A CLASS OF ITERATION METHODS FOR A STRONGLY MONOTONE OPERATOR EQUATION AND APPLICATION TO FINITE ELEMENT APPROXIMATE SOLUTION OF NONLINEAR ELLIPTIC BOUNDARY VALUE PROBLEM*

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1. Introduction

Let H be a real Hilbert space with inner product $(\cdot, \cdot)$ and reduced norm $\|\cdot\|$. An operator $A: H\rightarrow H$ is said to be strongly monotone with modulus $\alpha>0$ if

$$(Ax-Ay,\ x-y)\geqslant\alpha\|x-y\|^2,\qquad \forall x,\ y\in H.$$

It is said to be monotone if the inequality above is valid for $\alpha=0$. Furthermore, it is maximal monotone if it is monotone and its monotone proper extension does not exist.

Let A be a strongly monotone operator as above, and $b\in H$ a definite element. The present article is devoted to a study of the operator equation

$$Ax=b, \tag{1.1}$$

which is often deduced from practical problems in the field of differential equation, variational method and optimal control (cf. [5], [6] and [7]). The iteration schemes we will use is

$$x_{n+1}=x_n+t_n(b-Ax_n), \tag{1.2}$$

where $\{t_n\}$ is a parameter sequence of positive reals.

A special form of equation (1.1), $x+Bx=b$, where B is a monotone operator, has been discussed recently by several authors. Using the schemes (1.2), R. E. Bruck, Jr.[1] proved its local convergence on the assumption that the equation is solvable; W. G. Dotson, Jr.[2] assumed B to be nonexpansive; and You Zhao-yong[3] relaxed B into being Lipschitz continuous and then proved its global convergence. Instead of (1.2), O. Nevanlinna[4] applied the iteration

$$x_{n+1}=x_n+t_n(b-Ax_n+\theta_nx_n) \tag{1.3}$$

and, supposing that B is continuous and bounded and satisfies a linear growth condition respectively, proved the global convergence of (1.3). In his results, for a continuous B, it is required that at each iteration step the parameters t_n and θ_n be

* Received September 10, 1982.

chosen on an optimum condition which is concerned with the operator itself, and, for a bounded B, use of a so called reinitalization processes demanded.

However, in practice, the local convergence and the dependence of the choice of the parameter t_n upon the operator itself constantly cause inconvenience, and in many practical problems such as the nonlinear elliptic boundary value problem to be studied in section 3, neither the Lipschitz continuity nor the boundedness, nor even the general continuity, can be guaranteed for the operator. Therefore, we will try here to weaken the assumption of continuity, boundedness and linear growth condition on A (for equation (1.1)), and on this basis establish the global convergence of (1.2) with the choice of the parameter being independent of the operator. As an application, we will also try to provide a convenient and efficient iteration method for finite element approximate solution of the elliptic boundary value problem (1.3).

2. Global Convergence Theorems

We first introduce the following definitions:

Definition 1. *Let g: $H\to[0, \infty)$ be a functional which maps any bounded closed convex set in H into a bounded set in $[0, \infty)$. An operator A: $H\to H$ is said to be upper controlled by the functional g if*

$$\|Ax\|\leqslant g(x)$$

is valid for every $x\in H$.

Definition 2. *Let $\varphi:[0, \infty)\to[0, \infty)$ be a continuous real function with the property $\varphi(0)=0$. An operator A: $H\to H$ is said to have the continuity of the upper controlled function φ provided*

$$\|Ax-Ay\|\leqslant\varphi(\|x-y\|)$$

is valid for every x, $y\in H$.

