2, \cdots, N, \tag{7}$$

where $f_j^n = f(x_j, t_n, z_j^n)$ and $\Delta_+ z_j = z_{j+1} - z_j$, $\Delta_- z_j = z_j - z_{j-1}$. The finite difference boundary conditions corresponding to the nonlinear mutual boundary conditions (4) are as follows:

$$\frac{u_1^n - u_0^n}{h} = \frac{\psi(u_1^n, v_1^n, w_1^n; u_{J-1}^n, v_{J-1}^n, w_{J-1}^n) - \psi(u_1^{n-1}, v_1^n, w_1^n; u_{J-1}^n, v_{J-1}^n, w_{J-1}^n)}{u_1^n - u_1^{n-1}},$$

$$\frac{v_1^n - v_0^n}{h} = \frac{\psi(u_1^{n-1}, v_1^n, w_1^n; u_{J-1}^n, v_{J-1}^n, w_{J-1}^n) - \psi(u_1^{n-1}, v_1^{n-1}, w_1^n; u_{J-1}^n, v_{J-1}^n, w_{J-1}^n)}{v_1^n - v_1^{n-1}},$$

$$\frac{w_1^n - w_0^n}{h} = \frac{\psi(u_1^{n-1}, v_1^{n-1}, w_1^n; u_{J-1}^n, v_{J-1}^n, w_{J-1}^n) - \psi(u_1^{n-1}, v_1^{n-1}, w_1^{n-1}; u_{J-1}^n, v_{J-1}^n, w_{J-1}^n)}{w_1^n - w_1^{n-1}},$$

$$-\frac{u_J^n - u_{J-1}^n}{h} = \frac{\psi(u_1^{n-1}, v_1^{n-1}, w_1^{n-1}; u_{J-1}^n, v_{J-1}^n, w_{J-1}^n) - \psi(u_1^{n-1}, v_1^{n-1}, w_1^{n-1}; u_{J-1}^{n-1}, v_{J-1}^n, w_{J-1}^n)}{u_{J-1}^n - u_{J-1}^{n-1}},$$

$$-\frac{v_J^n - v_{J-1}^n}{h} = \frac{\psi(u_1^{n-1}, v_1^{n-1}, w_1^{n-1}; u_{J-1}^{n-1}, v_{J-1}^n, w_{J-1}^n) - \psi(u_1^{n-1}, v_1^{n-1}, w_1^{n-1}; u_{J-1}^{n-1}, v_{J-1}^{n-1}, w_{J-1}^n)}{v_{J-1}^n - v_{J-1}^{n-1}},$$

$$-\frac{w_J^n - w_{J-1}^n}{h} = \frac{\psi(u_1^{n-1}, v_1^{n-1}, w_1^{n-1}; u_{J-1}^{n-1}, v_{J-1}^{n-1}, w_{J-1}^n) - \psi(u_1^{n-1}, v_1^{n-1}, w_1^{n-1}; u_{J-1}^{n-1}, v_{J-1}^{n-1}, w_{J-1}^{n-1})}{w_{J-1}^n - w_{J-1}^{n-1}}, \tag{8$_1$}$$

where $n=1, 2, \cdots, N$. Denote (8) for brevity by

$$\frac{\Delta_+ z_0^n}{h} = \widetilde{\text{grad}}_0\, \psi(z_1^n, z_{J-1}^n),$$

$$-\frac{\Delta_- z_J^n}{h} = \widetilde{\text{grad}}_1\, \psi(z_1^n, z_{J-1}^n). \tag{8$_2$}$$

The finite difference initial condition is

$$z_j^0 = \varphi_j, \quad j=0, 1, \cdots, J, \tag{9}$$

where $\varphi_j = \varphi(x_j)$, $j=0, 1, \cdots, J$.

Symbol "$\cdot$" denotes the scalar product of two three-dimensional vectors. For the discrete functions $\{u_j\}$ and $\{v_j\}$, we take the notations:

$$(u\cdot v)_h = \sum_{j=0}^{J} (u_j \cdot v_j) h \quad \text{and} \quad \|u\|_h^2 = (u\cdot u)_h.$$

§ 2

Now we are going to prove the existence of solution z_j^n $(j=0, 1, \cdots, J)$ for the nonlinear finite difference system (7), (8) and (9), where z_j^{n-1} $(j=0, 1, \cdots, J)$ are regarded as known vectors, $n=1, 2, \cdots, N$.

Lemma 1. *Under the conditions* (I) *and* (II) *for sufficiently small Δt the nonlinear finite difference system* (7), (8) *and* (9) *has at least one solution* z_j^n $(j=0, 1, \cdots, J;$ $n=0,1, \cdots, N)$.

