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ON THE CONVERGENCE OF WEISSMAN-TAYLOR ELEMENT FOR REISSNER-MINDLIN PLATE

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Abstract. In this paper, we study the Weissman-Taylor rectangular element for the Reissner-Mindlin plate [12] model and provide a convergence analysis for the transverse displacement and the rotation. We show that the element is stable and locking free, thereby improve the results of [8] and [9].

Key Words. Reissner-Mindlin plate, Locking-free, Weissman-Taylor element

1. Introduction

The Reissner-Mindlin plate model is widely used by engineers. A direct finite element approximation often yields poor results due to the shear locking, namely, the numerical solution is significantly smaller than the exact one. The development of general procedures to overcome this drawback is an active research area. Many methods have been proposed so far. However, a rigorous convergence and stability proof is missing for most of these methods, even if numerical tests show that they work properly. This is the case for the rectangular element proposed by Weissman and Taylor[12]. The element was analyzed in [8] and [9]. Nevertheless, whether the element is locking free is unclear in the previous analysis.

In this paper, we show that the transverse displacement and the rotation are convergent uniformly with respect to the plate thickness for the rectangular Weissman-Taylor element. Therefore the element is locking-free. For simplicity, we consider only a square mesh. However, the analysis is valid for a rectangular mesh as well.

The paper is organized as follows. The Reissner-Mindlin plate model is reviewed in Section 2; the Weissman-Taylor element is introduced in Section 3; the error analysis is presented in Section 4; and finally, a conclusion is given in Section 5.

Throughout the paper, C denotes a genetic constant, which is not necessarily the same at different places. However, C is independent of the mesh size h and the plate thickness t. We shall use standard notations of the Sobolev space.

2. Reissner-Mindlin Plate Model

Let Ω be a rectangle representing the mid-surface of the plate. Assume that the plate is clamped along the boundary $\partial\Omega$. Let ω and $\boldsymbol{\phi}$ denote the transverse displacement and the rotation, respectively, which are determined by the following

Problem 2.1. *Find* $(\boldsymbol{\phi}, \omega) \in \boldsymbol{H}_0^1(\Omega) \times H_0^1(\Omega)$*, such that*

$$(1) \qquad a(\boldsymbol{\phi}, \boldsymbol{\psi}) + \lambda t^{-2}(\nabla\omega - \boldsymbol{\phi}, \nabla v - \boldsymbol{\psi}) = (g, v), \quad \forall(\boldsymbol{\psi}, v) \in \boldsymbol{H}_0^1(\Omega) \times H_0^1(\Omega).$$

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Here g is the scaled transverse loading, t is the plate thickness, $\lambda = E\kappa/(2+2\nu)$ is the shear modulus, E is the Young's modulus, ν is the Poisson ratio and κ is the shear correction factor. The bilinear form a is defined by $a(\boldsymbol{\eta},\boldsymbol{\psi}) = (\mathcal{C}\mathcal{E}\boldsymbol{\eta},\mathcal{E}\boldsymbol{\psi})$, here $\mathcal{C}\tau$ is defined for any 2×2 symmetric matrix τ as

$$\mathcal{C}\tau := \frac{E}{12(1-\nu^2)}\left[(1-\nu)\tau + \nu\,\mathrm{tr}(\tau)\boldsymbol{I}\right].$$

Introducing the shear strain

$$\boldsymbol{\gamma} := \lambda t^{-2}(\nabla\omega - \boldsymbol{\phi})$$

as an independent variable, we get the following mixed problem

Problem 2.2. *Find* $(\boldsymbol{\phi},\omega,\boldsymbol{\gamma}) \in \boldsymbol{H}_0^1(\Omega)\times H_0^1(\Omega)\times \boldsymbol{L}^2(\Omega)$, *such that*

$$a(\boldsymbol{\phi},\boldsymbol{\psi}) + (\boldsymbol{\gamma},\nabla v - \boldsymbol{\psi}) = (g,v), \qquad \forall(\boldsymbol{\psi},v)\in \boldsymbol{H}_0^1(\Omega)\times H_0^1(\Omega), \tag{2}$$

$$\lambda^{-1}t^2(\boldsymbol{\gamma},\boldsymbol{s}) - (\nabla\omega - \boldsymbol{\phi},\boldsymbol{s}) = 0, \qquad \forall \boldsymbol{s}\in \boldsymbol{L}^2(\Omega). \tag{3}$$

The existence and uniqueness of the solution of Problem 2.2 and the following regularity result can be found in [4, 9].

Lemma 2.3. *Let* $(\boldsymbol{\phi},\omega,\boldsymbol{\gamma}) \in \boldsymbol{H}_0^1(\Omega)\times H_0^1(\Omega)\times \boldsymbol{L}^2(\Omega)$ *be the solution of Problem 2.2, then the following regularity estimates hold*

$$\|\boldsymbol{\phi}\|_2 + \|\boldsymbol{\gamma}\|_0 \le C\|g\|_{-1}, \tag{4}$$

$$\|\omega\|_2 \le C(\|g\|_{-1} + t^2\|g\|_0), \qquad t\|\boldsymbol{\phi}\|_3 \le C\|g\|_0, \tag{5}$$

$$\|\boldsymbol{\gamma}\|_{\boldsymbol{H}(\mathrm{div})} \le C\|g\|_0, \qquad t\|\boldsymbol{\gamma}\|_1 \le C(\|g\|_{-1} + t\|g\|_0). \tag{6}$$

3. Finite element approximation

Let $\mathcal{T}_h$ be a uniform square partition of the domain Ω with the mesh size h, which is the refinement of a coarser partition $\mathcal{T}_{2h}$ with the mesh size $2h$. Let F_K be the affine mapping from the reference square $\hat{K} = [-1,1]^2$ onto the element K, which is defined by

$$F_K(\xi,\eta) = (x_k + h\xi, y_k + h\eta),$$

where (x_k,y_k) is the center of K. Denote $\hat{v}(\xi,\eta) = v(x_k+h\xi, y_k+h\eta)$.

