ON STABILITY AND CONVERGENCE FOR DISCRETE-DISCONTINUOUS FINITE ELEMENT METHOD*

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Abstract

In this paper, we deal with the discrete-discontinuous finite element method for solving the time-dependent neutron transport equation in two-dimensional planar geometry. Its stability and convergence are proved. The numerical results are given. Compared with SN method it is of higher accuracy and superconvergence.

The discrete-ordinate method[4] (DSN method) is an effective method for solving neutron transport equations. Its computing process is simpler and the amount of program and calculation is less than that of other methods. And its error, as can be demonstrated, is at most of order 2. But solving some physical problems, a more accurate approximate solution, and so a solution of higher accuracy with less store, are desired. Therefore, it is natural to adopt the finite element method in solving neutron transport problems. Although the variational method (Ritz method) can be used, the complexity of the formula and large amount of program and calculation have impeded its application. On the other hand, while the Galerkin method can be used in the finite element method for its simpler computing process and program, it requires to solve of a large system of linear algebraic equations and will give rise to more difficulties. As the discontinuous finite element method has the advantages of both DSN and FEM, it provides a better way to solving multidimensional transport problems.

The discontinuous finite element method, where the angular flux is assumed to be given by a low-order polynomial in each mesh, has been used to solve the discrete-ordinate equations. It has been considered in [1, 2, 3] for solving the simplified steady neutron transport equations.

In this paper, we describe the basic steps of this method for solving time-dependent neutron transport equations in two-dimensional planar geometry. Some estimations of solution and error are given and the stability and superconvergency of the method are proved. Here, Crank-Nicholson central difference approximation is used for the time variable, while the discrete ordinate approximation for the angular variables.

Furthermore, we have used the above method to calculate many numerical examples for one-dimensional slab problem. The results demonstrate higher accuracy, faster rate of convergence and higher efficiency.

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1. Numerical Method

We consider the initial-boundary value problem for the time-dependent neutron transport equation in two-dimensional planar geometry:

$$\begin{cases} A(\varphi) \equiv \frac{1}{v_g}\frac{\partial \varphi_g}{\partial t} + \boldsymbol{\Omega}\cdot\operatorname{grad}\varphi_g + \alpha_g\varphi_g = S_g(\varphi_g) + F_g, \text{ in } D = B_{xy}\times Q_\Omega \times E_t, \\ \varphi_g(t, x, y, \mu, \nu)\big|_{t=0} = \varphi_g^0(x, y, \mu, \nu), \\ \varphi_g(t, x, y, \mu, \nu) = 0, \text{ if } (x, y)\in\Gamma,\ \boldsymbol{\Omega}\cdot\boldsymbol{n}_\Gamma < 0; \\ \varphi_g(t, x, y, \mu, \nu)\big|_{x=0} = \varphi_g(t, x, y, -\mu, \nu)\big|_{x=0}, \\ \varphi_g(t, x, y, \mu, \nu)\big|_{y=0} = \varphi_g(t, x, y, \mu, -\nu)\big|_{y=0}, \end{cases} \tag{1.1}$$

where

$$\boldsymbol{\Omega}\cdot\operatorname{grad}\varphi_g = \mu\frac{\partial \varphi_g}{\partial x} + \nu\frac{\partial\varphi_g}{\partial y},$$

the function $\varphi_g(t, x, y, \mu, \nu)$ represents the flux of g-group neutron at the point (t, x, y) in the angular direction $\boldsymbol{\Omega} = (\mu, \nu)$, α_g is the nuclear macroscopic total cross section, $S_g(\varphi)$ represents sources of neutrons due to scattering and fission, and F_g inhomogeneous source terms. Let us assume that the domains B_{xy}, Q_Ω, E_t take the forms: $B_{xy} = \{0 \leqslant x \leqslant X,\ 0\leqslant y\leqslant Y\}$, $Q_\Omega = \{0\leqslant \mu^2+\nu^2\leqslant 1\}$, $E_t = \{0\leqslant t\leqslant T\}$. Γ is the boundary of B_{xy} which is $x = X$ or $y = Y$. Denote by $\boldsymbol{n}_\Gamma$ the unit vector in the direction of outward normal to Γ.

We choose a suitable set of discrete direction and weights $\{\boldsymbol{\Omega}_{ms}, w_{ms}\}$, where $\boldsymbol{\Omega}_{ms} = (\mu_{ms}, \nu_{ms})$, $s = 1, 2, \cdots, N$, $m = 1, 2, \cdots, M_s$. For simplicity, we omit the group index g and restrict our discussion to one-group and isotropic scattering. Hence, the discrete-ordinate equations can be written as

$$\begin{cases} A_{ms}(\varphi_{ms}) = \frac{1}{v}\frac{\partial\varphi_{ms}}{\partial t} + \mu_{ms}\frac{\partial\varphi_{ms}}{\partial x} + \nu_{ms}\frac{\partial \varphi_{ms}}{\partial y} + \alpha\varphi_{ms} = S_M(\varphi) + F_{ms}, \text{ in } B_{xy}\times E_t, \\ \varphi_{ms}\big|_{t=0} = \varphi^0(x, y, \mu_{ms}, \nu_{ms}), \\ \varphi_{ms}\big|_\Gamma = 0, \text{ if } (x, y)\in\Gamma,\ \Omega_{ms}\cdot\boldsymbol{n}_\Gamma<0, \\ \varphi_{ms}(t, 0, y, \mu_{ms}, \nu_{ms}) = \varphi_{ms}(t, 0, y, -\mu_{ms}, \nu_{ms}), \\ \varphi_{ms}(t, x, 0, \mu_{ms}, \nu_{ms}) = \varphi_{ms}(t, x, 0, \mu_{ms}, -\nu_{ms}), \end{cases} \tag{1.2}$$

where $s = 1, 2, \cdots, N$, $m = 1, 2, \cdots, M_s$, $\varphi_{ms} = \varphi(t, x, y, \mu_{ms}, \nu_{ms})$, $S_M(\varphi) = \beta\sum_{m,s}\varphi_{ms}w_{ms}$, $\sum_{m,s} w_{ms} = 1$.

The boundary conditions can be written as

$$\begin{cases} \varphi_{ms}\big|_{x=X} = 0,\ s = 1, 2, \cdots, N,\ m = 1, 2, \cdots, \frac{M_s}{2}, \\ \varphi_{ms}\big|_{y=Y} = 0,\ s = 1, 2, \cdots, \frac{N}{2},\ m = 1, 2, \cdots, M_s, \\ \varphi_{ms}\big|_{x=0} = \varphi_{M_s+1-m,s}\big|_{x=0},\ s = 1, 2, \cdots, N,\ m = \frac{M_s}{2}+1, \cdots, M_s, \\ \varphi_{ms}\big|_{y=0} = \varphi_{m,N+1-s}\big|_{y=0},\ s = \frac{N}{2}+1, \cdots, N,\ m = 1, 2, \cdots, M_s. \end{cases} \tag{1.3}$$

Let $0 = x_0 < x_1 < \cdots < x_I = X$, $0 = y_0 < y_1 < \cdots < y_J = Y$, $0 = t_0 < t_1 < \cdots < t_{N_0} = T$ be the subdivisions of the intervals $[0, X]$, $[0, Y]$, $[0, T]$ respectively. Denote by J_h the set of all spacial meshes, that is,

$$J_h=\{D_{ij} \mid x_{i-1}\leqslant x\leqslant x_i,\ y_{j-1}\leqslant y\leqslant y_j,\ i=1, 2, \cdots, I,\ j=1, 2, \cdots, J\}. \quad (1.4)$$