Proof. Let us construct a continuous mapping T_λ of $3(J+1)$-dimensional Euclidean space $\mathbb{R}^{3(J+1)}$ into itself with the parameter $0\leqslant\lambda\leqslant1$ as follows: For any $y_j\in\mathbb{R}^3$ $(j=0, 1, \cdots, J)$, we define $z_j\in\mathbb{R}^3$ $(j=0, 1, \cdots, J)$ by

$$z_j=z_j^{n-1}+\lambda \frac{\Delta t}{h^2}(y_j\times \Delta_+\Delta_-y_j)+\lambda\Delta t f(x_j, t_n, y_j), \quad j=1, 2, \cdots, J-1 \tag{10}$$

and

$$\begin{aligned} z_0&=y_1-\lambda h \widetilde{\operatorname{grad}_0} \psi(y_1, y_{J-1}),\\ z_J&=y_{J-1}-\lambda h \widetilde{\operatorname{grad}_1} \psi(y_1, y_{J-1}), \end{aligned} \tag{11}$$

where $0\leqslant\lambda\leqslant1$. For the existence of the solutions of the finite difference system (7), (8) and (9), let us consider the bound for all possible solutions of the nonlinear systems

$$z_j=z_j^{n-1}+\lambda \frac{\Delta t}{h^2}(z_j\times \Delta_+\Delta_-z_j)+\lambda\Delta t f(x_j, t_n, z_j), \quad j=1, 2, \cdots, J-1 \tag{12}$$

and

$$\begin{aligned} z_0&=z_1-\lambda h \widetilde{\operatorname{grad}_0} \psi(z_1, z_{J-1}),\\ z_J&=z_{J-1}-\lambda h \widetilde{\operatorname{grad}_1} \psi(z_1, z_{J-1}) \end{aligned} \tag{13}$$

with the parameter $0\leqslant\lambda\leqslant1$.

For any fixed $j=1, 2, \cdots, J-1$, we have from (12)

$$|z_j|^2=z_j^{n-1}\cdot z_j+\lambda \frac{\Delta t}{h^2} z_j\cdot (z_j\times \Delta_+\Delta_-z_j)+\lambda\Delta t z_j\cdot f(x_j, t_n, z_j), \quad j=1, 2, \cdots, J-1.$$

It follows that

$$(1-2\lambda(b+\delta)\Delta t)|z_j|^2\leqslant|z_j^{n-1}|^2+\frac{\lambda\Delta t}{2\delta}|f(x_j, t_n, 0)|^2, \quad j=1, 2, \cdots, J-1.$$

As Δt is sufficiently small that $1-2b\Delta t>0$ and $\delta>0$ is so small that $1-2(b+\delta)\Delta t>0$, $|z_j|$ $(j=1, 2, \cdots, J-1)$ is uniformly bounded with respect to the parameter $0\leqslant\lambda\leqslant1$, that is,

$$|z_j|\leqslant C_1=\frac{\max\limits_j |z_j^{n-1}|+\sqrt{\dfrac{\Delta t}{2\delta}}\max\limits_{(x,t)\in Q_T}|f(x, t, 0)|}{\sqrt{1-2(b+\delta)\Delta t}},$$

where C_1 is a constant independent of $0\leqslant\lambda\leqslant1$ and $j=1, 2, \cdots, J-1$. From (13) it is clear that $|z_0|$ and $|z_J|$ are also uniformly bounded with respect to $0\leqslant\lambda\leqslant1$.

Then the lemma follows immediately from the fixed-point theorem.

§ 3

We turn to estimate the solutions z_j^n $(j=0, 1, \cdots, J;\ n=0, 1, \cdots, N)$ of the finite difference system (7), (8) and (9).

Taking the scalar product of the three-dimensional vector $z_j^n \Delta t$ and the three-dimensional finite difference vector equation (7), for $j=1, 2, \cdots, J-1$, we get

$$|z_j^n|^2 = z_j^{n-1}\cdot z_j^n + \Delta t z_j^n \cdot f_j^n, \quad j=1, 2, \cdots, J-1.$$