Define

$$\begin{aligned}
W_h &= \{v\in H_0^1(\Omega) \mid \hat{v}_{|_{\hat{K}}} \in Q_1(\hat{K})\ \forall K\in\mathcal{T}_h\},\\
B_{Nc} &= \{v\in L^2(\Omega) \mid \hat{v}_{|_{\hat{K}}} \in (1-\xi^2, 1-\eta^2)\ \forall K\in\mathcal{T}_h\},\\
\widehat{\boldsymbol{\Gamma}}_h &= \{\boldsymbol{\gamma}\in \boldsymbol{L}^2(\Omega) \mid \boldsymbol{\gamma}_{|_K} \in P_1(K)^2\ \forall K\in\mathcal{T}_h\},\\
\boldsymbol{\Gamma}_h^R &= \{\boldsymbol{\chi}\in \boldsymbol{L}^2(\Omega) \mid \hat{\boldsymbol{\chi}}_{|_{\hat{K}}} \in Q_{0,1}\times Q_{1,0}\ \forall K\in\mathcal{T}_h\},\\
\boldsymbol{\Gamma}_h &= \{\boldsymbol{\chi}\in \boldsymbol{H}_0(\mathrm{rot},\Omega) \mid \hat{\boldsymbol{\chi}}_{|_{\hat{K}}} \in Q_{0,1}\times Q_{1,0}\ \forall K\in\mathcal{T}_h\},
\end{aligned}$$

where $(1-\xi^2, 1-\eta^2)$ is the non-conforming bubble space generated by $1-\xi^2$ and $1-\eta^2$, $Q_{0,1} = (1,\eta), Q_{1,0} = (1,\xi)$.

Set

$$W_h^* = W_h \oplus B_{Nc}, \boldsymbol{V}_h^* = [W_h]^2 \oplus B_{Nc}^2, \boldsymbol{V}_h = [W_h]^2.$$

Let $\boldsymbol{R}_h$ be the usual Raviart-Thomas interpolation operator from $\boldsymbol{H}(\mathrm{rot},\Omega)$ to $\boldsymbol{\Gamma}_h$, which is locally defined by

$$\int_{e_i} (\boldsymbol{\psi} - \boldsymbol{R}_h\boldsymbol{\psi}) \cdot \boldsymbol{t} ds = 0, \qquad e_i \subset \partial K.$$

For the operator $\boldsymbol{R}_h$, we have the following properties(see [9] and references therein):

Lemma 3.1. *Let* Π *be the* L^2 *projection operator onto* $\boldsymbol{\Gamma}_h$, Π_0 *be the* L^2 *projection operator onto the piecewise constant space, then*

$$\boldsymbol{R}_h\boldsymbol{\psi} = \Pi\boldsymbol{\psi}, \quad \mathrm{rot}\,\boldsymbol{R}_h\boldsymbol{\psi} = \Pi_0\,\mathrm{rot}\,\boldsymbol{\psi}, \qquad \boldsymbol{\psi} \in \boldsymbol{V}_h,$$

$$\begin{aligned} \|\boldsymbol{\psi} - \boldsymbol{R}_h\boldsymbol{\psi}\|_0 &\leq Ch\|\boldsymbol{\psi}\|_1, \qquad \forall \boldsymbol{\psi} \in \boldsymbol{H}^1(\Omega) \cap \boldsymbol{H}_0(\mathrm{rot},\Omega), \\ \|\boldsymbol{\psi} - \boldsymbol{R}_h\boldsymbol{\psi}\|_{H^{-1}_{(\mathrm{rot})}} &\leq Ch^2(\|\boldsymbol{\psi}\|_1 + \|\,\mathrm{rot}\,\boldsymbol{\psi}\|_1), \qquad \forall \boldsymbol{\psi} \in \boldsymbol{H}^1(\Omega) \cap \boldsymbol{H}^1(\mathrm{rot}), \end{aligned}$$

where the norm $\|\cdot\|_{H^{-1}_{(\mathrm{rot})}}$ *is defined by*

$$\|\boldsymbol{\psi}\|_{H^{-1}_{(\mathrm{rot})}} = (\|\boldsymbol{\psi}\|^2_{-1} + \|\,\mathrm{rot}\,\boldsymbol{\psi}\|^2_{-1})^{1/2}.$$

The Weissman-Taylor element [12] can be stated as the solution of the following

Problem 3.2. *Find* $(\boldsymbol{\phi}_h, \omega_h, \boldsymbol{\gamma}_h) \in \boldsymbol{V}^*_h \times W^*_h \times \hat{\boldsymbol{\Gamma}}_h$, *such that*

$$\begin{aligned} a_h(\boldsymbol{\phi}_h, \boldsymbol{\psi}) + (\boldsymbol{\gamma}_h, \boldsymbol{\nabla}_h v - \boldsymbol{\psi}) &= (f, v^c), \ \forall (\boldsymbol{\psi}, v) \in \boldsymbol{V}^*_h \times W^*_h, \\ (\boldsymbol{\nabla}_h\omega_h - \boldsymbol{\phi}_h, \boldsymbol{s}) - \lambda^{-1}t^2(\boldsymbol{\gamma}_h, \boldsymbol{s}) &= 0, \ \forall \boldsymbol{s} \in \hat{\boldsymbol{\Gamma}}_h, \end{aligned}$$

where v^c *denotes the conforming part of* $v \in W^*_h$.

Due to the non-conforming bubble, the bilinear form $a(\cdot,\cdot)$ is replaced by $a_h(\cdot,\cdot)$, which is the same bilinear form computed element by element. Similarly, the gradient operator $\boldsymbol{\nabla}$ is replaced by gradient on each element, $\boldsymbol{\nabla}_h$. See [12] for details.