Let ∂D_{ij} be the boundary of D_{ij}. We set

$$\begin{cases} (\partial_- D_{ij})_{ms}=\{(x, y)\in\partial D_{ij} \mid \boldsymbol{\Omega}_{ms}\cdot\boldsymbol{n}\leqslant 0\}, \\ (\partial_+ D_{ij})_{ms}=\{(x, y)\in\partial D_{ij} \mid \boldsymbol{\Omega}_{ms}\cdot\boldsymbol{n}>0\}, \end{cases} \quad (1.5)$$

where $\boldsymbol{n}$ is the unit vector in the direction of outward normal to the boundary ∂D_{ij}. We consider the finite-dimensional space

$$V_h=\{u \mid u|_{D_{ij}}\in P_K(D_{ij}),\ \forall D_{ij}\in J_h\}, \quad (1.6)$$

where $P_K(D_{ij})$ denotes the subspace of all polynomials of the form

$$p(x, y)=\sum_{m,n=0}^{K} c_{mn}x^m y^n, \quad c_{mn}\in\mathbb{R}, \quad (x, y)\in D_{ij}.$$

It is of dimension $K^*=(K+1)^2$. Notice that in general a function $u\in V_h$ does not satisfy any continuity requirement at the interelement boundaries. For simplicity, we often omit the index m and s.

Thus, the finite element approximation of problem (1.2) can be stated as follows: Find a function $\varphi_h\in V_h$ such that for all $D_{ij}\in J_h$

$$-\int_{\partial_- D_{ij}} \boldsymbol{\Omega}\cdot\boldsymbol{n}(\varphi_h-\chi_h)\psi\, dl+\iint_{D_{ij}}(A_h(\varphi_h)-S_M(\varphi_h)-F)\psi\, d\mathscr{D}=0, \quad \forall\psi\in P_K(D_{ij}), \quad (1.7)$$

where

$$\chi_h=\begin{cases} 0, & \text{on } \partial_- D_{ij}\cap\Gamma_-, \\ \text{outward trace of } \varphi_h, & \text{on } \partial_- D_{ij}\backslash(\partial_- D_{ij}\cap\Gamma_-), \end{cases} \quad (1.8)$$

$$A_h(\varphi_h)=\frac{1}{v}\frac{D\varphi_h}{Dt}+\boldsymbol{\Omega}\cdot\operatorname{grad}\varphi_h+\alpha\varphi_h, \quad \frac{D\varphi_h}{Dt}=(\varphi_h^{n+1}-\varphi_h^n)/\Delta t.$$

We choose K^* collocation points on D_{ij}, on which we construct the subspace $P_K(D_{ij})$. Then, we introduce the basis $\{\psi_k\}$ $(k=1, 2, \cdots, K^*)$ for the space $P_K(D_{ij})$. Therefore, equation (1.7) is equivalent to the following equations

$$-\int_{\partial_- D_{ij}} \boldsymbol{\Omega}\cdot\boldsymbol{n}(\varphi_h-\chi_h)\psi_k\, dl+\iint_{D_{ij}}[A_h(\varphi_h)-S_M(\varphi_h)-F]\psi_k\, d\mathscr{D}=0, \quad k=1, 2, \cdots, K^*, \quad (1.9)$$

where

$$\varphi_h=\sum_{k=1}^{K^*}\varphi_{hk}\psi_k.$$

Clearly, $\varphi_{hk}=\varphi_h|_{x=x_k, y=y_k}$. Then we obtain a system of K^* linear algebraic equations. Using initial conditions, incoming boundary conditions and symmetric reflecting boundary conditions, one can proceed iteratively as DSN method, and hence calculate the interior fluxes mesh by mesh.

2. An Estimate of the Approximate Solution and Stability

Suppose that J_h does not depend on t, μ and ν, and denote by $L_2(B_{xy})$ the space of all real functions φ_h which are square integrable over B_{xy}. At time t and direction $(\mu, \nu)_{ms}$, we assume that $\varphi_h(t, x, y, \mu_{ms}, \nu_{ms})\in L_2(B_{xy})$ and define

$$\begin{cases} \|\varphi_h\|^2_{0,D_{ij};t,\Omega_{ms}} = \iint\limits_{D_{ij}} [\varphi_h(t, x, y, \mu_{ms}, \nu_{ms})]^2 d\mathscr{D}, \\ \|\varphi_h\|^2_{0,B_{xy};t,\Omega_{ms}} = \iint\limits_{B_{xy}} \varphi_h^2 d\mathscr{D} = \sum_{i,j} \|\varphi_h\|^2_{0,D_{ij};t,\Omega_{ms}} \end{cases} \tag{2.1}$$

and

$$\begin{cases} \|\varphi_h\|^2_{0,B_{xy}\times Q_\Omega;t} = \sum_{i,j,m,s} \|\varphi_h\|^2_{0,D_{ij};t,\Omega_{ms}} \cdot w_{ms}, \\ \|\varphi_h\|^2_{0,D} = \sum_{n,i,j,m,s} \|\varphi_h\|^2_{0,D_{ij};t,\Omega_{ms}} \cdot w_{ms} \Delta t^{n+\frac{1}{2}}. \end{cases} \tag{2.2}$$

In the following discussion, we omit the indexes t and Ω_{ms}.

As to the physical parameters, we assume

$$\begin{cases} \alpha = \alpha(x, y) \geqslant \alpha_0 \geqslant 0, \\ \beta = \beta(x, y) \leqslant \beta_1, \\ \dfrac{1}{v_1} \geqslant \dfrac{1}{v(x, y)} \geqslant \dfrac{1}{v_0} > 0. \end{cases} \tag{2.3}$$

Theorem 1. *Suppose that φ_h is a solution of (1.7), and that (2.3) and*

$$\Delta t < \frac{2}{a} \tag{2.4}$$

hold. Then φ_h satisfies

$$\|\varphi_h\|^2_{0,B_{xy}\times Q_\Omega}|_{t=T} + \|\varphi_h\|^2_{0,D} \leqslant C(\|\varphi_h\|^2_{0,B_{xy}\times Q_\Omega}|_{t=0} + \|F\|^2_{0,D}), \tag{2.5}$$

where

$$\begin{cases} C = \dfrac{e^{at}\max\left(\dfrac{1}{2v_1},\ 1\right)}{\min\left(\dfrac{1}{2v_0},\ \alpha_0 + \dfrac{a}{4v_0} - \beta_1 - 1\right)}, \\ a = \begin{cases} 4v_0(2+\beta_1-\alpha_0), & \alpha_0-\beta_1-1\leqslant 0, \\ 0, & \alpha_0-\beta_1-1>0. \end{cases} \end{cases} \tag{2.6}$$

Proof. We choose $\psi = \varphi_h e^{-at^{n+\frac{1}{2}}}$ in (1.7) and use

$$2(\varphi_h^{n+1}-\varphi_h^n)\bar{\varphi}_h e^{-at^{n+\frac{1}{2}}} = (\varphi_h^{n+1})^2 e^{-at^{n+1}} - (\varphi_h^n)^2 e^{-at^n} + (1-e^{-\frac{1}{2}a\Delta t})[(\varphi_h^{n+1})^2 + e^{\frac{a}{2}\Delta t}(\varphi_h^n)^2] e^{-at^{n+\frac{1}{2}}}, \tag{2.7}$$

$$\iint\limits_{D_{ij}} (\boldsymbol{\Omega}\cdot \operatorname{grad}\varphi_h)\varphi_h e^{-at^{n+\frac{1}{2}}} d\mathscr{D} = \frac{1}{2}\int_{\partial D_{ij}} \boldsymbol{\Omega}\cdot \boldsymbol{n}\varphi_h^2 e^{-at^{n+\frac{1}{2}}} dl, \tag{2.8}$$

$$(\varphi_h - \chi_h)\varphi_h = \frac{1}{2}(\varphi_h^2 - \chi_h^2 + (\varphi_h-\chi_h)^2) \tag{2.9}$$

