CANONICAL BOUNDARY ELEMENT METHOD FOR PLANE ELASTICITY PROBLEMS*

YU DE-HAO (余德浩)
(Computing Center, Academia Sinica, Beijing, China)

Abstract

In this paper, we apply the canonical boundary reduction, suggested by Feng Kang[1], to the plane elasticity problems, find the expressions of canonical integral equations and Poisson integral formulas in some typical domains. We also give the numerical method for solving these equations together with their convergence and error estimates. Coupling with classical finite element method, this method can be applied to other domains.

Introduction

The plane elasticity problems include the plane stress problem and plane strain problem. They have an unified mathematical formulation[2].

Taking displacements $u_1(x_1, x_2)$ and $u_2(x_1, x_2)$ in directions x_1 and x_2 as basic unknown functions, we can give the expressions of strain ε_{ij}, $i, j=1, 2$, and stress σ_{ij}, $i, j=1, 2$. Consider the equilibrium equations with traction boundary condition

$$\begin{cases} (\lambda+2\mu)\ \mathrm{grad\ div}\ \boldsymbol{u}-\mu\ \mathrm{rot\ rot}\ \boldsymbol{u}=0, & \text{in } \Omega, \\ \sum_{j=1}^{2} \sigma_{ij} n_j = g_i, \quad i=1, 2, & \text{on } \Gamma, \end{cases} \tag{1}$$

where Ω is a domain with smooth boundary Γ, λ and μ are Lamè coefficients, (n_1, n_2) are the outward normal direction cosines on Γ. Let

$$V(\Omega) = H^1(\Omega)^2,$$

$$\mathscr{R} = \left\{ \boldsymbol{v} \in V(\Omega) \,\middle|\, \boldsymbol{v} = \begin{bmatrix} c_1 - c_3 x_2 \\ c_2 + c_3 x_1 \end{bmatrix},\ c_1,\ c_2,\ c_3 \in R \right\},$$

where $H^1(\Omega)$ is the Sobolev space, then the boundary value problem (1) is equivalent to the variational problem

$$\begin{cases} \text{Find } \boldsymbol{u} \in V(\Omega) \text{ such that} \\ D(\boldsymbol{u}, \boldsymbol{v}) = F(\boldsymbol{v}), \quad \forall \boldsymbol{v} \in V(\Omega), \end{cases} \tag{2}$$

where

$$\begin{aligned} D(\boldsymbol{u}, \boldsymbol{v}) &= \iint_\Omega \sum_{i,j=1}^{2} \sigma_{ij}(\boldsymbol{u})\, \varepsilon_{ij}(\boldsymbol{v})\, dp \\ &= \iint_\Omega \left\{ 2\mu \sum_{i,j=1}^{2} \varepsilon_{ij}(\boldsymbol{u})\, \varepsilon_{ij}(\boldsymbol{v}) + \lambda \sum_{k=1}^{2} \varepsilon_{kk}(\boldsymbol{u})\, \varepsilon_{kk}(\boldsymbol{v}) \right\} dp, \end{aligned}$$

$$F(\boldsymbol{v}) = \int_\Gamma \boldsymbol{g} \cdot \boldsymbol{v}\, ds.$$

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$D(\boldsymbol{u}, \boldsymbol{v})$ is a semi-positive definite symmetric bilinear form, and $D(\boldsymbol{v}, \boldsymbol{v})=0$ if and only if $\varepsilon_{ij}(\boldsymbol{v})=0$, $i, j=1, 2$, and if and only if $\boldsymbol{v}\in\mathscr{R}$.

The sufficient and necessary condition for having solution of the variational problem (2) is

$$F(\boldsymbol{v})=0, \quad \forall \boldsymbol{v}\in\mathscr{R}.$$

Thus we obtain the consistency condition of boundary traction as follows

$$\begin{cases} \int_\Gamma g_i\, ds=0, \quad i=1, 2, \\ \int_\Gamma (x_1g_2-x_2g_1)\, ds=0. \end{cases} \tag{3}$$

From now on we always assume that (3) is satisfied.

Let

$$V^{\cdot}(\Omega)=V(\Omega)/\mathscr{R},$$
$$F^{\cdot}(\boldsymbol{v}^{\cdot})=F(\boldsymbol{v}), \quad \boldsymbol{v}\in\boldsymbol{v}^{\cdot},$$
$$D^{\cdot}(\boldsymbol{u}^{\cdot}, \boldsymbol{v}^{\cdot})=D(\boldsymbol{u}, \boldsymbol{v}), \quad \boldsymbol{u}\in\boldsymbol{u}^{\cdot}, \quad \boldsymbol{v}\in\boldsymbol{v}^{\cdot},$$

we consider the variational problem (2) in the quotient space $V^{\cdot}(\Omega)$, i.e.

$$\begin{cases} \text{Find } \boldsymbol{u}^{\cdot}\in V^{\cdot}(\Omega) \text{ such that} \\ D^{\cdot}(\boldsymbol{u}^{\cdot}, \boldsymbol{v}^{\cdot})=F^{\cdot}(\boldsymbol{v}^{\cdot}), \quad \forall \boldsymbol{v}^{\cdot}\in V^{\cdot}(\Omega). \end{cases} \tag{4}$$

Using the Korn's inequality[4], we have

Proposition 1. $D^{\cdot}(\boldsymbol{v}^{\cdot}, \boldsymbol{v}^{\cdot})\geqslant\alpha\|\boldsymbol{v}^{\cdot}\|^2_{V^{\cdot}}, \forall \boldsymbol{v}^{\cdot}\in V^{\cdot}(\Omega)$,

where α is a positive constant.