By the static condensation procedure, Problem 3.2 can be converted into the following

Problem 3.3. [8] *Find* $(\boldsymbol{\phi}^c_h, \omega^c_h, \boldsymbol{\gamma}_h) \in \boldsymbol{V}_h \times W_h \times \boldsymbol{\Gamma}^R_h$, *such that*

$$\mathcal{A}_h(\boldsymbol{\phi}^c_h, \omega^c_h, \boldsymbol{\gamma}_h; \boldsymbol{\psi}, v, \boldsymbol{s}) = (f, v), \ \forall (\boldsymbol{\psi}, v, \boldsymbol{s}) \in \boldsymbol{V}_h \times W_h \times \boldsymbol{\Gamma}^R_h, \tag{7}$$

where $\boldsymbol{\phi}^c_h$ and ω^c_h are the conforming parts of $\boldsymbol{\phi}_h$ and ω_h, respectively, and

$$\begin{aligned} \mathcal{A}_h(\boldsymbol{\phi}_h, \omega_h, \boldsymbol{\gamma}_h; \boldsymbol{\psi}, v, \boldsymbol{s}) &= a_h(\boldsymbol{\phi}_h, \boldsymbol{\psi}) - \beta_2 h^2 \sum_{K\in\mathcal{T}_h} (A\boldsymbol{\phi}_h, A\boldsymbol{\psi}) \\ &+ (\boldsymbol{\gamma}_h, \boldsymbol{\nabla} v - \boldsymbol{\psi}) + \beta_1 h^2 \sum_{K\in\mathcal{T}_h} (\boldsymbol{\gamma}_h, A\boldsymbol{\psi}) + \lambda^{-1}t^2(\boldsymbol{\gamma}_h, \boldsymbol{s}) \\ &- (\boldsymbol{\nabla}\omega_h - \boldsymbol{\phi}_h, \boldsymbol{s}) + \beta_3 h^2 \sum_{K\in\mathcal{T}_h} (\boldsymbol{\Pi}_0\boldsymbol{\gamma}_h, \boldsymbol{s}) - \beta_1 h^2 \sum_{K\in\mathcal{T}_h} (A\boldsymbol{\phi}_h, \boldsymbol{s}), \end{aligned}$$

where A is the linear operator induced by $a(\cdot,\cdot)$ in the natural way and Π_0 is the L^2-orthogonal projection operator onto the piecewise constant function space, $\beta_i, i = 1, 2, 3$ are constants which can be written as

$$\beta_1 = \frac{2}{3}H^{-1}, \ \beta_2 = \frac{2H^{-1}(1-\nu+2\nu^2)}{3(1+\nu)^2}, \ \beta_3 = \frac{H^{-1}(3-\nu)}{3(1-\nu)}$$

with $H = \frac{E}{12(1-\nu^2)}$. The well-posedness of Problem3.3 can be found in [8] based on the following

Lemma 3.4. *There exists a positive constant* α*, such that*

$$a_h(\boldsymbol{\psi},\boldsymbol{\psi}) - \beta_2 h^2 \sum_{K\in\mathcal{T}_h}(A\boldsymbol{\psi}, A\boldsymbol{\psi}) \geq \alpha\|\boldsymbol{\psi}\|_1^2,\ \forall \boldsymbol{\psi}\in \boldsymbol{V}_h.$$

For the analysis, we need the following auxiliary problem, which is often called in literature the *MITC4* element or *Bathe-Dvorkin* element for the R-M plate[2].

Problem 3.5. *Find* $(\boldsymbol{\phi}_I, \omega_I)\in \boldsymbol{V}_h\times W_h$*, such that*

$$\begin{aligned} a(\boldsymbol{\phi}_I,\boldsymbol{\psi}) \quad &+ \quad (\boldsymbol{\gamma}_I, \nabla v - \boldsymbol{R}_h\boldsymbol{\psi}) = (g, v), \qquad \forall(\boldsymbol{\psi}, v)\in \boldsymbol{V}_h\times W_h,\\ \boldsymbol{\gamma}_I \quad &= \quad \lambda t^{-2}(\nabla\omega_I - \boldsymbol{R}_h\boldsymbol{\phi}_I). \end{aligned}$$

For our rectangular meshes, we have the following error estimates[7].

Lemma 3.6. *Let* $(\boldsymbol{\phi},\omega)$ *and* $(\boldsymbol{\phi}_I,\omega_I)$ *be the solutions of Probelm2.1 and Problem3.5 respectively, then*

$$\|\boldsymbol{\phi}-\boldsymbol{\phi}_I\|_1 + \|\omega-\omega_I\|_1 + t\|\boldsymbol{\gamma}-\boldsymbol{\gamma}_I\|_0 \leq Ch(\|g\|_{-1} + t\|g\|_0).$$

4. Error Analysis

In this section, we give the error analysis of Weissman-Taylor element. First, we introduce the following result from [8].

Lemma 4.1. *Under the hypotheses on the uniform square partition* $\mathcal{T}_h$*,*

$$\boldsymbol{R}_h\boldsymbol{\psi}_h = \Pi_h\boldsymbol{\psi}_h, \qquad \forall\boldsymbol{\psi}_h\in\boldsymbol{V}_h,$$

where $\Pi_h : \boldsymbol{H}_0^1(\Omega)\to \boldsymbol{\Gamma}_h^R$ *is the usual* L^2*-projection.*

Remark 4.2. *Lemma 4.1 can be regarded as a direct consequence of Lemma 3.1 and the definition of space* $\boldsymbol{\Gamma}_h^R$*.*

Then, we have the following theorem concerning the energy error estimate.

