SEVERAL ABSTRACT ITERATIVE SCHEMES FOR SOLVING THE BIFURCATION AT SIMPLE EIGENVALUES*

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Abstract

In this paper we consider the nonlinear operator equation $\lambda x = Lx + G(\lambda, x)$, where L is a closed linear operator of $X \to X$, X is a real Banach space, with a simple eigen value $\lambda_0 \neq 0$. We discretize its Liapunov-Schmidt bifurcation equation instead of the original nonlinear operator equation and estimate the approximating order of our approximate solution to the genuine solution. Our method is more convenient and more accurate. Meanwhile we put forward several abstract Newton-type iterative schemes, which are more efficient for practical computation, and get the result of their super-linear convergence.

1. Introduction

We consider the nonlinear operator equation in a real Banach space X

$$\lambda x = Lx + G(\lambda, x), \tag{1}$$

where L is a closed linear operator of $X \to X$ with a real simple eigenvalue $\lambda_0 \neq 0$, such that $\lambda_0 I - L$ is a Fredholm operator of index zero, and $G(\lambda, x)$ is a twice continuously differentiable operator of a neighborhood of $(\lambda_0, \theta) \in R \times X \to X$, $G(\lambda, \theta) \equiv \theta$. $G(\lambda, x) = O(\|x\|^2)$, $G_\lambda(\lambda, x) = O(\|x\|^2)$ hold uniformly for λ near λ_0. It is well known that λ_0 is a bifurcation point from the trivial solution of (1) (see [1]). That is, in the neighborhood of (λ_0, θ) there exists $(\lambda(\varepsilon), x(\varepsilon)) \neq (\lambda, \theta)$ which satisfies (1). Moreover

$$\lambda(0) = \lambda_0; \quad x(\varepsilon) = \varepsilon(u_0 + v(\varepsilon)); \quad u_0 \in N(\lambda_0 I - L);$$
$$v(0) = \theta; \quad v(\varepsilon) \in R(\lambda_0 I - L).$$

Here $N(\lambda_0 I - L)$ and $R(\lambda_0 I - L)$ denote the null space and the range of the operator $(\lambda_0 I - L)$ respectively, $\lambda(\varepsilon)$ and $v(\varepsilon)$ are continuously differentiable functions of ε.

In order to compute the bifurcation solution near the simple eigenvalue λ_0, the usual method is first to discretize the original nonlinear operator equation (1), and then to solve the finite dimensional bifurcation problem. Convergence of this method was proved by Atkinson[2], and Weiss[3]. When the eigenvalue λ_0 and its corresponding eigenelement were known in advance, Westreich and Varol[4] proposed an abstract iterative scheme as follows. Let Q be the canonical projection of X onto $N(\lambda_0 I - L)$ and let $Q^* = I - Q$. By means of the Liapunov-Schmidt method, the nonlinear operator equation (1) can be expressed in the equivalent form

* Received November 9, 1982.

$$\begin{cases} \lambda s v = sLv + Q^* G(\lambda, s(u_0+v)), \\ \lambda s u_0 = sLu_0 + QG(\lambda, s(u_0+v)), \end{cases} \tag{2}$$

where $v \in R(\lambda_0 I - L)$. Let ϕ be the bounded linear functional such that $\phi[u_0]=1$ and $\phi[v]=0$ for $v \in R(\lambda_0 I - L)$. Then $Qx = \phi[x]u_0$ for $x \in X$. For example, if w_0 is an eigenelement of L^* (the adjoint of L) corresponding to λ_0 and w_0 satisfies the normality condition $\langle u_0, w_0\rangle = 1$, $\phi[x]$ can be expressed by $\langle x, w_0\rangle$. Therefore, (1) can also be expressed in the other equivalent form

$$\begin{cases} v = s^{-1}(\lambda I - Q^* L)^{-1} Q^* G(\lambda, s(u_0+v)), \\ \lambda = \lambda_0 + s^{-1}\phi[G(\lambda, s(u_0+v))]. \end{cases} \tag{3}$$

The operator $\lambda I - L$ restricted to $R(\lambda_0 I - L)$ has a uniformly bounded inverse for all λ near λ_0.

The following simple iterative scheme was proposed by Westreich and Varol[4] to solve the bifurcation equation (3)

$$I_1: \quad v^0(s) = \theta, \quad \lambda^0(s) = \lambda_0,$$
$$(\lambda^k(s) I - Q^* L) v^{k+1}(s) = Q^* G(\lambda^k(s), su_0 + v^k(s)),$$
$$\lambda^{k+1}(s) = \lambda_0 + s^{-1}\phi[G(\lambda^k(s), su_0 + v^k(s))].$$

They proved that $\lambda^k(s)$ and $v^k(s)$ converge to the solutions $\lambda(s)$ and $v(s)$ of the bifurcation equation (3) respectively. However, its convergent rate is only linear. In practical numerical computation, the discretization of the original problem is still needed. They pointed out that under some condition the limit of the bifurcation solution of the discretized problem is the solution of the original bifurcation equation (3). However, they did not give the approximating order.