Then by Lax-Milgram Lemma we obtain immediately

Proposition 2. The variational problem (4) has one and only one solution, and the solution depends on given traction continuously.

1. Canonical Boundary Reduction

Take the plane Cartesian coordinates. Define a differential operator L and a differential boundary operator β as follows:

$$L=\begin{bmatrix} a\dfrac{\partial^2}{\partial x^2}+b\dfrac{\partial^2}{\partial y^2} & (a-b)\dfrac{\partial^2}{\partial x\partial y} \\ (a-b)\dfrac{\partial^2}{\partial x\partial y} & b\dfrac{\partial^2}{\partial x^2}+a\dfrac{\partial^2}{\partial y^2} \end{bmatrix},$$

$$\beta=\begin{bmatrix} an_1\dfrac{\partial}{\partial x}+bn_2\dfrac{\partial}{\partial y} & (a-2b)n_1\dfrac{\partial}{\partial y}+bn_2\dfrac{\partial}{\partial x} \\ (a-2b)n_2\dfrac{\partial}{\partial x}+bn_1\dfrac{\partial}{\partial y} & an_2\dfrac{\partial}{\partial y}+bn_1\dfrac{\partial}{\partial x} \end{bmatrix}_\Gamma, \tag{5}$$

where $a=\lambda+2\mu$, $b=\mu$, then the boundary value problem (1) can be written as

$$\begin{cases} L\boldsymbol{u}=0, & \text{in } \Omega, \\ \beta\boldsymbol{u}=\boldsymbol{g}, & \text{on } \Gamma. \end{cases} \tag{6}$$

We have Green's formula

$$\iint_\Omega \boldsymbol{v}\cdot L\boldsymbol{u}\, dp=\int_\Gamma \boldsymbol{v}\cdot\beta\boldsymbol{u}\, ds-D(\boldsymbol{u}, \boldsymbol{v}) \tag{7}$$

and second Green's formula

$$\iint_{\Omega}(\boldsymbol{v}\cdot L\boldsymbol{u}-\boldsymbol{u}\cdot L\boldsymbol{v})\,dp=\int_{\Gamma}(\boldsymbol{v}\cdot\beta\boldsymbol{u}-\boldsymbol{u}\cdot\beta\boldsymbol{v})\,ds. \tag{8}$$

Because of $L\boldsymbol{v}=0$ and $\beta\boldsymbol{v}=0$ for $\boldsymbol{v}\in\mathscr{R}$, we have

$$\int_{\Gamma}\boldsymbol{v}\cdot\boldsymbol{g}\,ds=0,\quad\forall\boldsymbol{v}\in\mathscr{R}$$

when $\boldsymbol{u}$ satisfies (6). It is just the consistency condition (3).

Let $G(x, y; x', y')$ be the Green matrix for plane elasticity problem in Ω, from (8) we can obtain the Poisson integral formula

$$\boldsymbol{u}=\int_{\Gamma}(\beta'G)\cdot\boldsymbol{u}\,ds', \tag{9}$$

where β' is the corresponding differential boundary operator with respect to variables x' and y'. Thus the solution of a plane elasticity problem in Ω with displacement boundary condition is given by (9). Affect (9) by β, we obtain

$$\boldsymbol{g}=\beta\boldsymbol{u}=\int_{\Gamma}(\beta\beta'G)^{(-0)}\cdot\boldsymbol{u}\,ds'\equiv\mathscr{K}\gamma\boldsymbol{u}. \tag{10}$$

It is the canonical integral equation. The superscript (-0) denotes the limit from internal side of Γ. γ: $H^1(\Omega)^2\to H^{\frac{1}{2}}(\Gamma)^2$ is the trace operator. The integral operator $\mathscr{K}:\gamma\boldsymbol{u}\to\beta\boldsymbol{u}$ of $[H^{\frac{1}{2}}(\Gamma)]^2\to[H^{-\frac{1}{2}}(\Gamma)]^2$ is a semi-positive definite self-adjoint pseudodifferential operator of order 1. Then the plane elasticity problem with traction boundary condition, i.e. the problem (6), is reduced to the canonical integral equation (10).

Let

$$\bar{D}(\gamma\boldsymbol{u},\ \gamma\boldsymbol{v})=\int_{\Gamma}\gamma\boldsymbol{v}\cdot\mathscr{K}\gamma\boldsymbol{u}\,ds,$$

$$F_0(\gamma\boldsymbol{v})=\int_{\Gamma}\boldsymbol{g}\cdot\gamma\boldsymbol{v}\,ds,$$

then the canonical integral equation (10) is equivalent to the following variational problem

$$\begin{cases}\text{Find } \boldsymbol{u}_0\in H^{\frac{1}{2}}(\Gamma)^2 \text{ such that}\\ \bar{D}(\boldsymbol{u}_0,\ \boldsymbol{v}_0)=F_0(\boldsymbol{v}_0),\quad\forall\boldsymbol{v}_0\in H^{\frac{1}{2}}(\Gamma)^2.\end{cases} \tag{11}$$