Theorem 4.3. *Let* $(\boldsymbol{\phi},\omega,\boldsymbol{\gamma})$ *and* $(\boldsymbol{\phi}_h,\omega_h,\boldsymbol{\gamma}_h)$ *be the solutions of Problem 2.2 and Problem 3.2 respectively, then*

$$\|\boldsymbol{\phi}-\boldsymbol{\phi}_h\|_{1,h} + \|\omega-\omega_h\|_{1,h} + h\|\Pi_0\boldsymbol{\gamma}-\Pi_0\boldsymbol{\gamma}_h\|_0 + t\|\boldsymbol{\gamma}-\boldsymbol{\gamma}_h\|_0 \leq Ch(\|g\|_{-1}+t\|g\|_0).$$

Proof. First, by Lemma 2.3, Lemma 3.6 and the inverse estimate, we have

$$\begin{aligned} |\boldsymbol{\phi}_I|_{2,h} \quad &\leq \quad |\boldsymbol{\phi}_I - \Pi_1\boldsymbol{\phi}|_{2,h} + |\Pi_1\boldsymbol{\phi}|_{2,h}\\ &\leq \quad Ch^{-1}|\boldsymbol{\phi}_I-\Pi_1\boldsymbol{\phi}|_1 + C\|\boldsymbol{\phi}\|_2\\ &\leq \quad Ch^{-1}|\boldsymbol{\phi}_I-\boldsymbol{\phi}| + Ch^{-1}|\boldsymbol{\phi}-\Pi_1\boldsymbol{\phi}|_1 + C\|\boldsymbol{\phi}\|_2\\ &\leq \quad C(\|g\|_{-1}+t\|g\|_0), \end{aligned} \tag{8}$$

where Π_1 is the usual bilinear Lagrange interpolation operator.

Set

$$\varepsilon_{\boldsymbol{\phi}} = \boldsymbol{\phi}_h^c - \boldsymbol{\phi}_I;\quad \varepsilon_\omega = \omega_h^c - \omega_I;\quad \varepsilon_{\boldsymbol{\gamma}} = \boldsymbol{\gamma}_h - \Pi_0\boldsymbol{\gamma}.$$

From Lemma 3.4, we obtain

$$\alpha(\|\varepsilon_{\boldsymbol{\phi}}\|_1^2 + h^2\|\Pi_0\varepsilon_{\boldsymbol{\gamma}}\|_0^2 + t^2\|\varepsilon_{\boldsymbol{\gamma}}\|_0^2) \leq \mathcal{A}_h(\varepsilon_{\boldsymbol{\phi}},\varepsilon_\omega,\varepsilon_{\boldsymbol{\gamma}};\varepsilon_{\boldsymbol{\phi}},\varepsilon_\omega,\varepsilon_{\boldsymbol{\gamma}}). \tag{9}$$

Taking into account Problem 2.2 and Problem 3.3, we have the following decomposition,

$$\mathcal{A}_h(\varepsilon_{\boldsymbol{\phi}},\varepsilon_\omega,\varepsilon_{\boldsymbol{\gamma}};\varepsilon_{\boldsymbol{\phi}},\varepsilon_\omega,\varepsilon_{\boldsymbol{\gamma}}) = A_1 + A_2 + \cdots + A_8,$$

where

$$A_1 = h^2\beta_2\sum_{K\in\mathcal{T}_h}(A\boldsymbol{\phi}_I, A\varepsilon_{\boldsymbol{\phi}}),\quad A_2 = -h^2\beta_1\sum_{K\in\mathcal{T}_h}(\Pi_0\boldsymbol{\gamma}, A\varepsilon_{\boldsymbol{\phi}}),$$

$$A_3 = -h^2\beta_3 \sum_{K\in\mathcal{T}_h} (\Pi_0\boldsymbol{\gamma}, \varepsilon_{\boldsymbol{\gamma}}), \quad A_4 = h^2\beta_1 \sum_{K\in\mathcal{T}_h} (A\boldsymbol{\phi}_I, \varepsilon_{\boldsymbol{\gamma}}),$$

$$A_5 = a(\boldsymbol{\phi} - \boldsymbol{\phi}_I, \varepsilon_{\boldsymbol{\phi}}), \quad A_6 = (\boldsymbol{\gamma} - \Pi_0\boldsymbol{\gamma}, \nabla\varepsilon_\omega - \varepsilon_{\boldsymbol{\phi}})$$

$$A_7 = \lambda^{-1}t^2(\boldsymbol{\gamma} - \Pi_0\boldsymbol{\gamma}, \varepsilon_{\boldsymbol{\gamma}}), \quad A_8 = (\nabla(\omega_I - \omega) - (\boldsymbol{\phi}_I - \boldsymbol{\phi}), \varepsilon_{\boldsymbol{\gamma}}).$$

Let us estimate the above terms one by one.

By virtue of (8), using the inverse estimate again, the first term A_1 can be bounded as

$$| A_1 | \le Ch \mid \boldsymbol{\phi}_I \mid_{2,h} \|\varepsilon_{\boldsymbol{\phi}}\|_1 \le Ch^2(\|g\|_{-1}^2 + t^2\|g\|_0^2) + a_1\|\varepsilon_{\boldsymbol{\phi}}\|_1^2.$$

Applying Cauchy-Schwarz inequality, we can bound terms A_2 and A_3 as

$$| A_2 | \le Ch\|\Pi_0\boldsymbol{\gamma}\|_0\|\varepsilon_{\boldsymbol{\phi}}\|_1 \le Ch^2\|\Pi_0\boldsymbol{\gamma}\|_0^2 + a_2\|\varepsilon_{\boldsymbol{\phi}}\|_1^2,$$

$$| A_3 | \le Ch^2\|\Pi_0\boldsymbol{\gamma}\|_0\|\Pi_0\varepsilon_{\boldsymbol{\gamma}}\|_0 \le Ch^2\|\Pi_0\boldsymbol{\gamma}\|_0^2 + a_3h^2\|\Pi_0\varepsilon_{\boldsymbol{\gamma}}\|_0^2.$$

Using Cauchy-Schwarz inequality, (8) and the inverse estimate once more, we proceed as

$$| A_4 | \le Ch^2 \mid \boldsymbol{\phi} \mid_{2,h} \|\Pi_0\varepsilon_{\boldsymbol{\gamma}}\|_0 \le Ch^2(\|g\|_{-1}^2 + t^2\|g\|_0^2) + a_4h^2\|\Pi_0\varepsilon_{\boldsymbol{\gamma}}\|_0^2.$$