In this paper, we have two purposes. First, we propose several Newton-type iterative schemes for solving bifurcation equations and point out that, their convergent rate is super-linear, they may raise computational efficiency. Second, we directly discretize its Liapunov-Schmidt bifurcation equation (3) instead of the original nonlinear operator equation. Starting from this point of view, we obtain the estimation of the approximating order of our approximate solution to the genuine solution and prove that when λ_0 and u_0 are known in advance, the solution of the discretized bifurcation equation is more accurate and more convenient than the solution of the approximate finite dimensional bifurcation problem.

2. Several Iterative Schemes

In order to simplify our notations, we denote

$$G(\lambda^k(s), s(u_0+v^k(s))) = G^k, \quad \|G^k\| = O(s^2);$$
$$G_\lambda(\lambda^k(s), s(u_0+v^k(s))) = G^k_\lambda, \quad \|G^k_\lambda\| = O(s^2);$$
$$G_u(\lambda^k(s), s(u_0+v^k(s))) = G^k_u, \quad \|G^k_u\| = O(|s|).$$

The equivalent form of the bifurcation equation (3) is

$$\begin{cases} (\lambda I - Q^* L) v = s^{-1} Q^* G(\lambda, s(u_0+v)), \\ \lambda = \lambda_0 + s^{-1}\phi[G(\lambda, s(u_0+v))]. \end{cases} \tag{4}$$

(A) The Newton iterative scheme

N_1:
$$v^0(\varepsilon)=\theta,\quad \lambda^0(\varepsilon)=\lambda_0,$$
$$(\lambda^k(\varepsilon)I-Q^*L)v^{k+1}(\varepsilon)+(\lambda^{k+1}(\varepsilon)-\lambda^k(\varepsilon))v^k(\varepsilon)$$
$$=\varepsilon^{-1}\{Q^*G^k+Q^*G^k_\lambda(\lambda^{k+1}(\varepsilon)-\lambda^k(\varepsilon))+\varepsilon Q^*G^k_u(v^{k+1}(\varepsilon)-v^k(\varepsilon))\},\tag{5}$$
$$\lambda^{k+1}(\varepsilon)=\lambda_0+\varepsilon^{-1}\phi[G^k+G^k_\lambda(\lambda^{k+1}(\varepsilon)-\lambda^k(\varepsilon))+\varepsilon G^k_u(v^{k+1}(\varepsilon)-v^k(\varepsilon))].\tag{6}$$

We have

Theorem 1. *Suppose, in addition to the hypotheses in the introduction,* $\|G_{\lambda\lambda}(\lambda, x)\|$, $\|G_{\lambda u}(\lambda, x)\|$ *and* $\|G_{uu}(\lambda, x)\|$ *are bounded uniformly near* (λ_0, θ). *Then the sequence defined by the Newton iterative scheme* N_1 *converges quadratically to the solution of the bifurcation equation* (4).

Proof. First of all, we prove that the mapping defined by N_1 is a contraction mapping which maps the set $S(\lambda_0, E)\times T(\theta, M)$ into itself. Here $S(\lambda_0, E)=\{\lambda\,|\,|\lambda-\lambda_0|\leqslant E|\varepsilon|\}$ and $T(\theta, M)=\{x\,|\,\|x\|\leqslant M|\varepsilon|\}$, and the constant E and M will be chosen later.

By (6) we get
$$(\lambda^{k+1}(\varepsilon)-\lambda^k(\varepsilon))-\varepsilon^{-1}\phi[G^k_\lambda](\lambda^{k+1}(\varepsilon)-\lambda^k(\varepsilon))$$
$$=\lambda_0-\lambda^k(\varepsilon)+\varepsilon^{-1}\phi[G^k]+\phi[G^k_uv^{k+1}(\varepsilon)]-\phi[G^k_uv^k(\varepsilon)].$$

Therefore
$$\lambda^{k+1}(\varepsilon)-\lambda^k(\varepsilon)=(1-\varepsilon^{-1}\phi[G^k_\lambda])^{-1}\phi[G^k_uv^{k+1}(\varepsilon)]+(1-\varepsilon^{-1}\phi[G^k_\lambda])^{-1}$$
$$\cdot\{(\lambda_0-\lambda^k(\varepsilon))+\varepsilon^{-1}\phi[G^k]-\phi[G^k_uv^k(\varepsilon)]\}.\tag{7}$$