Can easily prove

Proposition 3. If $\boldsymbol{u}$ satisfies $L\boldsymbol{u}=0$, then

$$\bar{D}(\gamma\boldsymbol{u},\ \gamma\boldsymbol{v})=D(\boldsymbol{u},\ \boldsymbol{v}).$$

We also consider the variational problem (11) in the quotient space

$$V^{\cdot}(\Gamma)\equiv H^{\frac{1}{2}}(\Gamma)^2/\gamma\mathscr{R},$$

$$\begin{cases}\text{Find } \boldsymbol{u}_0^{\cdot}\in V^{\cdot}(\Gamma) \text{ such that}\\ \bar{D}^{\cdot}(\boldsymbol{u}_0^{\cdot},\ \boldsymbol{v}_0^{\cdot})=F_0(\boldsymbol{v}_0^{\cdot}),\quad\forall\boldsymbol{v}_0^{\cdot}\in V^{\cdot}(\Gamma),\end{cases} \tag{12}$$

where

$$\bar{D}^{\cdot}(\boldsymbol{u}_0^{\cdot},\ \boldsymbol{v}_0^{\cdot})=\bar{D}(\boldsymbol{u}_0,\ \boldsymbol{v}_0),\quad \boldsymbol{u}_0\in\boldsymbol{u}_0^{\cdot},\ \boldsymbol{v}_0\in\boldsymbol{v}_0^{\cdot},$$

$$F_0^{\cdot}(\boldsymbol{v}_0^{\cdot})=F_0(\boldsymbol{v}_0),\quad \boldsymbol{v}_0\in\boldsymbol{v}_0^{\cdot}.$$

From Proposition 3 we can easily obtain the $V^{\cdot}(\Gamma)$-ellipticity of the bilinear form

$\bar{D}'(\cdot, \cdot)$ and then we have

Proposition 4. The variational problem (12) has one and only one solution, and the solution depends on given traction continuously.

Now we are going to find the expressions of Poisson integral formulas and canonical integral equations of plane elasticity problems in some typical domains.

(i) Ω is the upper half-plane.

Let $U(s, y)=\int_{-\infty}^{\infty} e^{-2\pi xsi}\boldsymbol{u}(x, y)\,dx\equiv\mathscr{F}(\boldsymbol{u})$ is the Fourier transform of $\boldsymbol{u}$, $\boldsymbol{U}=(U_1, U_2)^T$, $\boldsymbol{u}=(u_1, u_2)^T$. Affect $L\boldsymbol{u}=0$ by Fourier transformation. We obtain

$$\begin{cases}(2\pi si)^2aU_1+b\dfrac{\partial^2}{\partial y^2}U_1+2\pi si(a-b)\dfrac{\partial}{\partial y}U_2=0,\\(2\pi si)^2bU_2+a\dfrac{\partial^2}{\partial y^2}U_2+2\pi si(a-b)\dfrac{\partial}{\partial y}U_1=0.\end{cases}$$

Then

$$\begin{cases}U_1=\left[\left(1-\dfrac{a-b}{a+b}|2\pi s|y\right)U_{10}-\dfrac{a-b}{a+b}i2\pi syU_{20}\right]e^{-|2\pi s|y},\\U_2=\left[\left(1+\dfrac{a-b}{a+b}|2\pi s|y\right)U_{20}-\dfrac{a-b}{a+b}i2\pi syU_{10}\right]e^{-|2\pi s|y}.\end{cases}\tag{13}$$

Because of $\boldsymbol{n}=(0, -1)^T$ on Γ,

$$\begin{bmatrix}g_1\\g_2\end{bmatrix}=\begin{bmatrix}-b\left(\dfrac{\partial u_1}{\partial y}+\dfrac{\partial u_2}{\partial x}\right)\\-(a-2b)\dfrac{\partial u_1}{\partial x}-a\dfrac{\partial u_2}{\partial y}\end{bmatrix}_{y=0}.$$

Taking its Fourier transform and substituting (13) for U, we obtain

$$\begin{bmatrix}\mathscr{F}(g_1)\\\mathscr{F}(g_2)\end{bmatrix}=\begin{bmatrix}\dfrac{2ab}{a+b}2\pi|s| & -\dfrac{2b^2}{a+b}i2\pi s\\\dfrac{2b^2}{a+b}i2\pi s & \dfrac{2ab}{a+b}2\pi|s|\end{bmatrix}\begin{bmatrix}U_{10}\\U_{20}\end{bmatrix}.$$

Then taking its inverse Fourier transform, we have

$$\boldsymbol{g}=\begin{bmatrix}-\dfrac{2ab}{(a+b)\pi x^2} & -\dfrac{2b^2}{a+b}\delta'(x)\\\dfrac{2b^2}{a+b}\delta'(x) & -\dfrac{2ab}{(a+b)\pi x^2}\end{bmatrix}*\boldsymbol{u}_0.\tag{14}$$

It is just the desired canonical integral equations. Here $*$ denotes the convolution.