From the definitions above, an operator having the continuity of an upper controlled function φ is also upper controlled by the functional $g(x)=\varphi(\|x-y^0\|+\|Ay^0\|)$ where y^0 is arbitrary in H. For an operator A that satisfies respectively the boundedness condition (i.e., A maps bounded sets into bounded sets), and linear growth condition $\|Ax\|\leqslant c(1+\|x\|)$ (one of the assumptions of [4]), there naturally exist functionals $g(x)=\|Ax\|$ and $g(x)=c(1+\|x\|)$ such that A is upper controlled by g. Also, an operator satisfying the Lipschitz condition $\|Ax-Ay\|\leqslant L\|x-y\|$ (the assumption in [3]) has the continuity of the upper controlled function $\varphi(t)=Lt$, and an operator which is upper controlled by a functional g may be discontinuous.

Set

$$R(x)=1/\alpha(\|b-Ax\|), \tag{2.1}$$

$$U(x, \delta)=\{y\in H \mid \|y\|\leqslant(R(x)+\delta)^{1/2}\}, \tag{2.2}$$

$$M(x)=\sup\{g(y) \mid \|y\|\leqslant(R(x)+\delta)^{1/2}+R(0)\}, \tag{2.3}$$

where $x\in H$ is arbitrary and δ is a positive real number.

Theorem 1. *Suppose that the strongly monotone operator A is upper controlled by a functional g and equation (1.1) is solvable. For any initial value $x_0\in H$, choose a sequence of positive reals $\{\bar{t}_n\}$ satisfying*

1) $0<\bar{t}_n<1,\ \bar{t}_0^2=R(x_0)(R(x_0)-1),\quad n=1, 2, \cdots,$

2) *the series* $\sum_{(n)}\bar{t}_n$ *diverges but* $\sum_{(n)}\bar{t}_n^2$ *converges.*

Set
$$\delta=\sum_{(n)}\bar{t}_n^2,$$
$$t_n=\min\{1,\ 1/2\alpha,\ \bar{t}_n/2M(x_0)\}.$$

Then the sequence x_n produced by (1.2) *with t_n chosen as above converges to the unique solution of equation* (1.1).

Proof. Let x^* be the unique solution of (1.1) (the uniqueness follows from the strong monotonicity of A). x^* satisfies the following identity
$$x^*=x^*+t_n(b-Ax^*)$$
for each integer n. We first show $x_n-x^*\in U(x_0,\delta)$ for each n. For $n=0$, it is apparent from the definition of strong monotonicity of A that
$$\alpha\|x_0-x^*\|^2\leqslant(Ax^*-Ax_0,\ x^*-x_0)\leqslant(b-Ax_0,\ x^*-x_0)=\|b-Ax\|\,\|x^*-x_0\|.$$
By the choice of $\bar{t}_0$ and (2.1) we have
$$\|x_0-x^*\|\leqslant1/\alpha\|b-Ax\|=R(x_0)\leqslant(R(x_0)+\bar{t}_0^2)^{1/2}\leqslant(R(x_0)+\delta)^{1/2}. \tag{2.4}$$
Thus $x_0-x^*\in U(x_0,\ \delta)$. If we assume that $x_n-x^*\in U(x_0,\ \delta)$ has been proved and $\|x_n-x^*\|$ satisfies
$$\|x_n-x^*\|\leqslant\Big(R(x_0)+\sum_{k=0}^{n-1}\bar{t}_k^2\Big)^{1/2}, \tag{2.5}$$
it follows from the iteration (1.2) that
$$\begin{aligned}\|x_{n+1}-x^*\|^2&\leqslant\|x_n-x^*\|^2-2t_n(Ax_n-Ax^*,\ x_n-x^*)+t_n^2\|Ax_n-Ax^*\|^2\\ &\leqslant(1-2\alpha t_n)\|x_n-x^*\|^2+t_n^2\|Ax_n-Ax^*\|^2.\end{aligned} \tag{2.6}$$
Noting that substituting 0 for x_0 in (2.4) gives
$$\|x^*\|\leqslant R(0)\leqslant R(0)+(R(x_0)+\delta)^{1/2}$$
and that, by assumption (2.5),
$$\|x_n\|\leqslant\|x_n-x^*\|+\|x^*\|\leqslant\Big(R(x_0)+\sum_{k=0}^{n-1}\bar{t}_k^2\Big)^{1/2}+R(0)\leqslant(R(x_0)+\delta)^{1/2}+R(0),$$
we see that $\|Ax^*\|$ and $\|Ax_n\|$ are both bounded above by $M(x_0)$. Therefore, together with the definition of t_n, (2.6) implies
$$\begin{aligned}\|x_{n+1}-x^*\|^2&\leqslant(1-2\alpha t_n)\|x_n-x^*\|^2+4M^2(x_0)t_n^2\\ &\leqslant\Big(R(x_0)+\sum_{k=0}^{n-1}\bar{t}_k^2\Big)+\bar{t}_n^2=R(x_0)+\sum_{k=0}^{n}\bar{t}_k^2<R(x_0)+\delta,\end{aligned} \tag{2.7}$$
that is, $\|x_{n+1}-x^*\|\leqslant\Big(R(x_0)+\sum_{k=0}^{n}\bar{t}_k^2\Big)^{1/2}<(R(x_0)+\delta)^{1/2}$ and $x_{n+1}-x^*\in U(x_0,\ \delta)$. By the principle of induction, the assertion holds for each integer n.