Rewriting (5) yields
$$[\lambda^k(\varepsilon)I-Q^*L-Q^*G^k_u]v^{k+1}(\varepsilon)+(\lambda^{k+1}(\varepsilon)-\lambda^k(\varepsilon))(v^k(\varepsilon)-\varepsilon^{-1}Q^*G^k_\lambda)$$
$$=\varepsilon^{-1}Q^*G^k-Q^*G^k_uv^k(\varepsilon).\tag{8}$$

Substituting $\lambda^{k+1}(\varepsilon)-\lambda^k(\varepsilon)$ in (8) by the right of (7), we see that
$$[\lambda^k(\varepsilon)I-Q^*L-Q^*G^k_u]v^{k+1}(\varepsilon)+(v^k(\varepsilon)-\varepsilon^{-1}Q^*G^k_\lambda)(1-\varepsilon^{-1}\phi[G^k_\lambda])^{-1}\phi[G^k_uv^{k+1}(\varepsilon)]$$
$$=\varepsilon^{-1}Q^*G^k-Q^*G^k_uv^k(\varepsilon)-(v^k(\varepsilon)-\varepsilon^{-1}Q^*G^k_\lambda)(1-\varepsilon^{-1}\phi[G^k_\lambda])^{-1}$$
$$\cdot\{(\lambda_0-\lambda^k(\varepsilon))+\varepsilon^{-1}\phi[G^k]-\phi[G^k_uv^k(\varepsilon)]\}.$$

Thus
$$v^{k+1}(\varepsilon)+[\lambda^k(\varepsilon)I-Q^*L-Q^*G^k_u]^{-1}(v^k(\varepsilon)-\varepsilon^{-1}Q^*G^k_\lambda)(1-\varepsilon^{-1}\phi[G^k_\lambda])^{-1}\phi[G^k_uv^{k+1}(\varepsilon)]$$
$$=[\lambda^k(\varepsilon)I-Q^*L-Q^*G^k_u]^{-1}\{\varepsilon^{-1}Q^*G^k-Q^*G^k_uv^k(\varepsilon)-(v^k(\varepsilon)-\varepsilon^{-1}Q^*G^k_\lambda)$$
$$\cdot(1-\varepsilon^{-1}\phi[G^k_\lambda])^{-1}+((\lambda_0-\lambda^k(\varepsilon))+\varepsilon^{-1}\phi[G^k]-\phi[G^k_uv^k(\varepsilon)])\}.$$

By the triangle inequality we have
$$\|v^{k+1}(\varepsilon)\|-\|(\lambda^k(\varepsilon)I-Q^*L-Q^*G^k_u)^{-1}\|\cdot(\|v^k(\varepsilon)\|+\|\varepsilon^{-1}Q^*G^k_\lambda\|)$$
$$\cdot(1-|\varepsilon^{-1}\phi[G^k_\lambda]|)^{-1}\cdot\|\phi\|\cdot\|G^k_u\|\cdot\|v^{k+1}(\varepsilon)\|$$
$$\leqslant\|(\lambda^k(\varepsilon)I-Q^*L-Q^*G^k_u)^{-1}\|\cdot\{\|\varepsilon^{-1}Q^*G^k\|+\|Q^*G^k_u\|\cdot\|v^k(\varepsilon)\|$$
$$+(\|v^k(\varepsilon)\|+\|\varepsilon^{-1}Q^*G^k_\lambda\|)(1-|\varepsilon^{-1}\phi[G^k_\lambda]|)^{-1}$$
$$\cdot(|\lambda_0-\lambda^k(\varepsilon)|+|\varepsilon^{-1}\phi[G^k]|+|\phi[G^k_uv^k(\varepsilon)]|)\}.$$

From now on we will use the same C to denote different constant which are independent of k. When ε is small enough, we can obtain
$$\|(\lambda^k(\varepsilon)I-Q^*L-Q^*G^k_u)^{-1}\|\leqslant\|(\lambda_0I-Q^*L)^{-1}\|+C|\varepsilon|,\quad \|\varepsilon^{-1}Q^*G^k_\lambda\|\leqslant C|\varepsilon|,\tag{11}$$
$$|\varepsilon^{-1}\phi[G^k_\lambda]|\leqslant C|\varepsilon|,\quad (1-|\varepsilon^{-1}\phi[G^k_\lambda]|)^{-1}\leqslant 1+C|\varepsilon|,\quad \|G^k_u\|\leqslant C|\varepsilon|,$$

$$\|Q^*G_u^k\|\leqslant C|s|,\ \|s^{-1}Q^*G_\lambda^k\|\leqslant C|s|,\ |s^{-1}\phi[G^k]|\leqslant C|s|.$$

Noticing that $\|v^k(s)\|=O(|s|)$, $|\lambda_0-\lambda^k(s)|=O(|s|)$ and s is small enough, we get the inequality

$$1-\|(\lambda^k(s)I-Q^*L-Q^*G_u^k)^{-1}\|\cdot(\|v^k(s)\|+\|s^{-1}Q^*G_\lambda^k\|)$$
$$\cdot(1-|s^{-1}\phi[G_\lambda^k]|)^{-1}\cdot\|\phi\|\cdot\|G_u^k\|>\frac{3}{4}.$$

Let $M=2\sup\limits_{|s|<s_0}(\|s^{-2}Q^*G\|\cdot\|(\lambda_0I-Q^*L)^{-1}\|)$, here s_0 is a small enough positive number. When s is small enough, by $\|v^k(s)\|\leqslant M|s|$ we deduce $\|v^{k+1}(s)\|\leqslant M|s|$. That is, the mapping defined by the Newton iterative scheme N_1 maps $T(\theta,\ M)$ into itself.