(where $\bar{\varphi}_h = \varphi_h = \varphi_h^{n+\frac{1}{2}} = \frac{1}{2}(\varphi_h^n + \varphi_h^{n+1})$). We obtain

$$\frac{1}{2}\int_{\partial_+ D_{ij}} \boldsymbol{\Omega}\cdot\boldsymbol{n}\varphi_h^2 e^{-at^{n+\frac{1}{2}}} dl + \frac{1}{2}\int_{\partial_- D_{ij}} \boldsymbol{\Omega}\cdot\boldsymbol{n}\chi_h^2 e^{-at^{n+\frac{1}{2}}} dl + \frac{1}{2}\int_{\partial_- D_{ij}} -\boldsymbol{\Omega}\cdot\boldsymbol{n}(\varphi_h-\chi_h)^2 e^{-at^{n+\frac{1}{2}}} dl$$
$$+ \iint\limits_{D_{ij}} \alpha\varphi_h^2 e^{-at^{n+\frac{1}{2}}} d\mathscr{D} - \iint\limits_{D_{ij}} (S_M(\varphi_h)+F)\varphi_h e^{-at^{n+\frac{1}{2}}} d\mathscr{D} + \iint\limits_{D_{ij}} \frac{2}{2v\Delta t}\{(\varphi_h^{n+1})^2 e^{-at^{n+1}}$$
$$- (\varphi_h^n)^2 e^{-at^n} + (1-e^{-\frac{1}{2}a\Delta t})[(\varphi_h^{n+1})^2 + e^{\frac{a}{2}\Delta t}(\varphi_h^n)^2] e^{-at^{n+\frac{1}{2}}}\} d\mathscr{D} = 0. \tag{2.10}$$

Multiplying (2.10) by $w_{ms}\Delta t^{n+\frac{1}{2}}$ and summing over n, i, j, m, s, we get

$$\sum_{n,i,j,m,s} w_{ms}\Delta t^{n+\frac{1}{2}}\cdot\frac{1}{2}\Big[\int_{\partial_+D_{ij}}\boldsymbol{\Omega}\cdot\boldsymbol{n}\,\varphi_h^2e^{-\alpha t}dl+\int_{\partial_-D_{ij}}+\boldsymbol{\Omega}\cdot\boldsymbol{n}\,\chi_h^2e^{-\alpha t}dl$$
$$+\int_{\partial_-D_{ij}}-\boldsymbol{\Omega}\cdot\boldsymbol{n}(\varphi_h-\chi_h)^2e^{-\alpha t}dl+2\iint_{D_{ij}}\alpha\varphi_h^2e^{-\alpha t}d\mathscr{D}\Big]$$
$$+\sum_{i,j,m,s} w_{ms}\iint_{D_{ij}}\frac{1}{2v}(\varphi_h)^2\Big|_{t=T}e^{-\alpha T}d\mathscr{D}$$
$$+\sum_{n,i,j,m,s} w_{ms}\Delta t\iint_{D_{ij}}\frac{1-e^{-\frac{1}{2}\alpha\Delta t}}{2v\Delta t}[(\varphi_h^{n+1})^2+e^{\frac{\alpha}{2}\Delta t}(\varphi_h^n)^2]e^{-\alpha t}d\mathscr{D}$$
$$=\sum_{i,j,m,s} w_{ms}\iint_{D_{ij}}\frac{1}{2v}(\varphi_h)^2|_{t=0}\,d\mathscr{D}+\sum_{n,i,j,m,s} w_{ms}\Delta t\iint_{D_{ij}}[S_M(\varphi_h)+F]\varphi_he^{-\alpha t}d\mathscr{D}. \quad (2.11)$$

First, we write (2.11) in the form $L_1+L_2+\cdots+L_6=r_1+r_2+r_3$, where L_i ($i=1,\cdots,6$) is the i-th term on the left, and r_i ($i=1, 2, 3$) the i-th term on the right. From definition (1.8), eliminating the term of outward trace, we have

$$L_1+L_2=\sum_{n=1}^{N_t}\sum_{s=\frac{N}{2}+1}^{N}\sum_{m=1}^{M_s}\sum_{i=1}^{I}\int_{\partial_+D_{ij}}\boldsymbol{\Omega}\cdot\boldsymbol{n}\,\varphi_h^2e^{-\alpha t}dl$$
$$+\sum_{n=1}^{N_t}\sum_{s=1}^{N}\sum_{m=\frac{M_s}{2}+1}^{M_s}\sum_{j=1}^{J}\int_{\partial_+D_{ij}}\boldsymbol{\Omega}\cdot\boldsymbol{n}\,\varphi_h^2e^{-\alpha t}dl>0. \quad (2.12)$$

Clearly,

$$L_3\geqslant 0, \quad (2.13)$$
$$L_4\geqslant\alpha_0\sum_{n,i,j,m,s} w_{ms}\Delta t^{n+\frac{1}{2}}\iint_{D_{ij}}\varphi_h^2e^{-\alpha t^{n+\frac{1}{2}}}d\mathscr{D}, \quad (2.14)$$
$$L_5\geqslant\frac{e^{-\alpha T}}{2v_0}\|\varphi_h\|^2_{0,B_{xy}\times Q_\Omega}|_{t=T}. \quad (2.15)$$

Using inequalities

$$\begin{cases}(1-e^{-\frac{1}{2}\alpha\Delta t})/\Delta t\geqslant\dfrac{\alpha}{4},\\ (\varphi_h^{n+1})^2+e^{\frac{\alpha}{2}\Delta t}(\varphi_h^n)^2\geqslant 2(\bar{\varphi}_h)^2.\end{cases} \quad (2.16)$$

we obtain

$$L_6\geqslant\frac{\alpha}{4v_0}\sum_{n,i,j,m,s} w_{ms}\Delta t^{n+\frac{1}{2}}\iint_{D_{ij}}\varphi_h^2e^{-\alpha t^{n+\frac{1}{2}}}d\mathscr{D}. \quad (2.17)$$

Similarly, we have

$$r_1\leqslant\frac{1}{2v_1}\|\varphi_h\|^2_{0,B_{xy}\times Q_\Omega}|_{t=0}, \quad (2.18)$$
$$r_2=\sum_{n,i,j}\Delta t^{n+\frac{1}{2}}\iint_{D_{ij}}\beta\Big(\sum_{m,s}(\varphi_h)_{ms}w_{ms}\Big)^2e^{-\alpha t^{n+\frac{1}{2}}}d\mathscr{D}$$
$$\leqslant\beta_1\sum_{n,i,j,m,s} w_{ms}\Delta t^{n+\frac{1}{2}}\iint_{D_{ij}}\varphi_h^2e^{-\alpha t^{n+\frac{1}{2}}}d\mathscr{D}, \quad (2.19)$$
$$r_3\leqslant\|F\|^2_{0,D}+\sum_{n,i,j,m,s} w_{ms}\Delta t^{n+\frac{1}{2}}\iint_{D_{ij}}\varphi_h^2e^{-\alpha t^{n+\frac{1}{2}}}d\mathscr{D}. \quad (2.20)$$

By (2.11)—(2.20), we obtain

$$\frac{e^{-\alpha T}}{2v_0}\|\varphi_h\|^2_{0,B_{xy}\times Q_\Omega}|_{t=T}+\Big(\alpha_0+\frac{\alpha}{4v_0}-\beta_1-1\Big)\sum_{n,i,j,m,s} w_{ms}\Delta t^{n+\frac{1}{2}}\iint_{D_{ij}}\varphi_h^2e^{-\alpha t^{n+\frac{1}{2}}}d\mathscr{D}$$
$$\leqslant\frac{1}{2v_1}\|\varphi_h\|^2_{0,B_{xy}\times Q_\Omega}|_{t=0}+\|F\|^2_{0,D}. \quad (2.21)$$

Obviously, when a satisfies condition (2.6), (2.5) holds. This completes the proof of the theorem.