Since from the condition (I) and

$$z_j^n \cdot f_j^n = z_j^n \cdot (f(x_j, t_n, z_j^n) - f(x_j, t_n, 0)) + z_j^n \cdot f(x_j, t_n, 0),$$

the above equation can be replaced by the following iterative forms

$$|z_j^n|^2 \leqslant \frac{|z_j^{n-1}|^2 + \frac{\Delta t}{2\delta}|f(x_j, t_n, 0)|^2}{1-2(b+\delta)\Delta t}, \quad j=1, 2, \cdots, J-1,$$

where Δt is sufficiently small that $1-2b\Delta t>0$ and $\delta>0$ is chosen so small that $1-2b\Delta t > 2\delta\Delta t$. Hence we have

$$|z_j^n|^2 \leqslant (1-2(b+\delta)\Delta t)^{-n}\left\{|z_j^0|^2 + \frac{1}{4\delta(b+\delta)} \max_{(x,t)\in Q_T} |f(x, t, 0)|^2\right\},$$

where $j=1, 2, \cdots, J-1$ and $n=1, 2, \cdots, N$. Then there is a constant C_2 independent of Δt and h, such that

$$|z_j^n| \leqslant C_2, \quad j=1, 2, \cdots, J-1;\ n=1, 2, \cdots, N.$$

It is evident that $|z_0^n|$ and $|z_J^n|$ are also uniformly bounded with respect to Δt and h from the nonlinear finite difference boundary conditions (8). Hence the following lemma is valid.

Lemma 2. *Under the conditions* (I), (II) *and* (III), *for sufficiently small Δt that $1-2b\Delta t>0$, the finite difference solutions z_j^n $(j=0, 1, \cdots, J;\ n=0, 1, \cdots, N)$ of the nonlinear finite difference system* (7), (8) *and* (9) *are uniformly bounded with respect to Δt and h, i.e.,*

$$|z_j^n| \leqslant K_1, \quad j=0, 1, \cdots, J;\ n=0, 1, \cdots, N, \tag{14}$$

where K_1 is a constant independent of Δt and h.

For the estimation of $\frac{\Delta_+ z_j^n}{h}$ $(j=0, 1, \cdots, J-1;\ n=0, 1, \cdots, N)$, we make the scalar product of $\Delta_+\Delta_- z_j^n \frac{\Delta t}{h}$ with the system (7) and then sum up the resulting relations for $j=1, 2, \cdots, J-1$. Then we have

$$\frac{1}{h}\sum_{j=1}^{J-1} \Delta_+\Delta_- z_j^n \cdot (z_j^n - z_j^{n-1}) = \frac{\Delta t}{h}\sum_{j=1}^{J-1} \Delta_+\Delta_- z_j^n \cdot f_j^n, \quad n=1, 2, \cdots, N, \tag{15}$$

where $\Delta_+\Delta_- z_j^n \cdot (z_j^n \times \Delta_+\Delta_- z_j^n) = 0$.

For the left hand part of this equality (15),

$$\frac{1}{h}\sum_{j=1}^{J-1} \Delta_+\Delta_- z_j^n \cdot (z_j^n - z_j^{n-1}) = -\frac{1}{h}\sum_{j=1}^{J-2} \Delta_+ z_j^n \cdot \Delta_+(z_j^n - z_j^{n-1})$$
$$-\frac{1}{h}\Delta_+ z_0^n \cdot (z_1^n - z_1^{n-1}) + \frac{1}{h}\Delta_- z_J^n \cdot (z_{J-1}^n - z_{J-1}^{n-1}).$$

Substituting the nonlinear finite difference boundary expressions (8) into the right part of the above equality and regarding that

$$\frac{\Delta_+ z_0^n}{h}\cdot(z_1^n-z_1^{n-1})-\frac{\Delta_- z_J^n}{h}\cdot(z_{J-1}^n-z_{J-1}^{n-1})$$
$$=\widetilde{\mathrm{grad}_0}\,\psi(z_1^n,\ z_{J-1}^n)\cdot(z_1^n-z_1^{n-1})+\widetilde{\mathrm{grad}_1}\,\psi(z_1^n,\ z_{J-1}^n)\cdot(z_{J-1}^n-z_{J-1}^{n-1})$$
$$=\psi(z_1^n,\ z_{J-1}^n)-\psi(z_1^{n-1},\ z_{J-1}^{n-1}),$$

we have

$$\frac{1}{h}\sum_{j=1}^{J-1}\Delta_+\Delta_- z_j^n\cdot(z_j^n-z_j^{n-1})=-\sum_{j=1}^{J-2}\left|\frac{\Delta_+ z_j^n}{h}\right|^2 h+\sum_{j=1}^{J-2}\frac{\Delta_+ z_j^n}{h}\cdot\frac{\Delta_+ z_j^{n-1}}{h}h$$
$$-\psi(z_1^n,\ z_{J-1}^n)+\psi(z_1^{n-1},\ z_{J-1}^{n-1}). \tag{16}$$