On the other hand, by (6)

$$(\lambda^{k+1}(s)-\lambda_0)(1-s^{-1}\phi[G_\lambda^k])=s^{-1}\phi[G^k]+s^{-1}\phi[G_\lambda^k](\lambda_0-\lambda^k(s))$$
$$+\phi[G_u^k(v^{k+1}(s)-v^k(s))].$$

Let $E=2\sup\limits_{|s|<s_0}\left|\frac{\phi[G]}{s^2}\right|$. We again choose s small enough to satisfy $1-|s^{-1}\phi[G_\lambda^k]|>\frac{3}{4}$. Thus by $|\lambda^k(s)-\lambda_0|<E|s|$ we deduce $|\lambda^{k+1}(s)-\lambda_0|\leqslant E|s|$. That is, the mapping defined by the Newton iterative scheme N_1 maps $S(\lambda_0,\ E)$ into itself. Thus we have proved that the mapping defined by the Newton iterative scheme N_1 maps $S(\lambda_0,\ E)\times T(\theta,\ M)$ into itself.

Next we consider the rate of convergence. The original bifurcation equation is

$$(\lambda(s)I-Q^*L)sv(s)=Q^*G(\lambda(s),\ s(u_0+v(s))), \tag{9}$$
$$\lambda(s)=\lambda_0+s^{-1}\phi[G(\lambda(s),\ s(u_0+v(s)))]. \tag{10}$$

The left hand side of (9) is equal to $(\lambda^k(s)I-Q^*L)sv(s)+sv^k(s)(\lambda(s)-\lambda^k(s))+(\lambda(s)-\lambda^k(s))s(v(s)-v^k(s))$. The right hand side of (9) is equal to $Q^*(G^k+G_\lambda^k(\lambda(s)-\lambda^k(s))+G_u^ks(v(s)-v^k(s)))+\frac{1}{2}Q^*\overline{G}_{uu}^k(s(v(s)-v^k(s)))^2+Q^*\overline{G}_{\lambda u}^k(\lambda(s)-\lambda^k(s))(s(v(s)-v^k(s)))+\frac{1}{2}Q^*\overline{G}_{\lambda\lambda}^k(\lambda(s)-\lambda^k(s))^2$. Subtracting (5) $\times s$ from (9) yields

$$(\lambda(s)-\lambda^{k+1}(s))sv^k(s)+(\lambda^k(s)I-Q^*L)s(v(s)-v^{k+1}(s))$$
$$+(\lambda(s)-\lambda^k(s))s(v(s)-v^k(s))$$
$$=Q^*G_\lambda^k(\lambda(s)-\lambda^{k+1}(s))+Q^*G_u^ks(v(s)-v^{k+1}(s))$$
$$+\frac{1}{2}Q^*\overline{G}_{uu}^k(s(v(s)-v^k(s)))^2+Q^*\overline{G}_{\lambda u}^k(\lambda(s)-\lambda^k(s))(s(v(s)-v^k(s)))$$
$$+\frac{1}{2}Q^*\overline{G}_{\lambda\lambda}^k(\lambda(s)-\lambda^k(s))^2.$$

We choose s small enough to satisfy $\|(\lambda^k(s)I-Q^*L)^{-1}QG_u^k\|\leqslant\frac{1}{2}$. Therefore

$$\|s(v(s)-v^{k+1}(s))\|\leqslant2\{\|Q^*G^k-sv^k(s)\|\cdot|\lambda(s)-\lambda^{k+1}(s)|$$
$$+\frac{1}{2}\|Q^*\|\cdot\|\overline{G}_{uu}^k\|\cdot\|s(v(s)-v^k(s))\|^2+(1+\|Q^*\|\cdot\|G_{\lambda u}^k\|)$$
$$\cdot|\lambda(s)-\lambda^k(s)|\cdot\|s(v(s)-v^k(s))\|+\frac{1}{2}\|Q^*\|$$
$$\cdot\|\overline{G}_{\lambda\lambda}^k\|\cdot|\lambda(s)-\lambda^k(s)|^2\}. \tag{11}$$