Corollary 1. Under the same assumptions as in Theorem 1, the approximate solution of (1.7) is stable with respect to the initial-value and the right-hand term F of (1.1) in the domain D.

3. Error Estimates

Suppose that the solution of (1.1) satisfies

$$\varphi\in L_\infty(C^3(E_t)\times H^{K+1}(B_{xy})\times C^0(Q_\Omega)), \tag{3.1}$$

where for a given integer $r\geqslant 0$,

$$H^r(B_{xy})=\{u\,|\,\partial^\alpha u\in L_2(B_{xy}),\ |\alpha|\leqslant r\} \tag{3.2}$$

is a usual Sobolev space with the norms

$$\begin{cases}\|u\|^2_{r,D_{ij}}=\sum\limits_{|\alpha|\leqslant r}\|\partial^\alpha u\|^2_{0,D_{ij}},\\ \|u\|^2_{r,D}=\sum\limits_{n,i,j,m,s}\|u\|^2_{r,D_{ij}}w_{ms}\Delta t^{n+\frac12}.\end{cases} \tag{3.3}$$

In (3.2) and (3.3), $\alpha=(\alpha_1,\alpha_2)$ is a multi-index, $|\alpha|=\alpha_1+\alpha_2$, and

$$\partial^\alpha=\left(\frac{\partial}{\partial x}\right)^{\alpha_1}\left(\frac{\partial}{\partial y}\right)^{\alpha_2}.$$

We also use the following semi-norm

$$|u|^2_{r,D_{ij}}=\sum_{|\alpha|=r}\|\partial^\alpha u\|^2_{0,D_{ij}}. \tag{3.4}$$

Lemma 3.1. *Assume that φ_h is the solution of (1.7) and φ is the exact solution of (1.1). Then, for any $D_{ij}\in J_h$, $u\in P_K(D_{ij})$ and $\eta\in L_2(\partial_-D_{ij})$ we have the identity*

$$\begin{aligned}&\frac12\int_{\partial_+D_{ij}}\boldsymbol{\Omega}\cdot\boldsymbol{n}(\varphi_h-u)^2e^{-\alpha t}dl+\frac12\int_{\partial_-D_{ij}}\boldsymbol{\Omega}\cdot\boldsymbol{n}(\chi_h-\eta)^2e^{-\alpha t}dl\\&\quad+\frac12\int_{\partial_-D_{ij}}-\boldsymbol{\Omega}\cdot\boldsymbol{n}((\varphi_h-u)-(\chi_h-\eta))^2e^{-\alpha t}dl+\iint_{D_{ij}}\alpha(\varphi_h-u)^2e^{-\alpha t}d\mathscr{D}+R\\&=\int_{\partial_+D_{ij}}\boldsymbol{\Omega}\cdot\boldsymbol{n}(\varphi-u)(\varphi_h-u)e^{-\alpha t}dl\\&\quad+\int_{\partial_-D_{ij}}\boldsymbol{\Omega}\cdot\boldsymbol{n}(\varphi-\eta)(\varphi_h-u)e^{-\alpha t}dl+\iint_{D_{ij}}(\varphi-u)A_h^*(\varphi_h-u)e^{-\alpha t}d\mathscr{D}\\&\quad+\iint_{D_{ij}}(S_M((\varphi_h-u)-(\varphi-u))-R_s)(\varphi_h-u)e^{-\alpha t}d\mathscr{D}+R_t,\end{aligned} \tag{3.5}$$

where

$$\begin{cases}R_s=S(\varphi)-S_M(\varphi),\\ R=\iint\limits_{D_{ij}}\frac{1}{2v\Delta t}\{(w^{n+1})^2e^{-\alpha t^{n+1}}-(w^n)^2e^{-\alpha t^n}\\ \qquad+(1-e^{-\frac12\alpha\Delta t})[(w^{n+1})^2+e^{\frac{\alpha}{2}\Delta t}(w^n)^2]e^{-\alpha t}\}d\mathscr{D},\\ R_t=\iint\limits_{D_{ij}}\left(\frac1v\frac{\partial\varphi}{\partial t}-\frac1v\frac{Du}{Dt}\right)we^{-\alpha t}d\mathscr{D},\\ w=\varphi_h-u,\quad A_h^*=-\boldsymbol{\Omega}\cdot\text{grad}+\alpha.\end{cases} \tag{3.6}$$

Proof. Given $u\in P_K(D_{ij})$ and $\eta\in L_2(\partial_-D_{ij})$, we set

$$w=\varphi_h-u\in P_K(D_{ij}),\quad \zeta=\chi_h-\eta\in L_2(\partial_-D_{ij}). \tag{3.7}$$

Consider the expression

$$X_h=-\int_{\partial_-D_{ij}}\boldsymbol{\Omega}\cdot\boldsymbol{n}(w-\zeta)we^{-\alpha t}\,dl+\iint_{D_{ij}}w[A_h(w)-S_M(w)-F]e^{-\alpha t}\,d\mathscr{D}. \tag{3.8}$$

First, using equalities (2.7), (2.8) and (2.9), we obtain

$$\begin{aligned}X_h=&\frac{1}{2}\int_{\partial_+D_{ij}}\boldsymbol{\Omega}\cdot\boldsymbol{n}w^2e^{-\alpha t}\,dl+\frac{1}{2}\int_{\partial_-D_{ij}}\boldsymbol{\Omega}\cdot\boldsymbol{n}\zeta^2e^{-\alpha t}\,dl+R\\&+\frac{1}{2}\int_{\partial_-D_{ij}}-\boldsymbol{\Omega}\cdot\boldsymbol{n}(w-\zeta)^2e^{-\alpha t}\,dl+\iint_{D_{ij}}\alpha w^2e^{-\alpha t}\,d\mathscr{D}\\&-\iint_{D_{ij}}(S_M(w)+F)we^{-\alpha t}\,d\mathscr{D}.\end{aligned} \tag{3.9}$$

On the other hand, using (1.7) and $\psi=we^{-\alpha t}$, we obtain

$$\begin{aligned}X_h=&\int_{\partial_-D_{ij}}\boldsymbol{\Omega}\cdot\boldsymbol{n}(\varphi-\eta)we^{-\alpha t}dl+\int_{\partial_+D_{ij}}\boldsymbol{\Omega}\cdot\boldsymbol{n}(\varphi-u)we^{-\alpha t}\,dl+\iint_{D_{ij}}(\varphi-u)A_h^*(w)e^{-\alpha t}\,d\mathscr{D}\\&+\iint_{D_{ij}}(S_M(u)-S(\varphi)-F)we^{-\alpha t}\,d\mathscr{D}+\iint_{D_{ij}}\left(\frac{1}{v}\frac{\partial\varphi}{\partial t}-\frac{1}{v}\frac{Du}{Dt}\right)we^{-\alpha t}\,d\mathscr{D},\end{aligned} \tag{3.10}$$

where $A_h^*=-\boldsymbol{\Omega}\cdot\text{grad}+\alpha$, φ is the exact solution of (1.1). Thus, by combining (3.9) and (3.10), we get the desired estimate.