Now we turn to consider the right hand part of the equality (15). This part can be written in the form

$$\frac{\Delta t}{h}\sum_{j=1}^{J-1}\Delta_+\Delta_- z_j^n\cdot f_j^n=-\frac{\Delta t}{h}\sum_{j=0}^{J-1}\Delta_+ z_j^n\cdot\Delta_+ f_j^n$$
$$-\Delta t\,\frac{\Delta_+ z_0^n}{h}\cdot f(0,\ t_n,\ z_0^n)+\Delta t\,\frac{\Delta_- z_J^n}{h}\cdot f(l,\ t_n,\ z_J^n).$$

From (8), we have

$$\left|\frac{\Delta_+ z_0^n}{h}\cdot f(0,\ t_n,\ z_0^n)-\frac{\Delta_- z_J^n}{h}\cdot f(l,\ t_n,\ z_J^n)\right|$$
$$=\left|\widetilde{\mathrm{grad}_0}\,\psi(z_1^n,\ z_{J-1}^n)\cdot f(0,\ t_n,\ z_0^n)+\widetilde{\mathrm{grad}_1}\,\psi(z_1^n,\ z_{J-1}^n)\cdot f(l,\ t_n,\ z_J^n)\right|\leqslant C_3,$$

where C_3 is a constant independent of Δt and h. On the other hand, we have

$$\sum_{j=0}^{J-1}\frac{\Delta_+ z_j^n}{h}\cdot\Delta_+ f_j^n=\sum_{j=0}^{J-1}\frac{\Delta_+ z_j^n}{h}\cdot[f(x_{j+1},\ t_n,\ z_{j+1}^n)-f(x_{j+1},\ t_n,\ z_j^n)]$$
$$+\sum_{j=0}^{J-1}\frac{\Delta_+ z_j^n}{h}\cdot[f(x_{j+1},\ t_n,\ z_j^n)-f(x_j,\ t_n,\ z_j^n)].$$

Since the Jacobi derivative matrix $f_z(x,\ t,\ z)$ of the vector function $f(x,\ t,\ z)$ has the property of semiboundedness (6),

$$\sum_{j=0}^{J-1}\frac{\Delta_+ z_j^n}{h}\cdot[f(x_{j+1},\ t_n,\ z_{j+1}^n)-f(x_j,\ t_n,\ z_j^n)]\leqslant b\left\|\frac{\Delta_+ z^n}{h}\right\|_h^2.$$

From the continuity of the vector function $f_x(x,\ t,\ z)$, we obtain

$$\left|\sum_{j=0}^{J-1}\frac{\Delta_+ z_j^n}{h}\cdot[f(x_{j+1},\ t_n,\ z_j^n)-f(x_j,\ t_n,\ z_j^n)]\right|\leqslant C_4\left\|\frac{\Delta_+ z^n}{h}\right\|_h^2+C_5,$$

where C_4 and C_5 are constants independent of Δt and h. At last for the right hand part of (15), we have

$$\left|\sum_{j=1}^{J-1}\left(\frac{\Delta_+\Delta_- z_j^n}{h^2}\cdot f_j^n\right)h\right|\leqslant C_6\sum_{j=1}^{J-2}\left|\frac{\Delta_+ z_j^n}{h}\right|^2 h+C_7, \tag{17}$$

where $C_6=b+C_4$ and

$$C_7\geqslant C_3+C_5+(b+C_4)h\{|\widetilde{\mathrm{grad}_0}\,\psi(z_1^n,\ z_{J-1}^n)|^2+|\widetilde{\mathrm{grad}_1}\,\psi(z_1^n,\ z_{J-1}^n)|^2\}.$$

Finally we obtain the iterative inequality

$$(1-2C_6\Delta t)W(n)\leqslant W(n-1)+2\Delta t C_7, \tag{18}$$

from the relations (15), (16) and (17), where

$$W(n)=\sum_{j=1}^{J-2}\left|\frac{\Delta_+z_j^n}{h}\right|^2h+2\psi(z_1^n,\ z_{J-1}^n).\tag{19}$$

Hence the iterative relation (18) implies the uniform boundedness of $W(n)$, as Δt is sufficiently small. Consequently $\left\|\frac{\Delta_+z^n}{h}\right\|_h$ $(n=0,\ 1,\ \cdots,\ N)$ is uniformly bounded with respect to Δt and h. Then the following lemma is valid.

Lemma 3. *Under the conditions* (I), (II) *and* (III), *the solution* z_j^n $(j=0,\ 1,\cdots,\ J;\ n=0,\ 1,\ \cdots,\ N)$ *of the finite difference system* (7), (8) *and* (9), *corresponding to the nonlinear mutual boundary problem* (4) *and* (5) *for the system* (2) *of ferromagnetic chain has the estimate*

$$\left\|\frac{\Delta_+z^n}{h}\right\|_h\leqslant K_2,\quad n=0,\ 1,\ \cdots,\ N,\tag{20}$$

where Δt is sufficiently small and K_2 is a constant independent of Δt and h.