Taking into account Lemma 3.6, we obtain

$$| A_5 | \le Ch(\|g\|_{-1} + t\|g\|_0)\|\varepsilon_{\boldsymbol{\phi}}\|_1 \le Ch^2(\|g\|_{-1}^2 + t^2\|g\|_0^2) + a_5\|\varepsilon_{\boldsymbol{\phi}}\|_1^2.$$

Now from Problem 3.3 and Lemma 4.1, we find that

$$\nabla\omega_h^c \mid_K = \boldsymbol{R}_h\boldsymbol{\phi}_h^c + \lambda^{-1}t^2\boldsymbol{\gamma}_h + C_K, \qquad \text{where } C_K \text{ is a constant.}$$

Thus we come to

$$\begin{aligned} A_6 &= (\boldsymbol{\gamma} - \Pi_0\boldsymbol{\gamma}, \nabla(\omega_h^c - \omega_I) - (\boldsymbol{\phi}_h^c - \boldsymbol{\phi}_I)) \\ &= (\boldsymbol{\gamma} - \Pi_0\boldsymbol{\gamma}, \boldsymbol{R}_h(\boldsymbol{\phi}_h^c - \boldsymbol{\phi}_I) - (\boldsymbol{\phi}_h^c - \boldsymbol{\phi}_I)) + \lambda^{-1}t^2(\boldsymbol{\gamma} - \Pi_0\boldsymbol{\gamma}, \boldsymbol{\gamma}_h - \boldsymbol{\gamma}_I) \\ &= A_6^1 + A_6^2. \end{aligned}$$

Owing to Lemma 3.1, we have

$$| A_6^1 | \le Ch\|\boldsymbol{\gamma} - \Pi_0\boldsymbol{\gamma}\|_0\|\varepsilon_{\boldsymbol{\phi}}\|_1 \le Ch^2\|\boldsymbol{\gamma}\|_0^2 + a_{6,1}\|\varepsilon_{\boldsymbol{\phi}}\|_1^2.$$

The key is to bound the second term

$$\begin{aligned} A_6^2 &= \lambda^{-1}t^2(\boldsymbol{\gamma} - \Pi_0\boldsymbol{\gamma}, \boldsymbol{\gamma}_h - \boldsymbol{\gamma}_I) \\ &= \lambda^{-1}t^2(\boldsymbol{\gamma} - \Pi_0\boldsymbol{\gamma}, \boldsymbol{\gamma}_h - \Pi_0\boldsymbol{\gamma}) \\ &\quad -\lambda^{-1}t^2(\boldsymbol{\gamma} - \Pi_0\boldsymbol{\gamma}, \boldsymbol{\gamma} - \Pi_0\boldsymbol{\gamma}) + \lambda^{-1}t^2(\boldsymbol{\gamma} - \Pi_0\boldsymbol{\gamma}, \boldsymbol{\gamma} - \boldsymbol{\gamma}_I). \end{aligned}$$

By virtue of Lemma 2.3 and Lemma 3.6, taking into account the error estimate for projection operator Π_0, we derive as

$$\begin{aligned} | A_6^2 | &\le Cht(\|g\|_{-1} + t\|g\|_0)\|\boldsymbol{\gamma}_h - \Pi_0\boldsymbol{\gamma}\|_0 + Ch^2(\|g\|_{-1}^2 + t^2\|g\|_0^2) \\ &\le Ch^2(\|g\|_{-1}^2 + t^2\|g\|_0^2) + a_{6,2}t^2\|\varepsilon_{\boldsymbol{\gamma}}\|_0^2. \end{aligned}$$

It is easy to see

$$| A_7 | = | \lambda^{-1}t^2(\boldsymbol{\gamma} - \Pi_0\boldsymbol{\gamma}, \varepsilon_{\boldsymbol{\gamma}}) | \le Ch^2(\|g\|_{-1}^2 + t^2\|g\|_0^2) + a_7t^2\|\varepsilon_{\boldsymbol{\gamma}}\|_0^2.$$

From the definition of operator $\boldsymbol{R}_h$, we can easily prove

$$(\boldsymbol{R}_h\boldsymbol{\phi}_I - \boldsymbol{\phi}_I, \varepsilon_{\boldsymbol{\gamma}}) = 0,$$

which, together with Lemma 4.1 and Lemma 3.6, imply

$$\begin{aligned} |\, A_8 \,| &= |\, (\nabla(\omega_I - \omega) - (\boldsymbol{\phi}_I - \boldsymbol{\phi}), \varepsilon_{\boldsymbol{\gamma}}) \,| \\ &= |\, ((\nabla\omega_I - \boldsymbol{R}_h\boldsymbol{\phi}_I) - (\nabla\omega - \boldsymbol{\phi}), \varepsilon_{\boldsymbol{\gamma}}) \,|=|\, \lambda^{-1}t^2(\boldsymbol{\gamma}_I - \boldsymbol{\gamma}, \varepsilon_{\boldsymbol{\gamma}}) \,| \\ &\leq Ch^2(\|g\|_{-1}^2 + t^2\|g\|_0^2) + a_8 t^2 \|\varepsilon_{\boldsymbol{\gamma}}\|_0^2. \end{aligned}$$

Let $a_1, \cdots, a_{6,1}, a_{6,2}, a_7, a_8$ be small enough and bring these inequalities together, we come to

$$\|\boldsymbol{\phi}_I - \boldsymbol{\phi}_h^c\|_1 + h\|\Pi_0(\boldsymbol{\gamma}_I - \boldsymbol{\gamma}_h)\|_0 + t\|(\boldsymbol{\gamma}_I - \boldsymbol{\gamma}_h)\|_0 \leq Ch(\|g\|_{-1} + t\|g\|_0). \tag{10}$$