For any $D_{ij}\in J_h$, there exists a biaffine invertible mapping which maps the reference element $\hat{D}=[-1\leqslant\xi\leqslant1,\ -1\leqslant\eta\leqslant1]$ onto D_{ij}. Given the reference element $\hat{D}$, we define $\hat{\Pi}$ to be the interpolation operator from $C^0(\hat{D})$ to $P_K(\hat{D})$, such that

$$\hat{\Pi}\hat{u}\big|_{\substack{\xi=\xi_k\\ \eta=\eta_l}}=\hat{u}\big|_{\substack{\xi=\xi_k\\ \eta=\eta_l}},\quad \forall\hat{u}\in C^0(\hat{D}),\quad l,k=1,2,\cdots,K+1, \tag{3.11}$$

where (ξ_k,η_l), k, $l=1,2,\cdots,K+1$, are given on $\hat{D}$. We define $\Pi_{D_{ij}}=\mathscr{L}(C^0(D_{ij});P_K(D_{ij}))$ by

$$\widehat{\Pi_{D_{ij}}u}=\hat{\Pi}\hat{u},\quad \forall u\in C^0(D_{ij}), \tag{3.12}$$

and $\Pi=\mathscr{L}(C_h^0(B_{xy});V_h)$ by

$$\Pi u|_{D_{ij}}=\Pi_{D_{ij}}u,\quad \forall D_{ij}\in J_h,\quad u\in C_h^0(B_{xy}), \tag{3.13}$$

where $\quad C_h^0(B_{xy})=\{u\,|\,u|_{D_{ij}}\in C^0(D_{ij}),\ \forall D_{ij}\in J_h\}.$

Furthermore, we introduce the following geometric parameters:

$$h(D_{ij})=\max(x_i-x_{i-1},\ y_j-y_{j-1}),\quad h_0(D_{ij})=\min(x_i-x_{i-1},\ y_j-y_{j-1}), \tag{3.14}$$

and assume that there exist two constants C_1 and C_2 independent of D_{ij}, such that

$$0<C_1\leqslant\frac{h(D_{ij})}{h_0(D_{ij})}\leqslant C_2,\ \text{for all}\ D_{ij}\in J_h. \tag{3.15}$$

Let $h=\max\limits_{i,j}h(D_{ij})$, $D_{ij}\in J_h$. Then, we can state some standard results which can be easily proved.

Lemma 3.2. *Assume that (3.15) holds. Then, there exists a constant $C>0$ independent of $D_{ij}\in J_h$, variables t, μ and ν such that for all $p\in L_\infty(P_K(D_{ij})\times C^0(E_t\times Q_\Omega))$ the estimates*

$$\begin{cases}|p|_{1,D_{ij}}\leqslant C(h(D_{ij}))^{-1}\|p\|_{0,D_{ij}},\\ \|p\|_{0,\partial D_{ij}}\leqslant C\,(h(D_{ij}))^{-\frac{1}{2}}\|p\|_{0,D_{ij}}\end{cases} \tag{3.16}$$

hold, where ∂D_{ij}^1 is any side of D_{ij} and

$$\|p\|_{0,\partial D_{ij}^1}=\left(\int_{\partial D_{ij}^1}|p|^2 dl\right)^{1/2}.$$

Lemma 3.3. *Assume that* (3.12) *and* (3.15) *hold. Then, there exists a constant $C>0$ independent of $D_{ij}\in J_h$, variables t, μ and ν such that for all $u\in L_\infty(H^{K+1}(D_{ij})\times C^0(E_t\times Q_\Omega))$ the estimates*

$$\begin{cases}|u-\Pi_{D_{ij}}u|_{m,D_{ij}}\leqslant C(h(D_{ij}))^{K+1-m}\|u\|_{K+1,D_{ij}}, & m=0,\ 1,\\ \|u-\Pi_{D_{ij}}u\|_{0,\partial D_{ij}^1}\leqslant C(h(D_{ij}))^{K+\frac{1}{2}}\|u\|_{K+1,D_{ij}}\end{cases}\tag{3.17}$$

hold, where ∂D_{ij}^1 is any side of D_{ij}.

Now we are able to prove the following results.

Theorem 2. *Assume that* (2.3), (2.4) *and* (3.15) *hold. Furthermore, let the solution φ of* (1.1) *satisfy* (3.1). *Then, there exists a constant $C>0$ independent of h, Δt, $\Delta\mu$, $\Delta\nu$, t, x, y, μ, ν such that*

$$\|\varphi_h-\varphi\|_{0,D}\leqslant C\Big(h^K\|\varphi\|_{K+1,D}+\max_D\left|\frac{\partial^3\varphi}{\partial t^3}\right|\Delta t^2+\|R_s\|_{0,D}\Big)\tag{3.18}$$

$$\Big[\sum_{n=1}^{N_t}\sum_{\Omega\cdot n>0}w_{ms}\Delta t^{n+\frac{1}{2}}\int_{\Gamma\cap\partial_+B_{xy}}\boldsymbol{\Omega}\cdot\boldsymbol{n}(\varphi_h-\Pi\varphi)^2dl\Big]^{1/2}$$
$$\leqslant C\Big(h^K\|\varphi\|_{K+1,D}+\max_D\left|\frac{\partial^3\varphi}{\partial t^3}\right|\Delta t^2+\|R_s\|_{0,D}\Big),\tag{3.19}$$

$$\Big[\sum_{n,i,j,m,s}w_{ms}\Delta t^{n+\frac{1}{2}}\int_{\partial_-D_{ij}}-\boldsymbol{\Omega}\cdot\boldsymbol{n}(\varphi_h-\chi_h)^2dl\Big]^{1/2}$$
$$\leqslant C\Big(h^K\|\varphi\|_{K+1,D}+\max_D\left|\frac{\partial^3\varphi}{\partial t^3}\right|\Delta t^2+\|R_s\|_{0,D}\Big),\tag{3.20}$$

where R_s can be seen in (3.6).

Proof. For any $D_{ij}\in J_h$, we define

$$\eta=\begin{cases}0, & \text{on } \partial_-D_{ij}\cap\Gamma_-,\\ \text{outward trace of }\Pi\varphi, & \text{on }\partial_-D_{ij}\backslash(\partial_-D_{ij}\cap\Gamma_-),\end{cases}\tag{3.21}$$

and let $u=\Pi\varphi$, $w=\varphi_h-\Pi\varphi$. Using Lemma 3.1, multiplying (3.5) by $w_{ms}\Delta t^{n+\frac{1}{2}}$ and summing over n, i, j, m, s, we get