Let us now consider the discrete function $s_j^n=\sum_{i=0}^{j}z_i^nh$ $(j=1,\ 2,\ \cdots,\ J;\ n=0,\ 1,\ \cdots,\ N)$[7]. From (7), we have

$$\frac{s_j^n-s_j^{n-1}}{\Delta t}=\sum_{i=0}^{j}\frac{1}{h}(z_i^n\times\Delta_+\Delta_-z_i^n)+\sum_{i=0}^{j}hf(x_i,\ t_n,\ z_i^n),\tag{21}$$

where by direct calculation,

$$\sum_{i=0}^{j}\frac{1}{h}(z_i^n\times\Delta_+\Delta_-z_i^n)=z_j^n\times\frac{\Delta_+z_j^n}{h}-z_0^n\times\frac{\Delta_+z_0^n}{h}.$$

Then (21) becomes

$$\frac{s_j^n-s_j^{n-1}}{\Delta t}=z_j^n\times\frac{\Delta_+z_j^n}{h}-z_0^n\times\widetilde{\mathrm{grad}}_0\,\psi(z_1^n,\ z_{J-1}^n)+\sum_{i=0}^{j}hf(x_i,\ t_n,\ z_i^n).$$

From this relation it is easy to verify the following lemma.

Lemma 4. *Under the conditions* (I), (II) *and* (III) *there is the estimation*

$$\left\|\frac{s^n-s^{n-1}}{\Delta t}\right\|_h\leqslant K_3,\quad n=1,\ 2,\ \cdots,\ N,\tag{22}$$

where $s_j=\sum_{i=1}^{j}z_ih$ $(j=1,\ 2,\ \cdots,\ J)$ and K_3 is a constant independent of Δt and h.

From the definition of s_j^n $(j=1,\ 2,\ \cdots,\ J;\ n=0,\ 1,\ \cdots,\ N)$ and the estimations obtained in Lemma 2 and Lemma 3, we know that $\|s^n\|_h$, $\left\|\frac{\Delta_+s^n}{h}\right\|_h$, $\left\|\frac{\Delta_+\Delta_-s^n}{h^2}\right\|_h$ and $\left\|\frac{s^n-s^{n-1}}{\Delta t}\right\|_h$ $(n=1,\ 2,\ \cdots,\ N)$ are uniformly bounded with respect to Δt and h. Using the interpolation formulas of difference quotients for the discrete functions in [9, 10], we have

$$\begin{aligned}&|\Delta_+\Delta_-s_j^n|\leqslant K_4h^{\frac{3}{2}},\quad j=2,\ 3,\ \cdots,\ J-1;\ n=0,\ 1,\ \cdots,\ N,\\&|\Delta_+s_j^n-\Delta_+s_j^{n-1}|\leqslant K_5\Delta t^{\frac{1}{4}}h,\quad j=1,\ 2,\ \cdots,\ J-1;\ n=1,\ 2,\ \cdots,\ N,\end{aligned}\tag{23}$$

where K_4 and K_5 are constants independent of Δt and h.

Lemma 5. *Under the conditions* (I), (II) *and* (III), *the solution* z_j^n $(j=0,1,\cdots,\ J;\ n=0,\ 1,\ \cdots,\ N)$ *of the finite difference system* (7), (8) *and* (9) *has the estimations*

$$|\Delta_+z_j^n|\leqslant K_4h^{\frac{1}{2}},\quad j=0,\ 1,\ \cdots,\ J-1;\ n=0,\ 1,\ \cdots,\ N\tag{24}$$

and

$$|z_j^n - z_j^{n-1}| \leqslant K_5 \Delta t^{\frac{1}{4}}, \quad j=0, 1, \cdots, J; n=1, 2, \cdots, N, \tag{25}$$

where K_4 and K_5 are constants independent of Δt and h.

§ 4

At begin of this section let us define the weak solution of the nonlinear mutual boundary problem (4) and (5) for the system (2) of ferro-magnetic chain. The purpose of this section is to establish the existence theorem of the weak solution by the way of passing limit for the finite difference solution z_j^n ($j=0, 1, \cdots, J$; $n=0, 1, \cdots, N$) of the system (7), (8) and (9) as Δt and h tend to zero.