On the other hand, the right hand side of (10) is equal to

$$\lambda_0+s^{-1}\phi[G^k+G^k_\lambda(\lambda(s)-\lambda^k(s))+G^k_u s(v(s)-v^k(s))+\frac{1}{2}\overline{G}^k_{\lambda\lambda}(\lambda(s)-\lambda^k(s))^2$$
$$+\overline{G}^k_{\lambda u}(\lambda(s)-\lambda^k(s))(s(v(s)-v^k(s)))+\frac{1}{2}\overline{G}^k_{uu}(s(v(s)-v^k(s)))].$$

Subtracting (6) from (10) yields

$$\lambda(s)-\lambda^{k+1}(s)=s^{-1}\phi[G^k_\lambda(\lambda(s)-\lambda^{k+1}(s))+G^k_u s(v(s)-v^{k+1}(s))$$
$$+\frac{1}{2}\overline{G}^k_{\lambda\lambda}(\lambda(s)-\lambda^k(s))^2+\overline{G}^k_{\lambda u}(\lambda(s)-\lambda^k(s))(s(v(s)-v^k(s)))$$
$$+\frac{1}{2}\overline{G}^k_{uu}(s(v(s)-v^k(s)))^2].$$

We again choose s small enough to satisfy $|s|^{-1}\cdot\|\phi\|\cdot\|G^k_\lambda\|\leqslant\frac{1}{2}$, therefore

$$|\lambda(s)-\lambda^{k+1}(s)|\leqslant 2\cdot\|\phi\|\{|s|^{-1}\cdot\|G^k_u\|\cdot\|s(v(s)-v^{k+1}(s))\|$$
$$+\frac{1}{2}\|\overline{G}^k_{\lambda\lambda}\|\cdot|\lambda(s)-\lambda^k(s)|^2+\|\overline{G}^k_{\lambda u}\|\cdot|\lambda(s)-\lambda^k(s)|$$
$$\cdot\|s(v(s)-v^k(s))\|+\frac{1}{2}\|\overline{G}^k_{uu}\|\cdot\|s(v(s)-v^k(s))\|^2\}. \tag{12}$$

Combining (12) with (11) yields

$$\|s(v(s)-v^{k+1}(s))\|\leqslant Cs^2\|s(v(s)-v^{k+1}(s))\|+C\{|\lambda(s)-\lambda^k(s)|^2$$
$$+2|\lambda(s)-\lambda^k(s)|\cdot\|s(v(s)-v^k(s))\|+\|s(v(s)-v^k(s))\|^2\}.$$

Note that

$$|\lambda(s)-\lambda^k(s)|\cdot\|s(v(s)-v^k(s))\|\leqslant\frac{1}{2}\{|\lambda(s)-\lambda^k(s)|^2+\|s(v(s)-v^k(s))\|^2\}$$

and s is small enough, then we get

$$\|s(v(s)-v^{k+1}(s))\|\leqslant C\{|\lambda(s)-\lambda^k(s)|^2+\|s(v(s)-v^k(s))\|^2\}. \tag{13}$$

Combining (13) with (12) yields

$$|\lambda(s)-\lambda^{k+1}(s)|\leqslant C\{|\lambda(s)-\lambda^k(s)|^2+\|s(v(s)-v^k(s))\|^2\}. \tag{14}$$

Thus we obtain

$$|\lambda(s)-\lambda^{k+1}(s)|+\|s(v(s)-v^{k+1}(s))\|$$
$$\leqslant C\{|\lambda(s)-\lambda^k(s)|^2+\|s(v(s)-v^k(s))\|^2\}.$$

It means that the sequence defined by the Newton iterative scheme N_1 converges quadratically to the solution of the bifurcation equation (4). Q. E. D.

(B) The half-Newton iterative scheme

N_2:
$$v^0(s)=\theta,\quad \lambda^0(s)=\lambda_0,$$
$$(\lambda^k(s)I-Q^*L)sv^{k+1}(s)=Q^*G^k+Q^*G^k_u s(v^{k+1}(s)-v^*(s)),$$
$$\lambda^{k+1}(s)=\lambda_0+s^{-1}\phi[G^k+G^k_u s(v^{k+1}(s)-v^k(s))].$$

The estimation of its approximating order is

$$|\lambda(s)-\lambda^{k+1}(s)|+\|s(v(s)-v^{k+1}(s))\|$$
$$\leqslant C\{|s|\cdot|\lambda(s)-\lambda^k(s)|+\|s(v(s)-v^k(s))\|^2\}.$$

Its proof is similar and simpler. The advantage of the scheme N_2 is that the iterative equations in it need not solving simultaneously and it possesses higher convergent rate than the simple iterative scheme without increasing much more computing quantity. Moreover, according to different specific conditions, we can also propose several different half-Newton iterative schemes in order to solve the bifurcation equations conveniently and efficiently. We do not state them in detail.