$$\frac{1}{2}\sum_{n,i,j,m,s}w_{ms}\Delta t^{n+\frac{1}{2}}\Big\{\int_{\partial_+D_{ij}}\boldsymbol{\Omega}\cdot\boldsymbol{n}(\varphi_h-\Pi\varphi)^2e^{-\alpha t}dl+\int_{\partial_-D_{ij}}\boldsymbol{\Omega}\cdot\boldsymbol{n}(\chi_h-\eta)^2e^{-\alpha t}dl$$
$$+\int_{\partial_-D_{ij}}-\boldsymbol{\Omega}\cdot\boldsymbol{n}[(\varphi_h-\Pi\varphi)-(\chi_h-\eta)]^2e^{-\alpha t}dl+\iint_{D_{ij}}2\alpha(\varphi_h-\Pi\varphi)^2e^{-\alpha t}d\mathscr{D}\Big\}$$
$$+\sum_{i,j,m,s}w_{ms}\iint_{D_{ij}}\frac{1}{2v}(w^{N_t})^2e^{-\alpha T}d\mathscr{D}+\sum_{n,i,j,m,s}w_{ms}\Delta t^{n+\frac{1}{2}}\iint_{D_{ij}}\frac{1-e^{-\frac{1}{2}\alpha\Delta t}}{2v\Delta t}$$
$$\times[(w^{n+1})^2+e^{\frac{\alpha}{2}\Delta t}(w^n)^2]\,e^{-\alpha t}d\mathscr{D}$$
$$=\sum_{i,j,m,s}w_{ms}\iint_{D_{ij}}\frac{1}{2v}w^2\Big|_{t=0}d\mathscr{D}+\sum_{n,i,j,m,s}w_{ms}\Delta t^{n+\frac{1}{2}}\Big\{\int_{\partial_-D_{ij}}\boldsymbol{\Omega}\cdot\boldsymbol{n}(\varphi-\eta)(\varphi_h-\Pi\varphi)e^{-\alpha t}dl$$
$$+\int_{\partial_+D_{ij}}\boldsymbol{\Omega}\cdot\boldsymbol{n}(\varphi-u)(\varphi_h-\Pi\varphi)e^{-\alpha t}dl+\iint_{D_{ij}}(\varphi-u)A_h^*(\varphi_h-\Pi\varphi)e^{-\alpha t}d\mathscr{D}$$
$$+\iint_{D_{ij}}[S_M((\varphi_h-\Pi\varphi)-(\varphi-\Pi\varphi))-R_s](\varphi_h-\Pi\varphi)e^{-\alpha t}d\mathscr{D}+R_t\Big\}.\tag{3.22}$$

Consider both sides of (3.22) as in Theorem 1. Similarly, we get

$$L_1+L_2=\frac{1}{2}\sum_{n=1}^{N_t}\sum_{\Omega\cdot n>0}\int_{\Gamma\cap\partial_+B_{xy}}\Omega\cdot n(\varphi_h-\Pi\varphi)^2e^{-at}\,dl, \tag{2.12)*}$$

$$L_3\geqslant 0, \tag{2.13)*}$$

$$L_4\geqslant\alpha_0e^{-aT}\|\varphi_h-\Pi\varphi\|_{0,D}^2, \tag{2.14)*}$$

$$L_5\geqslant\frac{e^{-aT}}{2v_0}\|\varphi_h-\Pi\varphi\|_{0,B_{xy}\times Q_a}^2\big|_{t=T}, \tag{2.15)*}$$

$$L_6\geqslant\frac{ae^{-aT}}{4v_0}\|\varphi_h-\Pi\varphi\|_{0,D}^2, \tag{2.17)*}$$

$$r_1\leqslant\frac{1}{2v_1}\|\varphi_h-\Pi\varphi\|_{0,B_{xy}\times Q_a}^2\big|_{t=0}, \tag{2.18)*}$$

$$r_2\leqslant Ch^K\|\varphi\|_{K+1,D}\|\varphi_h-\Pi\varphi\|_{0,D}, \tag{3.23}$$

$$r_3\leqslant Ch^K\|\varphi\|_{K+1,D}\|\varphi_h-\Pi\varphi\|_{0,D}, \tag{3.24}$$

$$\begin{aligned} r_4&\leqslant\sum_{n,i,j,m,s}w_{ms}\Delt t^{n+\frac{1}{2}}e^{-at}\|\varphi-\Pi\varphi\|_{0,D_{ij}}(|\varphi_h-\Pi\varphi|_{1,D_{ij}}+\alpha_1\|\varphi_h-\Pi\varphi\|_{0,D_{ij}})\\ &\leqslant Ch^K\|\varphi_h-\Pi\varphi\|_{0,D}\|\varphi\|_{K+1,D}, \end{aligned} \tag{3.25}$$

$$r_5\leqslant\beta_1\|\varphi_h-\Pi\varphi\|_{0,D}^2, \tag{3.26}$$

$$r_6\leqslant Ch^{K+1}\|\varphi\|_{K+1,D}\|\varphi_h-\Pi\varphi\|_{0,D}, \tag{3.27}$$

$$r_7\leqslant C\Big(h^{K+1}\|\varphi_t\|_{K+1,D}+\Delta t^2\max_D\Big|\frac{\partial^3\varphi}{\partial t^3}\Big|\Big)\|\varphi_h-\Pi\varphi\|_{0,D}, \tag{3.28}$$

$$r_8\leqslant\|R_s\|_{0,D}\|\varphi_h-\Pi\varphi\|_{0,D}. \tag{3.29}$$

Thus, combining (2.12)*—(2.18)* and (3.23)—(3.29), we get

$$\begin{aligned} &\frac{e^{-aT}}{2}\sum_{n=1}^{N_t}\sum_{\Omega\cdot n>0}w_{ms}\Delta t^{n+\frac{1}{2}}\int_{\Gamma\cap\partial_+B_{xy}}\Omega\cdot n(\varphi_h-\Pi\varphi)^2\,dl\\ &\quad+\frac{e^{-aT}}{2}\sum_{n,i,j,m,s}w_{ms}\Delta t^{n+\frac{1}{2}}\int_{\partial_-D_{ij}}-\Omega\cdot n((\varphi_h-\Pi\varphi)-(\chi_h-\eta))^2\,dl\\ &\quad+\frac{e^{-aT}}{2v_0}\|\varphi_h-\Pi\varphi\|_{0,B_{xy}\times Q_a}^2\big|_{t=T}+\Big(\alpha_0+\frac{a}{4v_0}-\beta_1\Big)e^{-aT}\|\varphi_h-\Pi\varphi\|_{0,D}\\ &\leqslant\frac{1}{2v_1}\|\varphi_h-\Pi\varphi\|_{0,B_{xy}\times Q_a}^2\big|_{t=0}+C\Big(h^K\|\varphi\|_{K+1,D}+h^{K+1}\|\varphi_t\|_{K+1,D}\\ &\quad+\Delta t^2\max_D\Big|\frac{\partial^3\varphi}{\partial t^3}\Big|+\|R_s\|_{0,D}\Big)\|\varphi_h-\Pi\varphi\|_{0,D}. \end{aligned} \tag{3.30}$$

Besides, we have

$$\|\varphi_h-\varphi\|_{0,D}\leqslant\|\varphi_h-\Pi\varphi\|_{0,D}+\|\Pi\varphi-\varphi\|_{0,D}, \tag{3.31}$$

$$\varphi_h-\chi_h=[(\varphi_h-\Pi\varphi)-(\chi_h-\eta)]+(\Pi\varphi-\varphi)+(\varphi-\eta). \tag{3.32}$$

Thus, using (3.30), (3.17), (3.31) and (3.32), we obtain inequalities (3.18)—(3.20).