The Poisson integral formulas can be obtained from (13) by using the inverse Fourier transformation:

$$\boldsymbol{u}=\begin{bmatrix}P_{11} & P_{12}\\P_{21} & P_{22}\end{bmatrix}*\boldsymbol{u}_0,\quad y>0,\tag{15}$$

where

$$P_{11}=\frac{y}{\pi(x^2+y^2)}+\frac{(a-b)y(x^2-y^2)}{(a+b)\pi(x^2+y^2)^2},$$

$$P_{12}=P_{21}=\frac{2(a-b)xy^2}{(a+b)\pi(x^2+y^2)^2},$$

$$P_{22}=\frac{y}{\pi(x^2+y^2)}-\frac{(a-b)y(x^2-y^2)}{(a+b)\pi(x^2+y^2)^2}.$$

(14) and (15) also can be obtained from the Green matrix with respect to the upper

half-plane

$$G=\begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix},$$

where

$$\begin{aligned}
G_{11}=&\frac{a+b}{8\pi ab}\ln\frac{(x-x')^2+(y-y')^2}{(x-x')^2+(y+y')^2}\\
&+\frac{a-b}{4\pi ab}\left[\frac{(y-y')^2}{(x-x')^2+(y-y')^2}-\frac{(y+y')^2}{(x-x')^2+(y+y')^2}\right]\\
&+\frac{(a-b)^2}{2\pi ab(a+b)}yy'\frac{(y+y')^2-(x-x')^2}{[(x-x')^2+(y+y')^2]^2},\\
G_{12}=&\frac{a-b}{4\pi ab}\left[\frac{(y-y')(x-x')}{(x-x')^2+(y+y')^2}-\frac{(y-y')(x-x')}{(x-x')^2+(y-y')^2}\right]\\
&+\frac{(a-b)^2}{\pi ab(a+b)}yy'\frac{(y+y')(x-x')}{[(x-x')^2+(y+y')^2]^2},\\
G_{21}=&\frac{a-b}{4\pi ab}\left[\frac{(y-y')(x-x')}{(x-x')^2+(y+y')^2}-\frac{(y-y')(x-x')}{(x-x')^2+(y-y')^2}\right]\\
&-\frac{(a-b)^2}{\pi ab(a+b)}yy'\frac{(y+y')(x-x')}{[(x-x')^2+(y+y')^2]^2},\\
G_{22}=&\frac{a+b}{8\pi ab}\ln\frac{(x-x')^2+(y-y')^2}{(x-x')^2+(y+y')^2}\\
&+\frac{a-b}{4\pi ab}\left[\frac{(y+y')^2}{(x-x')^2+(y+y')^2}-\frac{(y-y')^2}{(x-x')^2+(y-y')^2}\right]\\
&+\frac{(a-b)^2}{2\pi ab(a+b)}yy'\frac{(y+y')^2-(x-x')^2}{[(x-x')^2+(y+y')^2]^2}.
\end{aligned}$$

(ii) Ω is the interior domain to the circle with radius R.

Let $\boldsymbol{u}=u_r\boldsymbol{e}_r+u_\theta\boldsymbol{e}_\theta$, the plane elasticity equations can be written as

$$\begin{bmatrix} a\frac{\partial^2}{\partial r^2}+\frac{a}{r}\frac{\partial}{\partial r}-\frac{a}{r^2}+\frac{b}{r^2}\frac{\partial^2}{\partial\theta^2} & \frac{a-b}{r}\frac{\partial^2}{\partial\theta\,\partial r}-\frac{a+b}{r^2}\frac{\partial}{\partial\theta} \\ \frac{a-b}{r}\frac{\partial^2}{\partial\theta\,\partial r}+\frac{a+b}{r^2}\frac{\partial}{\partial\theta} & b\frac{\partial^2}{\partial r^2}+\frac{b}{r}\frac{\partial}{\partial r}-\frac{b}{r^2}+\frac{a}{r^2}\frac{\partial^2}{\partial\theta^2} \end{bmatrix}\begin{bmatrix} u_r \\ u_\theta \end{bmatrix}=0.$$

Using the separation of variables, we can obtain the canonical integral equation

$$\begin{bmatrix} g_r(\theta) \\ g_\theta(\theta) \end{bmatrix}=\begin{bmatrix} K_{rr} & K_{r\theta} \\ K_{\theta r} & K_{\theta\theta} \end{bmatrix}*\begin{bmatrix} u_r(R,\theta) \\ u_\theta(R,\theta) \end{bmatrix},$$

where

$$\begin{aligned}
K_{rr}&=-\frac{ab}{(a+b)2\pi R\sin^2\theta/2}-\frac{2b^2}{R(a+b)}\delta(\theta)+\frac{a^2}{\pi R(a+b)},\\
K_{r\theta}&=-K_{\theta r}=-\frac{ab}{(a+b)\pi R}\operatorname{ctg}\frac{\theta}{2}-\frac{2b^2}{R(a+b)}\delta'(\theta),\\
K_{\theta\theta}&=-\frac{ab}{(a+b)2\pi R\sin^2\theta/2}-\frac{2b^2}{R(a+b)}\delta(\theta)+\frac{b^2}{\pi R(a+b)},
\end{aligned}\tag{16}$$

and the Poisson integral formula

$$\begin{bmatrix} u_r(r,\theta) \\ u_\theta(r,\theta) \end{bmatrix}=\begin{bmatrix} P_{rr} & P_{r\theta} \\ P_{\theta r} & P_{\theta\theta} \end{bmatrix}*\begin{bmatrix} u_r(R,\theta) \\ u_\theta(R,\theta) \end{bmatrix},\quad 0\leqslant r<R,$$