On the other hand, by the decomposition, we have $\boldsymbol{\phi}_h = \boldsymbol{\phi}_h^c + \boldsymbol{\phi}_h^{nc}$, where $\boldsymbol{\phi}_h^{nc}$ is the nonconforming part of $\boldsymbol{\phi}_h$. Proceeding along the same line of Theorem 2.1 of [9], applying the above estimate about the conforming part and the following inequality

$$\|\boldsymbol{\phi}_h^{nc}\|_0 \leq Ch\|\boldsymbol{\phi}_h^{nc}\|_{1,h}, \tag{11}$$

the nonconforming bubble can be bounded as

$$\begin{aligned} C &\quad \|\boldsymbol{\phi}_h^{nc}\|_{1,h}^2 \\ \leq &\quad a_h(\boldsymbol{\phi}_h^{nc}, \boldsymbol{\phi}_h^{nc}) = (\boldsymbol{\gamma}_h, \boldsymbol{\phi}_h^{nc}) - a_h(\boldsymbol{\phi}_h^c, \boldsymbol{\phi}_h^{nc}) = (\Pi_0\boldsymbol{\gamma}_h, \boldsymbol{\phi}_h^{nc}) - a_h(\boldsymbol{\phi}_h^c, \boldsymbol{\phi}_h^{nc}) \\ = &\quad (\Pi_0\boldsymbol{\gamma}_h, \boldsymbol{\phi}_h^{nc}) - a_h(\boldsymbol{\phi}_h^c - \boldsymbol{\phi}, \boldsymbol{\phi}_h^{nc}) + [(\boldsymbol{\gamma}, \boldsymbol{\phi}_h^{nc}) - a_h(\boldsymbol{\phi}, \boldsymbol{\phi}_h^{nc})] - (\boldsymbol{\gamma}, \boldsymbol{\phi}_h^{nc}) \\ \leq &\quad C\|\Pi_0\boldsymbol{\gamma}_h\|_0\|\boldsymbol{\phi}_h^{nc}\|_0 + C\|\boldsymbol{\phi}_h^c - \boldsymbol{\phi}\|_1\|\boldsymbol{\phi}_h^{nc}\|_{1,h} \\ &\quad + |\sum_{K\in\mathcal{T}_h}\int_{\partial K}\mathcal{C}\mathcal{E}\boldsymbol{\phi}\cdot\boldsymbol{n}\boldsymbol{\phi}_h^{nc}ds\,| + \|\boldsymbol{\gamma}\|_0\|\boldsymbol{\phi}_h^{nc}\|_0, \end{aligned}$$

which, together with (10) and (11), and owing to the consistency error estimate of the Wilson element [11] imply

$$\|\boldsymbol{\phi}_h^{nc}\|_{1,h} \leq Ch(\|g\|_{-1} + t\|g\|_0), \quad \|\boldsymbol{\phi}_h^{nc}\|_0 \leq Ch^2(\|g\|_{-1} + t\|g\|_0). \tag{12}$$

Then we obtain

$$\|\boldsymbol{\phi}_h - \boldsymbol{\phi}\|_{1,h} \leq \|\boldsymbol{\phi}_h^c - \boldsymbol{\phi}\|_1 + \|\boldsymbol{\phi}_h^{nc}\|_{1,h} \leq Ch(\|g\|_{-1} + t\|g\|_0). \tag{13}$$

By the triangle inequality and (10), taking into account Lemma 3.6, we have

$$\begin{aligned} &h\|\Pi_0(\boldsymbol{\gamma} - \boldsymbol{\gamma}_h)\|_0 + t\|(\boldsymbol{\gamma} - \boldsymbol{\gamma}_h)\|_0 \\ &\leq h\|\Pi_0(\boldsymbol{\gamma} - \Pi_0\boldsymbol{\gamma})\|_0 + h\|\Pi_0(\Pi_0\boldsymbol{\gamma} - \boldsymbol{\gamma}_h)\|_0 + t\|(\boldsymbol{\gamma} - \boldsymbol{\gamma}_I)\|_0 + t\|(\boldsymbol{\gamma}_I - \boldsymbol{\gamma}_h)\|_0 \\ &\leq Ch(\|g\|_{-1} + t\|g\|_0). \end{aligned} \tag{14}$$

As for the transverse displacement ω_h, from the second equation of Problem 3.2, we have the following decomposition

$$\nabla_h\omega_h - \nabla\omega = \lambda^{-1}t^2(\boldsymbol{\gamma}_h - \boldsymbol{\gamma}) + (\hat{\Pi}_h\boldsymbol{\phi}_h - \boldsymbol{\phi}),$$

where $\hat{\Pi}_h$ is the projection operator on $\hat{\boldsymbol{\Gamma}}_h$, therefore

$$\|\nabla_h\omega_h - \nabla\omega\|_0 \leq Ch(\|g\|_{-1} + t\|g\|_0), \tag{15}$$

which completes the proof. □

In order to analyze the L^2 error estimate of the rotation, we need the following auxiliary problem and its mixed formulation:

Problem 4.4. *Find* $(\boldsymbol{\phi}_d, \omega_d) \in \boldsymbol{H}_0^1(\Omega) \times H_0^1(\Omega)$, *such that*

$$a(\boldsymbol{\psi}, \boldsymbol{\phi}_d) + (\nabla v - \boldsymbol{\psi}, \boldsymbol{\sigma}) = (\boldsymbol{\phi} - \boldsymbol{\phi}_h^c, \boldsymbol{\psi}) \qquad \forall \boldsymbol{\psi} \in \boldsymbol{H}_0^1(\Omega), \tag{16}$$

$$\boldsymbol{\sigma} = \lambda t^{-2}(\nabla\omega_d - \boldsymbol{\phi}_d). \tag{17}$$