4. Superconvergence Estimates

In this section we shall prove that the method has a rate of superconvergence. On the reference $\hat{D}=[-1,1]^2$, (ξ_k,η_l), $k,l=1,2,\cdots,K+1$, are chosen as collocation points, where $\xi_1,\xi_2,\cdots,\xi_{K+1}$ denote the $K+1$ Gauss-Legendre quadrature abscissae on the interval $[-1, 1]$. Similarly, we define the interpolation operator (3.11)—(3.13) and the norm

$$\|u\|_{0,\infty,B_{xy}}=\sup_{B_{xy}}|u(t, x, y, \mu_{ms}, \nu_{ms})| \tag{4.1}$$

for the space $L_\infty(B_{xy})$. Given an integer $r\geqslant 0$, let

$$W_\infty^r(B_{xy})=\{u\in L_\infty(B_{xy})\,|\,\partial^\alpha u\in L_\infty(B_{xy}),\ |\alpha|\leqslant r\} \tag{4.2}$$

be the Sobolev space with the norm

$$\|u\|_{r,\infty,B_{xy}}=\max\{\|\partial^\alpha u\|_{0,\infty,B_{xy}};\ |\alpha|\leqslant r\} \tag{4.3}$$

and $L_\infty(W_\infty^r(B_{xy})\times C^0(E_t\times Q_\Omega))$ with the norm

$$\|u\|_{r,\infty,D}=\sum_{n,i,j,m,s} w_{ms}\Delta t^{n+\frac{1}{2}}\|u\|_{r,\infty,B_{xy}}. \tag{4.4}$$

We can easily prove the following lemmas.

Lemma 4.1. *Assume that (3.13), (3.15) hold. Then, there exists a constant $C>0$ for any $u\in L_\infty((H^{K+1}(D_{ij})\cap W_\infty^{K+1}(D_{ij}))\times C^0(E_t\times Q_\Omega))$ such that*

$$\|u-\Pi_{D_{ij}}u\|_{0,D_{ij}}\leqslant Ch^{K+1}\|u\|_{K+1,D_{ij}}, \tag{4.5}$$

$$\|u-\Pi_{D_{ij}}u\|_{0,\partial D_{ij}^1}\leqslant Ch^{K+\frac{3}{2}}\|u\|_{K+1,\infty,D_{ij}}, \tag{4.6}$$

where ∂D_{ij}^1 is any side of ∂D_{ij}.

Lemma 4.2. *Assume that the solution u of problem (1.1) belongs to $L_\infty(H^{K+2}(B_{xy})\times C^0(E_t\times Q_\Omega))$. Then, there exists a constant $C>0$ independent of $D_{ij}\in J_h$, variables t, μ and ν such that for all $w\in P_K(D_{ij})\times C^0(E_t\times Q_\Omega)$*

$$|Z_{D_{ij}}(u, w)|\leqslant C(h(D_{ij}))^{K+1}|u|_{K+2,D_{ij}}\|w\|_{0,D_{ij}}, \tag{4.7}$$

where

$$Z_{D_{ij}}(u, w)=\int_{\partial_+D_{ij}}\boldsymbol{\Omega}\cdot\boldsymbol{n}(u-\Pi u)w\,dl+\int_{\partial_-D_{ij}}\boldsymbol{\Omega}\cdot\boldsymbol{n}(u-\eta_h)w\,dl$$
$$-\iint_{D_{ij}}(\boldsymbol{\Omega}\cdot\operatorname{grad} w)(u-\Pi u)d\mathscr{D}, \tag{4.8}$$

where η_h is the outward trace of Πu.

Proof. See Lemma 8 in [2].

Theorem 3. *Assume that the solution φ of problem (1.1) belongs to $L_\infty(C^3(E_t)\times(H^{K+2}(B_{xy})\cap W_\infty^{K+1}(B_{xy}))\times C^0(Q_\Omega))$, and that hypotheses (2.3), (2.4) and (3.15) hold. Then, there exists a constant $C>0$ independent of h, Δt, $\Delta\mu$ and $\Delta\nu$ such that*

$$\|\varphi_h-\varphi\|_{0,D}\leqslant C(A_0h^{K+1}+B_0\Delta t^2+\|R_s\|_{0,D}), \tag{4.9}$$

$$\left[\sum_{n=1}^{N_t}\sum_{\Omega\cdot n>0} w_{ms}\Delta t^{n+\frac{1}{2}}\int_{\Gamma\cap\partial_+B_{xy}}\boldsymbol{\Omega}\cdot\boldsymbol{n}(\varphi_h-\varphi)^2\,dl\right]^{1/2}$$
$$\leqslant C(A_1h^{K+1}+B_0\Delta t^2+\|R_s\|_{0,D}), \tag{4.10}$$

$$\left[\sum_{n,i,j,m,s} w_{ms}\Delta t^{n+\frac{1}{2}}\int_{\partial_-D_{ij}}-\boldsymbol{\Omega}\cdot\boldsymbol{n}(\varphi_h-\chi_h)^2\,dl\right]^{1/2}$$
$$\leqslant C(A_1h^{K+1}+B_0\Delta t^2+\|R_s\|_{0,D}), \tag{4.11}$$

where

$$A_0=\|\varphi_t\|_{K+1,D}+\|\varphi\|_{K+2,D}+\|\varphi\|_{K+1,B_{xy}\times Q_\Omega}|_{t=0},$$

$$B_0=\max_D\left|\frac{\partial^3\varphi}{\partial t^3}\right|,$$

$$A_1=\|\varphi_t\|_{K+1,D}+\|\varphi\|_{K+2,D}+\|\varphi\|_{K+1,\infty,D}+\|\varphi\|_{K+1,B_{xy}\times Q_\Omega}|_{t=0}.$$

Proof. As in Theorem 2, we write (3.22) as

$$L_1+L_2+\cdots+L_6=r_1+r_2+\cdots+r_8. \tag{4.12}$$

Repeating the procedure of the proof in Theorem 2, we obtain (2.12)*—(2.18)* and (3.26)—(3.29). By Lemma 4.2,

$$\begin{aligned} r_2+r_3+r_4 &= \sum_{n,i,j,m,s} w_{ms}\Delta t^{n+\frac{1}{2}}\left[Z_{D_{ij}}(\varphi,\ \varphi_h-\Pi\varphi)+\iint_{D_{ij}}\alpha(\varphi_h-\Pi\varphi)(\varphi-\Pi\varphi)d\mathscr{D}\right]e^{-\alpha t} \\ &\leqslant Ch^{K+1}\|\varphi\|_{K+2,D}\|\varphi_h-\Pi\varphi\|_{0,D}. \end{aligned} \tag{4.13}$$

From (3.22), (2.12)*—(2.18)*, (3.26)—(3.29) and (4.13) we have

$$\begin{aligned} &\frac{1}{2}\sum_{n=1}^{N_0}\sum_{\Omega\cdot n>0} w_{ms}\Delta t\int_{\Gamma\cap\partial,B_{xy}}\Omega\cdot n(\varphi_h-\Pi\varphi)^2e^{-\alpha t}dl+\frac{1}{2}\sum_{n,i,j,m,s} w_{ms}\Delta t \\ &\times\int_{\partial-D_{ij}}-\Omega\cdot n[(\varphi_h-\Pi\varphi)-(\chi_h-\eta)]^2e^{-\alpha t}dl+\frac{e^{-\alpha T}}{2v_0}\|\varphi_h-\Pi\varphi\|^2_{0,\,B_{xy}\times Q_a}\Big|_{t=T} \\ &+\left(\alpha_0+\frac{a}{4v_0}-\beta_1\right)e^{-\alpha T}\|\varphi_h-\Pi\varphi\|^2_{0,D} \\ &\leqslant\frac{1}{2v_1}\|\varphi_h-\Pi\varphi\|^2_{0,B_{xy}\times Q_a}\big|_{t=0}+C\Big[h^{K+1}(\|\varphi\|_{K+2,D}+\|\varphi_t\|_{K+1,D}) \\ &+\Delta t^2\max_D\left|\frac{\partial^3\varphi}{\partial t^3}\right|+\|R_s\|_{0,D}\Big]\|\varphi_h-\Pi\varphi\|_{0,D}. \end{aligned} \tag{4.14}$$

The remaining part of the proof is the same. In fact, from (3.31), (3.32), (4.14) and Lemma 4.1 we have (4.9)—(4.11).