Definition. *The three-dimensional vector function $z(x, t) \in L_2(0, T; H^1(0, l)) \cap C(Q_T)$ is called the weak solution of the nonlinear mutual boundary problem (4) and (5) for the system (2) of ferro-magnetic chain, if the following integral relation*

$$\int_0^T \int_0^l [g_t(x, t) z(x, t) - g_x(x, t)(z(x, t) \times z_x(x, t)) + g(x, t) f(x, t, z(x, t))] dx\, dt$$
$$+ \int_0^l g(x, 0)\varphi(x) dx - \int_0^l g(l, t)(z(l, t) \times \text{grad}_1 \psi(z(0, t), z(l, t))) dt$$
$$- \int_0^l g(0, t)(z(0, t) \times \text{grad}_0 \psi(z(0, t), z(l, t))) dt = 0 \tag{26}$$

holds for any test function $g(x, t) \in C^{(1)}(Q_T)$ with the property $g(x, T) \equiv 0$.

Let $z_{h\Delta t}(x, t) = z_j^n$ and $\bar{z}_{h\Delta t}(x, t) = \dfrac{\Delta_+ z_j^n}{h}$ for $(x, t) \in Q_j^n = \{jh < x \leqslant (j+1)h; (n-1)\Delta t < t \leqslant n\Delta t\}$ ($j=0, 1, \cdots, J-1$; $n=1, 2, \cdots, N$). Then $z_{h\Delta t}(x, t)$ and $\bar{z}_{h\Delta t}(x, t)$ are three-dimensional piecewise constant vector functions in Q_T. By the results of Lemma 2 and Lemma 3, it is easy to verify the estimation

$$\max_{(x,t)\in Q_T} |z_{h\Delta t}(x, t)| + \sup_{0<t<T} \|\bar{z}_{h\Delta t}(\cdot, t)\|_{L_2(0,l)} \leqslant C_8,$$

where C_8 is a constant independent of Δt and h.

We can select a sequence $\{h_i, \Delta t_i\}$, such that when $i \to \infty$, $h_i^2 + \Delta t_i^2 \to 0$ and also $z_{h_i\Delta t_i}(x, t)$ and $\bar{z}_{h_i\Delta t_i}(x, t)$ converge weakly to $z(x, t)$ and $\bar{z}(x, t)$ in $L_p(0, T; L_2(0, l))$ respectively for $2 \leqslant p < \infty$. The norms of $z(x, t)$ and $\bar{z}(x, t)$ in $L_p(0, T; L_2(0, l))$ are uniformly bounded for $2 \leqslant p < \infty$. Hence we have

$$\sup_{0<t<T} \|z(\cdot, t)\|_{L_2(0,l)} + \sup_{0<t<T} \|\bar{z}(\cdot, t)\|_{L_2(0,l)} \leqslant C_9,$$

where C_9 is a constant independent of Δt and h.

This means that $z(x, t)$ and $\bar{z}(x, t)$ are the three-dimensional vector functions belonging to $L_\infty(0, T; L_2(0, l))$. It can be verified by the usual way that $z(x, t)$ has the generalized derivative $z_x(x, t) = \bar{z}(x, t)$.

Let us construct $z_{h\Delta t}^*(x, t)$ as follows: in every small rectangular grid Q_j^n ($j=0, 1, \cdots, J-1$; $n=1, 2, \cdots, N$), $z_{h\Delta t}^*(x, t)$ is obtained by the linear expansion in both directions from the values of the discrete function z_j^n at four corners of Q_j^n. From (25) and (24), $\{z_{h\Delta t}^*(x, t)\}$ is uniformly bounded and equicontinuous set of three-dimensional vector functions. As $h_i^2 + \Delta t_i^2 \to 0$, the sequence $\{z_{h_i\Delta t_i}^*(x, t)\}$ converges uniformly to a three-dimensional vector function $z^*(x, t)$ in Q_T. Evidently

$z^*(x, t) \in C^{(\frac{1}{2}, \frac{1}{4})}(Q_T)$. From the construction of the vector functions $z^*_{h\Delta t}(x, t)$ and $z_{h\Delta t}(x, t)$, we have

$$|z^*_{h\Delta t}(x, t) - z_{h\Delta t}(x, t)| \leqslant C_{10}(h^{\frac{1}{2}} + \Delta t^{\frac{1}{4}}),$$

where C_{10} is a constant independent of Δt and h. So $z^*(x, t) = z(x, t)$ in Q_T.

Now we turn to prove that $z(x, t)$ is a weak solution of the nonlinear mutual boundary problem (4) and (5) for the system (2) of ferro-magnetic chain.