where

$$P_{rr}=\frac{[2aR\cos\theta-(a-b)r](R^2-r^2)}{(a+b)2\pi R(r^2+R^2-2Rr\cos\theta)}$$
$$+\frac{(a-b)(R^2-r^2)(R^2\cos\theta-2Rr+r^2\cos\theta)}{(a+b)2\pi(R^2+r^2-2Rr\cos\theta)^2},$$
$$P_{r\theta}=\frac{b(R^2-r^2)\sin\theta}{(a+b)\pi(R^2+r^2-2Rr\cos\theta)}+\frac{(b-a)(R^2-r^2)^2\sin\theta}{2(a+b)\pi(R^2+r^2-2Rr\cos\theta)^2},\tag{17}$$
$$P_{\theta r}=-\frac{a(R^2-r^2)\sin\theta}{(a+b)\pi(R^2+r^2-2Rr\cos\theta)}+\frac{(b-a)(R^2-r^2)^2\sin\theta}{2(a+b)\pi(R^2+r^2-2Rr\cos\theta)^2},$$
$$P_{\theta\theta}=\frac{[2bR\cos\theta-(b-a)r](R^2-r^2)}{(a+b)2\pi R(R^2+r^2-2Rr\cos\theta)}+\frac{(b-a)(R^2-r^2)(R^2\cos\theta-2Rr+r^2\cos\theta)}{(a+b)2\pi(R^2+r^2-2Rr\cos\theta)^2}.$$

(iii) Ω is the exterior domain to the circle with radius R.

By using the same method, we also can obtain the canonical integral equation

$$\begin{bmatrix} g_r(\theta) \\ g_\theta(\theta) \end{bmatrix}=\begin{bmatrix} K_{rr} & K_{r\theta} \\ K_{\theta r} & K_{\theta\theta} \end{bmatrix}*\begin{bmatrix} u_r(R,\ \theta) \\ u_\theta(R,\ \theta) \end{bmatrix},$$

where

$$K_{rr}=K_{\theta\theta}=-\frac{ab}{(a+b)2\pi R\sin^2\theta/2}+\frac{2b^2}{(a+b)R}\,\delta(\theta)+\frac{ab}{\pi R(a+b)},$$
$$K_{r\theta}=-K_{\theta r}=-\frac{ab}{(a+b)\pi R}\operatorname{ctg}\frac{\theta}{2}+\frac{2b^2}{(a+b)R}\,\delta'(\theta),\tag{18}$$

and the Poisson integral formula

$$\begin{bmatrix} u_r(r,\ \theta) \\ u_\theta(r,\ \theta) \end{bmatrix}=\begin{bmatrix} P_{rr} & P_{r\theta} \\ P_{\theta r} & P_{\theta\theta} \end{bmatrix}*\begin{bmatrix} u_r(R,\ \theta) \\ u_\theta(R,\ \theta) \end{bmatrix},\quad r>R,$$

where

$$P_{rr}=\frac{[2br\cos\theta+(a-b)R](r^2-R^2)}{(a+b)2\pi r(R^2+r^2-2Rr\cos\theta)}+\frac{(a-b)(r^2-R^2)(R^2\cos\theta-2Rr+r^2\cos\theta)}{2(a+b)\pi(R^2+r^2-2Rr\cos\theta)^2},$$
$$P_{r\theta}=\frac{b(r^2-R^2)\sin\theta}{(a+b)\pi(R^2+r^2-2Rr\cos\theta)}+\frac{(a-b)(r^2-R^2)^2\sin\theta}{2(a+b)\pi(R^2+r^2-2Rr\cos\theta)^2},$$
$$P_{\theta r}=-\frac{a(r^2-R^2)\sin\theta}{(a+b)\pi(R^2+r^2-2Rr\cos\theta)}+\frac{(a-b)(r^2-R^2)^2\sin\theta}{2(a+b)\pi(R^2+r^2-2Rr\cos\theta)^2},\tag{19}$$
$$P_{\theta\theta}=\frac{[2ar\cos\theta+(b-a)R](r^2-R^2)}{(a+b)2\pi r(R^2+r^2-2Rr\cos\theta)}-\frac{(a-b)(r^2-R^2)(R^2\cos\theta-2Rr+r^2\cos\theta)}{2(a+b)\pi(R^2+r^2-2Rr\cos\theta)^2}.$$

The canonical integral equations (14), (16) and (18) also have been found by using the method of complex analysis, the detail will appear in another paper.

2. Numerical Solution and Its Convergence

Let Ω is the interior domain to the circle with radius R, the variational problem corresponding to the canonical integral equation (16) is

$$\begin{cases} \text{Find } (u_{r0},\ u_{\theta 0})\in H^{\frac{1}{2}}(\Gamma)^2 \quad \text{such that} \\ \bar{D}(u_{r0},\ u_{\theta 0};\ v_{r0},\ v_{\theta 0})=F(v_{r0},\ v_{\theta 0}),\quad \forall(v_{r0},\ v_{\theta 0})\in H^{\frac{1}{2}}(\Gamma)^2. \end{cases}\tag{20}$$

where

$$F(v_{r0},\ v_{\theta 0})=R\int_0^{2\pi}[g_r(\theta)v_{r0}(\theta)+g_\theta(\theta)v_{\theta 0}(\theta)]d\theta,$$
$$\bar{D}(u_{r0},\ u_{\theta 0};\ v_{r0},\ v_{\theta 0})=\int_0^{2\pi}(v_{r0},\ v_{\theta 0})\left\{R\begin{bmatrix} K_{rr} & K_{r\theta} \\ K_{\theta r} & K_{\theta\theta} \end{bmatrix}*\begin{bmatrix} u_{r0} \\ u_{\theta 0} \end{bmatrix}\right\}d\theta.$$

It is easily seen that the integral kernel is independent of R in fact.