Problem 4.5. *Find* $(\boldsymbol{\phi}_d, \omega_d, \boldsymbol{\sigma}) \in \boldsymbol{H}_0^1(\Omega) \times H_0^1(\Omega) \times \boldsymbol{L}^2(\Omega)$*, such that*

$$a(\boldsymbol{\psi}, \boldsymbol{\phi}_d) + (\nabla v - \boldsymbol{\psi}, \boldsymbol{\sigma}) = (\boldsymbol{\phi} - \boldsymbol{\phi}_h^c, \boldsymbol{\psi}) \quad \forall (\boldsymbol{\psi}, v) \in \boldsymbol{H}_0^1(\Omega) \times H_0^1(\Omega), \tag{18}$$

$$(\nabla \omega_d - \boldsymbol{\phi}_d, \boldsymbol{\delta}) - \lambda^{-1} t^2 (\boldsymbol{\delta}, \boldsymbol{\sigma}) = 0 \quad \forall \boldsymbol{\delta} \in \boldsymbol{L}^2(\Omega). \tag{19}$$

For the solution of Problem 4.5, we have the following regularity result

$$\|\boldsymbol{\phi}_d\|_2 + \|\omega_d\|_3 + \|\boldsymbol{\sigma}\|_0 + t\|\boldsymbol{\sigma}\|_1 + \|\boldsymbol{\sigma}\|_{H(\mathrm{div})} \le C \|\boldsymbol{\phi} - \boldsymbol{\phi}_h^c\|_0. \tag{20}$$

Let $(\boldsymbol{\phi}_d^I, \omega_d^I, \boldsymbol{\sigma}^I)$ be the finite element approximation of $(\boldsymbol{\phi}_d, \omega_d, \boldsymbol{\sigma})$ by the MITC4 method, owing to Lemma 3.6, we have

$$\|\boldsymbol{\phi}_d^I - \boldsymbol{\phi}_d\|_1 + \|\omega_d - \omega_d^I\|_1 + t\|\boldsymbol{\sigma} - \boldsymbol{\sigma}^I\|_0 \le Ch \|\boldsymbol{\phi} - \boldsymbol{\phi}_h^c\|_0. \tag{21}$$

Similarly,

$$\|\boldsymbol{\phi}_d^I\|_{2,h} \le C \|\boldsymbol{\phi} - \boldsymbol{\phi}_h^c\|_0. \tag{22}$$

Then, we have the following L^2 errror estimate for the Weissman-Taylor element.

Theorem 4.6. *Let* $(\boldsymbol{\phi}, \omega, \boldsymbol{\gamma})$ *and* $(\boldsymbol{\phi}_h, \omega_h, \boldsymbol{\gamma}_h)$ *be the solutions of Problem 2.2 and Problem 3.2, respectively, then*

$$\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_0 + \|\omega - \omega_h\|_0 \le Ch^2 \|g\|_0$$

Proof. Let $\boldsymbol{\psi} = \boldsymbol{\phi} - \boldsymbol{\phi}_h^c$, $v = \omega - \omega_h^c$ and $\boldsymbol{\delta} = \boldsymbol{\gamma} - \boldsymbol{\gamma}_h$. Taking into account Probelm 2.2 and Problem 3.2, we have

$$\begin{aligned}
\|\boldsymbol{\phi} - \boldsymbol{\phi}_h^c\|_0^2 &= a(\boldsymbol{\phi} - \boldsymbol{\phi}_h^c, \boldsymbol{\phi}_d - \boldsymbol{\phi}_d^I) + (\boldsymbol{\gamma} - \boldsymbol{\gamma}_h, \nabla(\omega_d - \omega_d^I) - (\boldsymbol{\phi}_d - \boldsymbol{\phi}_d^I)) \\
&\quad + (\nabla(\omega - \omega_h^c) - (\boldsymbol{\phi} - \boldsymbol{\phi}_h^c), \boldsymbol{\sigma}) - \lambda^{-1} t^2 (\boldsymbol{\gamma} - \boldsymbol{\gamma}_h, \boldsymbol{\sigma}) + a_h(\boldsymbol{\phi}_h^{nc}, \boldsymbol{\phi}_d^I) \\
&= B_1 + \cdots + B_5.
\end{aligned}$$

Owing to (13) and (21),

$$|B_1| \le Ch^2 (\|g\|_{-1} + t\|g\|_0) \|\boldsymbol{\phi} - \boldsymbol{\phi}_h^c\|_0.$$

Because $(\boldsymbol{\gamma}_h, \boldsymbol{\phi}_d^I - \boldsymbol{R}_h \boldsymbol{\phi}_d^I) = 0$, we obtain

$$B_2 = \lambda^{-1} t^2 (\boldsymbol{\gamma} - \boldsymbol{\gamma}_h, \boldsymbol{\sigma} - \boldsymbol{\sigma}^I) + (\boldsymbol{\gamma}, \boldsymbol{\phi}_d^I - \boldsymbol{R}_h \boldsymbol{\phi}_d^I) = B_{2,1} + B_{2,2}.$$

It is easy to see

$$|B_{2,1}| \le Ch^2 (\|g\|_{-1} + t\|g\|_0) \|\boldsymbol{\phi} - \boldsymbol{\phi}_h^c\|_0.$$

Proceeding along the same line of Lemma 5.9 [10] and using (22), we come to

$$|B_{2,2}| \le Ch^2 \|\boldsymbol{\gamma}\|_{H(\mathrm{div})} |\boldsymbol{\phi}_d^I|_{2,h} \le Ch^2 \|g\|_0 \|\boldsymbol{\phi} - \boldsymbol{\phi}_h^c\|_0.$$