(C) The simpler iterative scheme

N_3: $$v^0(s)=\theta,\quad \lambda^0(s)=\lambda_0,$$
$$(\lambda_0 I-Q^*L)sv^{k+1}(s)=(\lambda_0-\lambda^k(s))sv^k(s)+Q^*G(\lambda^k(s),\ s(u_0+v^k(s))),$$
$$\lambda^{k+1}(s)=\lambda_0+s^{-1}\phi[G(\lambda^k(s),\ s(u_0+v^k(s)))].$$

Because $\lambda_0 I-Q^*L$ do not vary with k, the computing quantity of solving iteratively $sv^{k+1}(s)$ and $\lambda^{k+1}(s)$ is much less than that of the scheme I_1 in [4].

3. The Approximating Order of the Solution of Discretized Bifurcation Equation to the Solution of the Original Bifurcation Equation

In practical computation we can not directly solve the bifurcation equation (4) by any iterative scheme. We discretize its bifurcation equation instead of the original nonlinear operator equation and then solve the discretized bifurcation equation by some of the above iterative schemes. Let

$$v_h(s)=s^{-1}(\lambda_h(s)I-Q_h^*L_h)^{-1}Q_h^*G_h(\lambda_h(s),\ s(u_0+v_h(s))),\tag{15}$$
$$\lambda_h(s)=\lambda_0+s^{-1}\phi_h[G_h(\lambda_h(s),\ s(u_0+v_h(s)))],\tag{16}$$

where Q_h^*, L_h, G_h and ϕ_h are the approximations to Q^*, L, G and ϕ respectively because they are discretized. The uniform boundedness of $(\lambda_h(s)I-Q_h^*L_h)^{-1}$ with respect to h near 0 can be deduced easily from the boundedness of $(\lambda(s)I-Q^*L)^{-1}$. Therefore

$$\begin{aligned}&\|(\lambda(s)I-Q^*L)^{-1}-(\lambda_h(s)I-Q_h^*L_h)^{-1}\|\\ &=\|(\lambda(s)I-Q^*L)^{-1}[(\lambda_h(s)I-Q_h^*L_h)-(\lambda(s)I-Q^*L)](\lambda_h(s)I-Q_h^*L_h)^{-1}\|\\ &\leqslant C(|\lambda_h(s)-\lambda(s)|+\|Q_h^*L_h-Q^*L\|).\end{aligned}$$

Subtracting (16) from the second formula of (3) yields

$$\begin{aligned}|\lambda(s)-\lambda_h(s)|&=|s|^{-1}\cdot|\phi[G(\lambda(s),\ s(u_0+v(s)))]\\ &\quad-\phi_h[G_h(\lambda_h(s),\ s(u_0+v_h(s)))]|\\ &\leqslant|s|^{-1}\{|\phi[G(\lambda(s),\ s(u_0+v(s)))]-\phi_h[G(\lambda(s),\ s(u_0+v(s)))]|\\ &\quad+|\phi_h[G(\lambda(s),\ s(u_0+v(s)))]-\phi_h[G_h(\lambda_h(s),\ s(u_0+v_h(s)))]|\}\\ &\leqslant|s|^{-1}\{\|\phi-\phi_h\|\cdot\|G(\lambda(s),\ s(u_0+v(s)))\|+\|\phi_h\|\\ &\quad\cdot\|G(\lambda(s),\ s(u_0+v(s)))-G_h(\lambda_h(s),\ s(u_0+v_h(s)))\|\}.\end{aligned}$$

Because

$$\begin{aligned}&\|G(\lambda(s),\ s(u_0+v(s)))-G_h(\lambda_h(s),\ s(u_0+v(s)))\|\\ &\leqslant\|G(\lambda(s),\ s(u_0+v(s)))-G(\lambda_h(s),\ s(u_0+v_h(s)))\|\\ &\quad+\|G(\lambda_h(s),\ s(u_0+v_h(s)))-G_h(\lambda_h(s),\ s(u_0+v_h(s)))\|\\ &\leqslant\|\overline{G}_\lambda\|\cdot|\lambda(s)-\lambda_h(s)|+\|\overline{G}_u\|\cdot\|s(v(s)-v_h(s))\|+s^2\|G-G_h\|,\end{aligned}$$

where the discretization error $\|G-G_h\|=\|G(\lambda_h,\ u_0+v_h)-G_h(\lambda_h,\ u_0+v_h)\|$, we obtain

$$|\lambda(\varepsilon)-\lambda_h(\varepsilon)|\leqslant C|\varepsilon|\{|\lambda(\varepsilon)-\lambda_h(\varepsilon)|+\|v(\varepsilon)-v_h(\varepsilon)\|+\|\phi-\phi_h\|+\|G-G_h\|\}. \tag{17}$$