Now divide the boundary into N and taking the piecewise linear basis functions $\{L_j(\theta)\}\subset H^{\frac{1}{2}}(\Gamma)$, where

$$L_j(\theta)=\begin{cases}\frac{N}{2\pi}(\theta-\theta_{j-1}), & \theta_{j-1}\leqslant\theta\leqslant\theta_j,\\ \frac{N}{2\pi}(\theta_{j+1}-\theta), & \theta_j\leqslant\theta\leqslant\theta_{j+1},\\ 0, & \text{otherwise},\end{cases}$$

$$\theta_j=\frac{j}{N}2\pi,\quad j=1,\cdots,N.$$

Let

$$u_{r0}(\theta)\approx U_{r0}(\theta)=\sum_{j=1}^{N}U_jL_j(\theta),$$

$$u_{\theta0}(\theta)\approx U_{\theta0}(\theta)=\sum_{j=1}^{N}V_jL_j(\theta).$$

From (20) we obtain a linear algebraic equations

$$\begin{bmatrix}Q_{11} & Q_{12}\\ Q_{21} & Q_{22}\end{bmatrix}\begin{bmatrix}U\\ V\end{bmatrix}=\begin{bmatrix}B\\ C\end{bmatrix},\tag{21}$$

where

$$U=[U_1,\cdots,U_N]^T,\quad V=[V_1,\cdots,V_N]^T,$$
$$B=[b_1,\cdots,b_N]^T,\quad C=[c_1,\cdots,c_N]^T,$$
$$Q_{lm}=[q_{ij}^{(lm)}]_{i,j=1,\cdots,N},\quad l,m=1,2,$$
$$b_i=R\int_0^{2\pi}g_r(\theta)L_i(\theta)d\theta,\quad c_i=R\int_0^{2\pi}g_\theta(\theta)L_i(\theta)d\theta,\quad i=1,\cdots,N,$$
$$q_{ij}^{(11)}=\overline{D}(L_j,0;L_i,0),\quad q_{ij}^{(12)}=\overline{D}(0,L_j;L_i,0),$$
$$q_{ij}^{(21)}=\overline{D}(L_j,0;0,L_i),\quad q_{ij}^{(22)}=\overline{D}(0,L_j;0,L_i),\quad i,j=1,\cdots,N.\tag{22}$$

Using the formulas in the sense of distributions

$$-\frac{1}{4\sin^2\theta/2}=\sum_{n=1}^{\infty}n\cos n\theta$$

and

$$\frac{1}{2}\operatorname{ctg}\frac{\theta}{2}=\sum_{n=1}^{\infty}\sin n\theta,$$

we can obtain

$$Q_{11}=\frac{2ab}{a+b}(a_0,a_1,\cdots,a_{N-1})-\frac{2\pi b^2}{3N(a+b)}(4,1,0,\cdots,0,1)$$
$$+\frac{4\pi a^2}{N^2(a+b)}(1,\cdots,1),$$
$$Q_{12}=-Q_{21}=\frac{2ab}{a+b}(0,d_1,\cdots,d_{N-1})-\frac{b^2}{a+b}(0,1,0,\cdots,0,-1),$$
$$Q_{22}=\frac{2ab}{a+b}(a_0,a_1,\cdots,a_{N-1})-\frac{2\pi b^2}{3N(a+b)}(4,1,0,\cdots,0,1)$$
$$+\frac{4\pi b^2}{N^2(a+b)}(1,\cdots,1),\tag{23}$$

where $(\alpha_1,\cdots,\alpha_N)$ denotes the circulant matrix produced by $\alpha_1,\cdots,\alpha_N$,

$$a_k=\frac{4N^2}{\pi^3}\sum_{j=1}^{\infty}\frac{1}{j^3}\sin^4\frac{j\pi}{N}\cos\frac{jk}{N}2\pi,$$

$$d_k=\frac{4N^2}{\pi^3}\sum_{j=1}^{\infty}\frac{1}{j^4}\sin^4\frac{j\pi}{N}\sin\frac{jk}{N}2\pi,\quad k=0,\ 1,\ \cdots. \tag{24}$$

It is easily seen that Q_{11} and Q_{22} are symmetric circulant matrices, Q_{12} and Q_{21} are antisymmetric circulant matrices, and

$$Q=\begin{bmatrix}Q_{11} & Q_{12}\\ Q_{21} & Q_{22}\end{bmatrix}$$

is a semi-positive definite symmetric matrix.

From Proposition 4 we obtain

Proposition 5. The system of linear equations (21) has an unique solution up to a vector in the space

$$\left\{\begin{bmatrix}U\\ V\end{bmatrix}\middle|\left(\sum_{j=1}^{N}U_jL_j(\theta),\ \sum_{j=1}^{N}V_jL_j(\theta)\right)\in\gamma\mathscr{R}\right\}.$$

Therefore, if we want get an unique solution, we must add some other conditions. We can solve (21) by direct or iterative method.