Let $g(x, t)$ be a test function and $g^n_j = g(x_j, t_n)$ for $j=0, 1, \cdots, J$; $n=0, 1, \cdots, N$. From the finite difference system (7), we have

$$\sum_{j=1}^{J-1}\sum_{n=1}^{N} g^{n-1}_j \frac{z^n_j - z^{n-1}_j}{\Delta t} h\Delta t = \sum_{j=1}^{J-1}\sum_{n=1}^{N} g^{n-1}_j \left(z^n_j \times \frac{\Delta_+\Delta_- z^n_j}{h^2}\right) h\Delta t + \sum_{j=1}^{J-1}\sum_{n=1}^{N} g^{n-1}_j f^n_j h\Delta t. \quad (27)$$

From the identities

$$g^{n-1}_j \frac{z^n_j - z^{n-1}_j}{\Delta t} = -z^n_j \frac{g^n_j - g^{n-1}_j}{\Delta t} + \frac{g^n_j z^n_j - g^{n-1}_j z^{n-1}_j}{\Delta t}$$

and

$$g^{n-1}_j\left(z^n_j \times \frac{\Delta_+\Delta_- z^n_j}{h^2}\right) = -\frac{\Delta_- g^{n-1}_j}{h}\left(z^n_{j-1} \times \frac{\Delta_+ z^n_{j-1}}{h}\right) + \frac{1}{h}\left\{ g^{n-1}_j\left(z^n_j \times \frac{\Delta_+ z^n_j}{h}\right) - g^{n-1}_{j-1}\left(z^n_{j-1} \times \frac{\Delta_+ z^n_{j-1}}{h}\right)\right\},$$

we have

$$\sum_{j=1}^{J-1}\sum_{n=1}^{N} g^{n-1}_j \frac{z^n_j - z^{n-1}_j}{\Delta t} h\Delta t = -\sum_{j=1}^{J-1}\sum_{n=1}^{N} \frac{g^n_j - g^{n-1}_j}{\Delta t} z^n_j h\Delta t - \sum_{j=1}^{J-1} g^0_j z^0_j h + \sum_{j=1}^{J-1} g^N_j z^N_j h$$

and

$$\begin{aligned}&\sum_{j=1}^{J-1}\sum_{n=1}^{N} g^{n-1}_j\left(z^n_j \times \frac{\Delta_+\Delta_- z^n_j}{h^2}\right) h\Delta t \\ &= -\sum_{j=1}^{J-1}\sum_{n=1}^{N} \frac{\Delta_- g^{n-1}_j}{h}\left(z^n_{j-1} \times \frac{\Delta_+ z^n_{j-1}}{h}\right) h\Delta t + \sum_{n=1}^{N} g^{n-1}_{J-1}\left(z^n_{J-1} \times \frac{\Delta_+ z^n_{J-1}}{h}\right)\Delta t \\ &\quad - \sum_{n=1}^{N} g^{n-1}_0\left(z^n_0 \times \frac{\Delta_+ z^n_0}{h}\right)\Delta t.\end{aligned}$$

Since $g(x, T) \equiv 0$, then $g^N_j = 0$ $(j=0, 1, \cdots, J)$. On account of the finite difference nonlinear boundary conditions (8), (27) becomes

$$\begin{aligned}&\sum_{j=1}^{J-1}\sum_{n=1}^{N} \frac{g^n_j - g^{n-1}_j}{\Delta t} z^n_j h\Delta t + \sum_{j=0}^{J-1} g^0_j z^0_j h - \sum_{j=1}^{J-1}\sum_{n=1}^{N} \frac{\Delta_- g^{n-1}_j}{h}\left(z^n_{j-1} \times \frac{\Delta_+ z^n_{j-1}}{h}\right) h\Delta t \\ &+ \sum_{j=1}^{J-1}\sum_{n=1}^{N} g^{n-1}_j f^n_j h\Delta t - \sum_{n=1}^{N} g^{n-1}_{J-1}\,(z^n_{J-1} \times \widetilde{\operatorname{grad}}_1 \psi(z^n_1, z^n_{J-1}))\,\Delta t \\ &- \sum_{n=1}^{N} g^{n-1}_0 (z^n_0 \times \widetilde{\operatorname{grad}}_0 \psi(z^n_1, z^n_{J-1}))\,\Delta t = 0.\end{aligned} \quad (28)$$