Subtracting (15) from the first formula of (3) yields

$$\begin{aligned}\|v(\varepsilon)-v_h(\varepsilon)\| &\leqslant |\varepsilon|^{-1}\|(\lambda(\varepsilon)I-Q^*L)^{-1}Q^*G(\lambda(\varepsilon),\ \varepsilon(u_0+v(\varepsilon)))\\ &\quad -(\lambda_h(\varepsilon)I-Q_h^*L_h)^{-1}Q_h^*G_h(\lambda_h(\varepsilon),\ \varepsilon(u_0+v_h(\varepsilon)))\|\\ &\leqslant |\varepsilon|^{-1}\{\|(\lambda(\varepsilon)I-Q^*L)^{-1}Q^*G(\lambda(\varepsilon),\ \varepsilon(u_0+v(\varepsilon)))\\ &\quad -(\lambda_h(\varepsilon)I-Q_h^*L_h)^{-1}Q^*G(\lambda(\varepsilon),\ \varepsilon(u_0+v(\varepsilon)))\|\\ &\quad +\|(\lambda_h(\varepsilon)I-Q_h^*L_h)^{-1}[Q^*G(\lambda(\varepsilon),\ \varepsilon(u_0+v(\varepsilon)))\\ &\quad -Q_h^*G_h(\lambda_h(\varepsilon),\ \varepsilon(u_0+v_h(\varepsilon)))]\|\}\\ &\leqslant C|\varepsilon|\{|\lambda(\varepsilon)-\lambda_h(\varepsilon)|+\|Q^*L-Q_h^*L_h\|+\|Q^*-Q_h^*\|\\ &\quad +\|v(\varepsilon)-v_h(\varepsilon)\|+\|G-G_h\|\}.\end{aligned} \tag{18}$$

When $|\varepsilon|\leqslant\frac{1}{3C}$, by (17) and (18) we obtain

$$\begin{aligned}|\lambda(\varepsilon)-\lambda_h(\varepsilon)| &\leqslant C|\varepsilon|\{\|\phi-\phi_h\|+\|G-G_h\|\}\\ &\quad +C\varepsilon^2\{\|Q^*L-Q_h^*L_h\|+\|Q^*-Q_h^*\|\},\\ \|v(\varepsilon)-v_h(\varepsilon)\| &\leqslant C|\varepsilon|\{\|Q^*L-Q_h^*L_h\|+\|Q^*-Q_h^*\|\\ &\quad +\|G-G_h\|\}+C\varepsilon^2\|\phi-\phi_h\|.\end{aligned}$$

Combining the above two inequalities we get

$$\begin{aligned}&|\lambda(\varepsilon)-\lambda_h(\varepsilon)|+\|v(\varepsilon)-v_h(\varepsilon)\|\\ &\quad\leqslant C|\varepsilon|\{\|\phi-\phi_h\|+\|G-G_h\|+\|Q^*L-Q_h^*L_h\|+\|Q^*-Q_h^*\|\}.\end{aligned}$$

So far we have proved

Theorem 2. *Let $G(\lambda, x)$ be continuously differentiable with respect to λ and x, $G(\lambda, \theta)\equiv\theta$, $\|G(\lambda, x)\|=O(\|x\|^2)$ and $\|G_\lambda(\lambda, x)\|=O(\|x\|^2)$. Then the solution $\{\lambda_h(\varepsilon), v_h(\varepsilon)\}$ of the discretized bifurcation equations* (15) *and* (16) *converges to the solution $\{\lambda(\varepsilon), v(\varepsilon)\}$ of the original bifurcation equation* (3). *Moreover, we have the estimation*

$$\begin{aligned}&|\lambda(\varepsilon)-\lambda_h(\varepsilon)|+\|v(\varepsilon)-v_h(\varepsilon)\|\\ &\quad\leqslant C|\varepsilon|\{\|\phi-\phi_h\|+\|G-G_h\|+\|Q^*L-Q_h^*L_h\|+\|Q^*-Q_h^*\|\}.\end{aligned}$$

Remark 1. By Theorem 2, we know that our method is more accurate, especially for ε near 0. However, in the error of the approximate solution by usual method to discretize the nonlinear operator equation first and then to solve the finite dimensional bifurcation problem, there are $|\lambda_0-\lambda_{0h}|$ and $\|u_0-u_{0h}\|$ which are the errors of the approximate eigenvalue and the approximate eigenelement in addition to the above discretization errors of the operator (see [2], [3]). Thus the computation stated in [4] is verified.

Remark 2. When λ_0 and u_0 are known in advance, our method saves the step of computing the approximate eigenvalue and the approximate eigenelement and decrease computations. In this way, our method makes computation of the bifurcation problem more convenient and more efficient. In order to understand the error of the approximate solution we take an example.