When Ω is the exterior domain to the circle with radius R, we also can obtain a system of linear equations in the form of (21) from the variational problem corresponding to (18), but where

$$Q_{11}=Q_{22}=\frac{2ab}{a+b}(a_0,\ a_1,\ \cdots,\ a_{N-1})+\frac{2\pi b^2}{3N(a+b)}(4,\ 1,\ 0,\ \cdots,\ 0,\ 1)$$

$$+\frac{4\pi ab}{N^2(a+b)}(1,\ \cdots,\ 1), \tag{25}$$

$$Q_{12}=-Q_{21}=\frac{2ab}{a+b}(0,\ d_1,\ \cdots,\ d_{N-1})+\frac{b^2}{a+b}(0,\ 1,\ 0,\ \cdots,\ 0,\ -1).$$

Here a_k, d_k, $k=0,\ 1,\ \cdots,\ N-1$ are given by (24).

Let $u_0(\theta)\in H^{\frac{1}{2}}(\Gamma)^2$ be the solution of canonical integral equation (16) and $U_0^{(N)}(\theta)\in S_N^2$ be the corresponding approximate solution found by canonical boundary element method. Let $\Pi: H^{\frac{1}{2}}(\Gamma)^2\to S_N^2$ be the interpolation operator and $\|\cdot\|_{\overline{D}}$ be the energy norm on $V^*(\Gamma)$ derived from $\overline{D}(\cdot,\ \cdot)$. By the methods used in classical theory on finite element method and in canonical boundary element method for harmonic and biharmonic boundary value problems[6], and by using some known results about solutions of plane elasticity problems[4], we can easily obtain following results.

Lemma 1. $\overline{D}(u_0-U_0^{(N)},\ V_0)=0,\ \forall V_0\in S_N^2;$

$$\|u_0-U_0^{(N)}\|_{\overline{D}}=\inf_{V_0\in S_N^2}\|u_0-V_0\|_{\overline{D}}.$$

Lemma 2. *The energy norm $\|\cdot\|_{\overline{D}}$ and the quotient norm $\|\cdot\|_{V^*(\Gamma)}$ are equivalent.*

Theorem 1 (Convergence). *If Π satisfies $\lim_{N\to\infty}\|v_0-\Pi v_0\|_{V(\Gamma)}=0,\ \forall v_0\in V(\Gamma)=[H^{\frac{1}{2}}(\Gamma)]^2$, then $\lim_{N\to\infty}\|u_0-U_0^{(N)}\|_{\overline{D}}=0$.*

Proof. Since Lemma 2, there exists a constant K, such that

$$\|v_0\|_{\overline{D}}\leqslant K\|v_0\|_{V(\Gamma)},\quad \forall v_0\in V(\Gamma),$$

and from the trace theorem, we have a constant T, such that

$$\|\gamma \boldsymbol{v}\|_{V(\Gamma)} \leqslant T\|\boldsymbol{v}\|_{V(\Omega)}, \quad \forall \boldsymbol{v} \in V(\Omega).$$

Because of $\boldsymbol{u}_0 \in V(\Gamma)$, there exists $\boldsymbol{u} \in V(\Omega)$, such that $\gamma \boldsymbol{u} = \boldsymbol{u}_0$.

For arbitrary $\varepsilon > 0$, we have $\boldsymbol{w} \in C^\infty(\bar{\Omega})^2$, such that $\|\boldsymbol{u}-\boldsymbol{w}\|_{V(\Omega)} \leqslant \frac{\varepsilon}{2KT}$. Let $\gamma \boldsymbol{w} = \boldsymbol{w}_0$, then

$$\|\boldsymbol{u}_0 - \boldsymbol{w}_0\|_D \leqslant KT\|\boldsymbol{u}-\boldsymbol{w}\|_{V(\Omega)} \leqslant \frac{\varepsilon}{2}.$$

Moreover, for fixed $\boldsymbol{w}_0$, there exists N_0, such that $\|\boldsymbol{w}_0 - \Pi \boldsymbol{w}_0\|_{V(\Gamma)} < \frac{\varepsilon}{2K}$ when $N > N_0$. Then $\|\boldsymbol{w}_0 - \Pi \boldsymbol{w}_0\|_D < \frac{\varepsilon}{2}$. Using Lemma 1, we obtain

$$\|\boldsymbol{u}_0 - \boldsymbol{U}_0^{(N)}\|_D = \inf_{\boldsymbol{V}_0 \in S_h} \|\boldsymbol{u}_0 - \boldsymbol{V}_0\|_D \leqslant \|\boldsymbol{u}_0 - \boldsymbol{w}_0\|_D + \|\boldsymbol{w}_0 - \Pi \boldsymbol{w}_0\|_D < \varepsilon.$$

Then $\lim_{N\to\infty} \|\boldsymbol{u}_0 - \boldsymbol{U}_0^{(N)}\|_D = 0$. The proof is complete.