This can be expressed in the integral form as

$$\begin{aligned}&\iint_{Q_T} \tilde{g}_{h\Delta t}(x, t) z_{h\Delta t}(x, t)\,dx\,dt + \int_0^l g_{h\Delta t}(x, 0)\varphi_h(x)\,dx \\ &- \iint_{Q_T} \bar{g}_{h\Delta t}(x-h, t-\Delta t)\,(z_{h\Delta t}(x-h, t) \times \bar{z}_{h\Delta t}(x-h, t))\,dx\,dt \\ &+ \iint_{Q_T} \bar{g}_{h\Delta t}(x, t-\Delta t) F_{h\Delta t}(x, t)\,dx\,dt\end{aligned}$$

$$-\int_0^T g_{h\Delta t}(l-h,\ t-\Delta t)\,(z_{h\Delta t}(l-h,\ t)\times G^1_{h\Delta t}(t))\,dt$$
$$-\int_0^T g_{h\Delta t}(0,\ t-\Delta t)\,(z_{h\Delta t}(0,\ t)\times G^0_{h\Delta t}(t))\,dt=0, \tag{29}$$

where $g_{h\Delta t}(x,\ t)$, $\bar g_{h\Delta t}(x,\ t)$ and $\tilde g_{h\Delta t}(x,\ t)$ are the appropriate piecewise constant functions, corresponding to the discrete functions g^n_j, $\frac{\Delta_+ g^n_j}{h}$ and $\frac{g^n_j-g^{n-1}_j}{\Delta t}$ $(j=0,\ 1,\ \cdots,\ J-1;\ n=1,\ 2,\ \cdots,\ N)$ defined as before respectively, $F_{h\Delta t}(x,\ t)=f^n_j=f(x_j,\ t_n,\ z^n_j)$ in Q^n_j $(j=0,\ 1,\ \cdots,\ J-1;\ n=1,\ 2,\ \cdots,\ N)$ and $\varphi_h(x)=\varphi_j$ in $(jh,\ (j+1)h]$ $(j=0,\ 1,\ \cdots,\ J-1)$ are two three-dimensional piecewise constant vector functions; furthermore $G^0_{h\Delta t}(t)=\widetilde{\mathrm{grad}}_0\,\psi(z^n_1,\ z^n_{J-1})$ and $G^1_{h\Delta t}(t)=\widetilde{\mathrm{grad}}_1\,\psi(z^n_1,\ z^n_{J-1})$ in $((n-1)\Delta t,\ n\Delta t]$ $(n=1,\ 2,\ \cdots,\ N)$ are also two three-dimensional piecewise constant vector functions. Since $g(x,\ t)$ is a smooth function, $g_{h\Delta t}(x,\ t)$, $\bar g_{h\Delta t}(x,\ t)$ and $\tilde g_{h\Delta t}(x,\ t)$ are uniformly convergent to $g(x,\ t)$, $g_x(x,\ t)$ and $g_t(x,\ t)$ in Q_T respectively as $h^2+\Delta t^2\to 0$ and $g_{h\Delta t}(x,\ 0)$ is uniformly convergent to $g(x,\ 0)$ in $[0,\ l]$ as $h\to 0$. When $h\to 0$, $\varphi_h(x)$ converges uniformly to $\varphi(x)$ in $[0,\ l]$. On the other hand $z_{h\Delta t}(x,\ t)$ and $F_{h\Delta t}(x,\ t)$ are uniformly convergent to $z(x,\ t)$ and $f(x,\ t,\ z(x,\ t))$ respectively in Q_T and $\bar z_{h\Delta t}(x,\ t)$ is weakly convergent to $z_x(x,\ t)$ in Q_T as $h_i^2+\Delta t_i^2\to 0$. When $h_i^2+\Delta t_i^2\to 0$, $G^0_{h\Delta t}(t)$ and $G^1_{h\Delta t}(t)$ are uniformly convergent to $\mathrm{grad}_0\,\psi(z(0,t),\ z(l,\ t))$ and $\mathrm{grad}_1\,\psi(z(0,\ t),\ z(l,\ t))$ in $[0,\ T]$ respectively. Hence passing limit as $h_i^2+\Delta t_i^2\to 0$, (29) tends to the integral relation (26) for any test function $g(x,\ t)\in C^1(Q_T)$ with $g(x,\ T)\equiv 0$. Therefore the three-dimensional vector function $z(x,\ t)$ is a weak solution of the nonlinear mutual boundary problem (4) and (5) for the system (2) of ferro-magnetic chain. Thus the following existence theorem is proved.

Theorem. *Suppose that the conditions* (I), (II) *and* (III) *are satisfied, the nonlinear mutual boundary problem* (4) *and* (5) *for the system* (2) *of ferro-magnetic chain has at least one three-dimensional vector weak solution* $z(x,\ t)\in L_\infty(0,\ T;\ H^1(0,\ l))\cap C^{(\frac{1}{2},\ \frac{1}{4})}(Q_T)$.

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