4. Example

Let
$$\begin{cases} y'' = \lambda(y+y^2), \\ y(0) = y(\pi) = 0. \end{cases}$$

Its equivalent integral equation is

$$y(t) = -\lambda \int_0^\pi H(t, s)[y(s)+y^2(s)]ds,$$

where

$$H(t, s) = \begin{cases} t(\pi - s), & t \leqslant s, \\ s(\pi - t), & t > s. \end{cases}$$

As well known, $\lambda = -1$ is a simple eigenvalue, whose corresponding normal eigenelement is $\sqrt{\frac{2}{\pi}} \sin t$. By selfadjointness $\sqrt{\frac{2}{\pi}} \sin t$ is also the normal eigenelement of the adjoint operator,

$$(Qf)(t) = \left(\frac{2}{\pi} \int_0^\pi \sin s\, f(s) ds\right) \sin t,$$

$$\phi[f] = \sqrt{\frac{2}{\pi}} \int_0^\pi \sin s\, f(s) ds.$$

Let $\mu = \frac{1}{\lambda}$. The Liapunov–Schmidt bifurcation equation is

$$\mu v(t) + (I-Q) \int_0^\pi H(t, s) v(s) ds = -(I-Q) \int_0^\pi H(t, s)(s \sin s + v(s))^2 ds,$$

$$\mu = -1 - s^{-1}\left(\frac{2}{\pi}\right) \int_0^\pi \left(\int_0^\pi H(t, s)(s \sin s + v(s))^2 ds\right) \sin t\, dt$$

$$= -1 - s^{-1}\left(\frac{2}{\pi}\right) \int_0^\pi [(s \sin s + v(s))^2 \int_0^\pi H(t, s) \sin t\, dt]\, ds$$

$$= -1 - s^{-1}\left(\frac{2}{\pi}\right) \int_0^\pi (s \sin s + v(s))^2 \sin s\, ds.$$

We divide the interval $[0, \pi]$ into $m = 2p$ equal subintervals. Let $t_j = j\pi/m$ and we replace the above integral operators by numerical integral operators obtained by the compound Simpson's quadrature rule. Thus we have

$$\begin{cases} A_m(\bar{\mu})\bar{v} = -(I-Q_m) H_m f_m(s, \bar{v}), \\ \bar{\mu} = -1 - (s\pi)^{-1} \cdot 2 \cdot \sum_{j=1}^{m-1} w_j (s \sin t_j + v_j)^2 \sin t_j, \end{cases} \quad (19)$$

where

$$w_j = \begin{cases} \frac{4\pi}{3m} & \text{if } j \text{ is even}, \\ \frac{2\pi}{3m} & \text{if } j \text{ is odd}, \end{cases}$$

$$\bar{v} = (v_1, \cdots, v_{m-1})^T, \quad v_j = v(t_j),$$

$$f_m(s, \bar{v}) = ((s \sin t_1 + v_1)^2, \cdots, (s \sin t_{m-1} + v_{m-1})^2),$$

$$H_m=\begin{pmatrix} w_1H(t_1, t_1), \cdots, w_jH(t_1, t_j), \cdots, w_{m-1}H(t_1, t_{m-1}) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ w_1H(t_i, t_1), \cdots, w_jH(t_i, t_j), \cdots, w_{m-1}H(t_i, t_{m-1}) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ w_1H(t_{m-1}, t_1), \cdots, w_jH(t_{m-1}, t_j), \cdots, w_{m-1}H(t_{m-1}, t_{m-1}) \end{pmatrix},$$

$$Q_m=\frac{2}{\pi}\begin{pmatrix} w_1\sin t_1\sin t_1, \cdots, w_j\sin t_1\sin t_j, \cdots, w_{m-1}\sin t_1\sin t_{m-1} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ w_1\sin t_i\sin t_1, \cdots, w_j\sin t_i\sin t_j, \cdots, w_{m-1}\sin t_i\sin t_{m-1} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ w_1\sin t_{m-1}\sin t_1, \cdots, w_j\sin t_{m-1}\sin t_j, \cdots, w_{m-1}\sin t_{m-1}\sin t_{m-1} \end{pmatrix},$$

$$A_m(\bar{\mu})=\bar{\mu}I+(I-Q_m)H_m.$$

Let $h=\frac{\pi}{m}$. We have

$$\|\phi-\phi_h\|=O(h^4),$$
$$\|G-G_h\|=\|Hf-H_mf_m\|=O(h^4),$$
$$\|Q^*L-Q_h^*L_h\|=\|(I-Q)H-(I-Q_m)H_m\|=O(h^4),$$
$$\|Q^*-Q_h^*\|=\|(I-Q)-(I-Q_m)\|=O(h^4).$$

According to Theorem 2, we obtain

$$|\mu(\varepsilon)-\bar{\mu}_h(\varepsilon)|+\|v(\varepsilon)-\bar{v}_h(\varepsilon)\|\leqslant C|\varepsilon|h^4.$$

In practical computation we use various Newton-type iterative schemes stated in this paper to solve the discretized bifurcation equation (19).

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