Theorem 2. *If* $\boldsymbol{u}_0 \in H^{k+1}(\Gamma)^2$, $k \geqslant 1$, Π *satisfies*

$$\|\boldsymbol{v}_0 - \Pi \boldsymbol{v}_0\|_{s,\Gamma} \leqslant Ch^{k+1-s}\|\boldsymbol{v}_0\|_{k+1,\Gamma}, \quad \forall \boldsymbol{v}_0 \in H^{k+1}(\Gamma)^2,\ s=0,\ 1,$$

then

$$\|\boldsymbol{u}_0 - \boldsymbol{U}_0^{(N)}\|_D \leqslant Ch^{k+\frac{1}{2}}\|\boldsymbol{u}_0\|_{k+1,\Gamma},$$

where

$$h = \frac{2\pi}{N}.$$

Proof. From Lemma 1, Lemma 2 and the interpolation inequality[5], we have

$$\|\boldsymbol{u}_0 - \boldsymbol{U}_0^{(N)}\|_D \leqslant C\|\boldsymbol{u}_0 - \Pi\boldsymbol{u}_0\|_{\frac{1}{2},\Gamma} \leqslant C\|\boldsymbol{u}_0 - \Pi\boldsymbol{u}_0\|_{0,\Gamma}^{\frac{1}{2}}\|\boldsymbol{u}_0 - \Pi\boldsymbol{u}\|_{1,\Gamma}^{\frac{1}{2}} \leqslant Ch^{k+\frac{1}{2}}\|\boldsymbol{u}_0\|_{k+1,\Gamma},$$

where we denote every constant by C.

This numerical method is also suitable for the exterior domain to a circle. If Ω is the interior or exterior to an arbitrary smooth closed curve Γ, we can use the canonical boundary element method coupled with the finite element method[7]. For example, Ω is a bounded domain. Draw in Ω a circle Γ' with radius R, dividing Ω into Ω_1 and Ω_2, where Ω_1 is a circular domain. Then the variational problem (2) is equivalent to

$$\begin{cases} \text{Find } \boldsymbol{u} \in V(\Omega_2) \quad \text{such that} \\ D_2(\boldsymbol{u}, \boldsymbol{v}) + \bar{D}_1(\gamma'\boldsymbol{u}, \gamma'\boldsymbol{v}) = F_0(\boldsymbol{v}), \quad \forall \boldsymbol{v} \in V(\Omega_2), \end{cases} \tag{26}$$

where $V(\Omega_2) = H^1(\Omega_2)^2$, $\gamma': V(\Omega_2) \to H^{\frac{1}{2}}(\Gamma')^2$ is a trace operator,

$$D_2(\boldsymbol{u}, \boldsymbol{v}) = \iint_{\Omega_2} \sum_{i,j=1}^{2} \sigma_{ij}(\boldsymbol{u})\varepsilon_{ij}(\boldsymbol{v})\,dp,$$

$$\bar{D}_1(\gamma'\boldsymbol{u}, \gamma'\boldsymbol{v}) = \int_{\Gamma'} \gamma'\boldsymbol{v} \cdot \mathscr{K}\gamma'\boldsymbol{u}\,ds,$$

$\mathscr{K}$ is the canonical integral operator for the plane elasticity problem in Ω_1, its expression is given by (16).

Now divide circle Γ' into N_1 and take a triangulation of Ω_2 such that its nodes on Γ' coincide with the dividing points of Γ'. Let the conforming condition

$$\gamma'\Pi\boldsymbol{v} = \Pi_0\gamma'\boldsymbol{v}, \quad \forall \boldsymbol{v} \in V(\Omega_2)$$

is satisfied, where Π and Π_0 are the interpolation operators corresponding to finite elements and canonical boundary elements respectively. Take the piecewise linear basis functions $\{L_j\}_{j=1,\cdots,N}$ in Ω_2, where $N=N_1+N_2$ and the subscripts $j=1, \cdots, N_1$ correspond to the nodes on Γ'. Let

$$u_r \approx U_r = \sum_{j=1}^{N} U_j L_j, \quad u_\theta \approx U_\theta = \sum_{j=1}^{N} V_j L_j,$$

from (26) we obtain following system of linear algebraic equations

$$\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} B \\ C \end{bmatrix}, \tag{27}$$

where

$$\begin{aligned} & b_i = \int_\Gamma g_r L_i \, ds, \quad c_i = \int_\Gamma g_\theta L_i \, ds, \quad i=1, \cdots, N, \\ & Q_{lm} = [q_{ij}^{(lm)}]_{i,j=1,\cdots,N}, \quad l, m=1, 2, \\ & q_{ij}^{(11)} = D_2(L_j, 0; L_i, 0) + \bar{D}_1(\gamma' L_j, 0; \gamma' L_i, 0), \\ & q_{ij}^{(12)} = D_2(0, L_j; L_i, 0) + \bar{D}_1(0, \gamma' L_j; \gamma' L_i, 0), \\ & q_{ij}^{(21)} = D_2(L_j, 0; 0, L_i) + \bar{D}_1(\gamma' L_j, 0; 0, \gamma' L_i), \\ & q_{ij}^{(22)} = D_2(0, L_j; 0, L_i) + \bar{D}_1(0, \gamma' L_j; 0, \gamma' L_i). \end{aligned} \tag{28}$$

In all formulas of (28), the first part can be obtained by classical finite element method, and the second part is given by (23)—(24), of course we must substitute N_1 for N.

The convergence and error estimates of the coupling approximate solution can be obtained by using the same method as in [7].

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