By virtue of the equation of Problem 3.3 and the second inequality of Lemma 3.1, we derive

$$\begin{aligned}
|B_3 + B_4| &= |-(\nabla \omega_h^c - \boldsymbol{\phi}_h^c, \boldsymbol{\sigma}) + \lambda^{-1} t^2 (\boldsymbol{\gamma}_h, \boldsymbol{\sigma})| \\
&= |-\beta_1 h^2 (\Pi_0 \boldsymbol{\gamma}_h, \boldsymbol{\sigma}) + \beta_2 h^2 \sum_{K \in \mathcal{T}_h} (A\boldsymbol{\phi}_h^c, \boldsymbol{\sigma})_K + (\boldsymbol{\phi}_h^c - \boldsymbol{R}_h \boldsymbol{\phi}_h^c, \boldsymbol{\sigma})| \\
&\le Ch^2 (\|\Pi_0 \boldsymbol{\gamma}_h\|_0 + |\boldsymbol{\phi}_h^c|_{2,h}) \|\boldsymbol{\sigma}\|_0 + \|\boldsymbol{\phi}_h^c - \boldsymbol{R}_h \boldsymbol{\phi}_h^c\|_{H^{-1}(\mathrm{rot})} \|\boldsymbol{\sigma}\|_{H(\mathrm{div})} \\
&\le Ch^2 (\|g\|_{-1} + t\|g\|_0) \|\boldsymbol{\phi} - \boldsymbol{\phi}_h^c\|_0.
\end{aligned}$$

Integrating by parts and using (22) and Theorem 4.3, the last term can be bounded as

$$\begin{aligned} \mid B_5 \mid &= |-(\boldsymbol{\phi}_h^{nc}, \operatorname{div} \mathcal{C}\mathcal{E}\boldsymbol{\phi}_d^I) + \sum_{K\in\mathcal{T}_h} \int_{\partial K} \boldsymbol{\phi}_h^{nc} \mathcal{C}\mathcal{E}\boldsymbol{\phi}_d^I \cdot \boldsymbol{n}| \\ &\leq C\|\boldsymbol{\phi}_h^{nc}\|_0 \|\boldsymbol{\phi}_d^I\|_{2,h} + Ch\|\boldsymbol{\phi}_h^{nc}\|_{1,h}\|\boldsymbol{\phi}_d^I\|_{2,h} \\ &\leq Ch^2(\|g\|_{-1} + t\|g\|_0)\|\boldsymbol{\phi} - \boldsymbol{\phi}_h^c\|_0. \end{aligned}$$

Bring these inequalities together, we come to

$$\|\boldsymbol{\phi} - \boldsymbol{\phi}_h^c\|_0 \leq Ch^2\|g\|_0. \tag{23}$$

The nonconforming bubble has been bounded in (12), which, together with (23) and triangle inequality, imply

$$\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_0 \leq Ch^2\|g\|_0. \tag{24}$$

For the L^2 error estimate of the transverse displacement, we need the following auxiliary problem: Find $z \in H_0^1(\Omega)$ and $z_h \in W_h$ such that

$$(\nabla z, \nabla s) = (\omega - \omega_h, s), \quad \forall s \in H_0^1(\Omega), \quad (\nabla z_h, \nabla s) = (\omega - \omega_h, s), \quad \forall s \in W_h.$$

By a routine way, we have

$$\|z - z_h\|_0 + h|z - z_h|_1 \leq Ch^2\|\omega - \omega_h\|_0.$$

Therefore,

$$\begin{aligned} \|\omega - \omega_h\|_0^2 &= (\omega - \omega_h, \omega - \omega_h) - (\nabla z, \nabla_h(\omega - \omega_h)) + (\nabla z, \nabla_h(\omega - \omega_h)) \\ &= J_1 + J_2. \end{aligned}$$

By the standard procedure of the nonconforming Wilson element[11], we get

$$|J_1| \leq Ch^2\|z\|_2\|\omega - \omega_h\|_{1,h} \leq Ch^2(\|g\|_{-1} + t\|g\|_0)\|\omega - \omega_h\|_0,$$

$$\begin{aligned} J_2 &= (\nabla z, \nabla_h\omega - \nabla_h\omega_h) \\ &= (\nabla z - \nabla z_h, \nabla_h\omega - \nabla_h\omega_h) + (\nabla z_h, \nabla_h\omega - \nabla_h\omega_h) \\ &= (\nabla z - \nabla z_h, \nabla_h\omega - \nabla_h\omega_h) + \lambda^{-1}t^2(\boldsymbol{\gamma} - \boldsymbol{\gamma}_h, \nabla z_h) \\ &\quad +(\boldsymbol{\phi} - \boldsymbol{R}_h\boldsymbol{\phi}_h^c, \nabla z_h) + (\Pi_h\boldsymbol{\phi}_h^{nc}, \nabla z_h) \\ &= J_2^1 + \cdots + J_2^4. \end{aligned}$$

Obviously,

$$|J_2^1 + J_2^4| \leq Ch^2(\|g\|_{-1} + t\|g\|_0)\|\omega - \omega_h\|_0, \quad J_2^2 = 0.$$

The last term can be estimated as

$$\begin{aligned} |J_2^3| &= \mid (\boldsymbol{\phi} - \boldsymbol{R}_h\boldsymbol{\phi}_h^c, \nabla z_h) \mid = \mid (\boldsymbol{\phi} - \boldsymbol{\phi}_h^c, \nabla z_h) \mid \\ &\leq Ch^2\|g\|_0\|z_h\|_1 \leq Ch^2|g\|_0\|\omega - \omega_h\|_0. \end{aligned}$$

Bring all these inequalities, we obtain

$$\|\omega - \omega_h\|_0 \leq Ch^2\|g\|_0,$$

which completes the proof. □

Remark 4.7. *We focused on the Weissman-Taylor element in the discussion. However, the analysis and error estimates hold also for other Wilson-type plate bending elements proposed in* [9].

5. Conclusion

In this paper, we have analyzed the rectangular Weissman-Taylor element for the Reissner-Mindlin plate and have proved that the element is locking free. Our analysis depends heavily on some orthogonal properties associated with rectangular meshes. These properties are not valid for general quadrilateral meshes. Therefore the locking free issue for the Weissman-Taylor element over quadrilateral meshes is still unsolved.

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