Corollary 2. Under the same assumptions as in Theorem 3, the solution of the discrete-discontinuous finite element equation (1.7) converges to the exact solution of problem (1.1) as mesh size h, Δt, $\Delta\mu$, $\Delta\nu$ tends to zero (assume that the numerical integration error $R_s=S(\varphi)-S_M(\varphi)\to0$ as $\Delta\mu\to0$ and $\Delta\nu\to0$).

5. Numerical Results and Analysis

For time-independent neutron transport problems in one-dimensional slab geometry, we give some numerical results of the discrete-discontinuous finite element method (E-method) and the SN method (D-metnod). We consider three problems:

(1) Accuracy: the rate of convergence of each method as the mesh size tends to zero.

(2) Computing time: for a given accuracy which method needs less computing time "cost".

(3) Storage: for a given accuracy which method needs less storage "cost".

Let us denote the computing time and storage for E-method and D-method by ΔT_D, S_D, ΔT_E and S_E respectively and define

$$\eta=\frac{\Delta T_D}{\Delta T_E},\quad \delta=\frac{S_D}{S_E}. \tag{5.1}$$

We construct exact solutions

$$\begin{cases}\varphi_1=\left[64\left(t_1-\frac{1}{4}\right)\left(t_1-\frac{1}{2}\right)\left(3-\frac{16}{3}t_1\right)+100\right]\mu^2, & 0\leqslant x\leqslant X_1,\\ \varphi_2=a(1-t_2)e^{bx}\mu^2, & X_1\leqslant x\leqslant X_2,\end{cases} \tag{5.2}$$

where $\mu\in D_\mu$, $a=44\Big/\left[\left(1-\frac{X_1}{X_2}\right)e^{bX_1}\right]$, $t_1=\frac{x}{X_1}$, $t_2=\frac{x}{X_2}$.

For this we give the following outside source

$$\begin{cases} F_1 = -\dfrac{64}{X_1}\mu^3\left(16t_1^2 - 14t_1 + \dfrac{35}{12}\right) + \left[64\left(t_1 - \dfrac{1}{4}\right)\left(t_1 - \dfrac{1}{2}\right)\left(3 - \dfrac{16}{3}t_1\right) + 100\right] \\ \qquad \times\left(\alpha_1\mu^2 + \dfrac{\beta_1}{3}\right), \quad 0 \leqslant x \leqslant X_1, \\ F_2 = ae^{bx}\left[\mu^3\left(b(1-t_2) - \dfrac{1}{X_2}\right) + (1-t_2)\left(\alpha_2\mu^2 - \dfrac{\beta_2}{3}\right)\right], \quad X_1 \leqslant x \leqslant X_2, \end{cases} \tag{5.3}$$

where parameters b, α_i, β_i, X_i $(i=1, 2)$ are given.

To evaluate the accuracy of each method on various meshes, we introduce two measures of error which we loosely refer to as error norms. In order to determine the accuracy of pointwise scalar flux values, we use the maximum norm, defined by

$$P_M = \max_{i,n,k}\left|\frac{(\varphi_h)_{n,i,k} - \varphi_{n,i,k}}{\varphi_{n,i,k}}\right|. \tag{5.4}$$

Here, φ is the exact solution and φ_h is some approximation. The second error norm which we use here is the "mean square" norm, defined by

$$P_A = \left[\sum_{n,i,k}\frac{h_i}{2}w_n w_k\left(\frac{(\varphi_h)_{n,i,k} - \varphi_{n,i,k}}{\varphi_{n,i,k}}\right)^2\right]^{1/2}. \tag{5.5}$$

Table 1 Accuracy analysis for calculation of the exact solution with $\alpha_1=10$, $\beta_1=1$, $\alpha_2=3$, $\beta_2=0.1$, $X_1=1$, $X_2=2$

I \ M (K, e)		1			2			3		
		P_A	P_M	ΔT	P_A	P_M	ΔT	P_A	P_M	ΔT
2	E_1	0.23	0.29	0.24	0.81×10^{-1}	0.11	0.36	0.38×10^{-2}	0.85×10^{-2}	0.39
	E_4									
4	E_1	0.11	0.14	0.38	0.12×10^{-1}	0.19×10^{-1}	0.53	0.44×10^{-3}	0.13×10^{-2}	0.64
	E_4	0.25	0.50	0.38	0.84×10^{-1}	0.19	0.53	0.17×10^{-1}	0.47×10^{-1}	0.63
10	E_1	0.33×10^{-1}	0.75×10^{-1}	0.73	0.12×10^{-2}	0.36×10^{-2}	1.04	0.21×10^{-4}	0.87×10^{-4}	1.39
	E_4	0.10	0.29	0.73	0.13×10^{-1}	0.46×10^{-1}	1.02	0.10 10^{-2}	0.44×10^{-2}	1.41
20	E_1	0.13×10^{-1}	0.41×10^{-1}	1.35	0.23×10^{-3}	0.91×10^{-3}	1.93	0.20×10^{-5}	0.11×10^{-4}	2.68
	E_4	0.45×10^{-1}	0.16	1.36	0.29×10^{-2}	0.13×10^{-1}	1.95	0.11×10^{-3}	0.60×10^{-3}	2.72
40	E_1	0.50×10^{-2}	0.22×10^{-1}	1.42	0.40×10^{-4}	0.23×10^{-3}	2.44	0.18×10^{-6}	0.13×10^{-5}	3.45
	E_4	0.18×10^{-1}	0.86×10^{-1}	1.71	0.57×10^{-3}	0.34×10^{-2}	2.43	0.11×10^{-4}	0.79×10^{-4}	3.48
60	D_1	0.31×10^{-2}	0.16×10^{-1}	1.12						
	D_4	0.12×10^{-1}	0.63×10^{-1}	1.12						
80	D_1	0.20×10^{-2}	0.12×10^{-1}	1.31						
	D_4	0.83×10^{-1}	0.48×10^{-1}	1.48						
98	D_1	0.15×10^{-2}	0.99×10^{-2}	1.83						
	D_4	0.62×10^{-1}	0.39×10^{-1}	1.81						

Table 1 provides an accuracy analysis for the calculation of the exact solution, where K is the degree of a polynomial, I is the total number of nets, M denotes a method, D is the DSN method, E_1 and E_2 denote a discrete-discontinuous finite element with $b=1$ and $b=4$ respectively, and ΔT is computing time. From Table 1, it is clear that the E-method is more computationally efficient than the D-method. For example, the D-method with 98 grids and $b=1$ yields $P_M=0.99\times10^{-2}$, $\Delta T_D=1.83$, while the E-method with 4 grids and 4 collocation points ($K=3$) gives $P_M=0.85\times10^{-2}$, $\Delta T_E=0.39$. Then, $\eta=\frac{1.83}{0.39}\doteq4.7$, $\delta=\frac{98\times4}{8\times4}\doteq12$. Thus, the E-method is much more efficient in respect of both the computing time and storage. Furthermore, from Table 1, it is seen that the error of the E-method with $K=3$ decreases by order of magnitude as the mesh size tends to zero, where as the error of the D-method only decreases by multiple. Hence, this method is superconvergent. We have also calculated a slab criticality problem. The result is satisfactory, too.

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