We now prove the convergence of x_n to x^*. Because, by recurrence on index n, (2.7) yields
$$\|x_{n+1}-x^*\|^2\leqslant\prod_{k=0}^{n}(1-2\alpha t_k)\|x_0-x^*\|^2+\Big[\sum_{j=0}^{n-1}\Big(\prod_{k=j}^{n-1}(1-2\alpha t_k)\bar{t}_j^2+\bar{t}_n^2\Big)\Big], \tag{2.8}$$
it is sufficient to justify that the right terms of (2.8) tend to zero as $n\to\infty$ respectively. By the hypothesis, the series $\sum_{(n)}\bar{t}_n$ is divergent, and so is the series $\sum_{(n)}t_n$. As a

result, $\prod_{k=0}^{n}(1-2\alpha t_k)\to 0$ $(n\to\infty)$ and, therefore, the first term of the right of (2.8) tends to zero. To show the last term also tends to zero, we let

$$a_{nj}=\prod_{k=j}^{n-1}(1-2\alpha t_k).$$

Clearly, $a_{nj}<1$ and $a_{nj}\to 0$ for any fixed j from the argument above. Given $\varepsilon>0$, we choose a positive integer m so that

$$\sum_{k=m+1}^{\infty}\bar{t}_k^2<\varepsilon/2$$

and, for such m, choose N so that $a_{nj}<\varepsilon/2$ whenever $n>N$ and $1\leqslant j\leqslant m$ (both are possible because, by hypothesis, $\sum_{(n)}\bar{t}_n^2$ is convergent and $a_{nj}\to 0$). Then, for any $n>\max\{N-1, m\}$, it follows that

$$\sum_{j=0}^{n-1}\left(\prod_{k=j}^{n-1}(1-2\alpha t_k)\bar{t}_j^2\right)+\bar{t}_n^2=\sum_{j=0}^{n-1}a_{nj}\bar{t}_j^2+\bar{t}_n^2\leqslant\delta\cdot\max\{a_{n1}, a_{n2}, \cdots, a_{nm}\}+\sum_{k=m+1}^{n}\bar{t}_k^2$$

$$<\delta\cdot\varepsilon/(2\delta)+\varepsilon/2=\varepsilon,$$

which shows that the last term of (2.8) tends to zero, too. The proof is completed.

Remark. O. Nevanlinna[4] proved the global convergence of (1.3) for the maximal monotone operator A on the assumption that it satisfies linear growth and boundedness respectively. In contrast, Theorem 1 is proved for the strongly monotone operator A; it does not require maximal monotonicity, but it uniformly relaxes the assumptions into control by a functional g. What is more, the reinitalization is dismissed from iteration (1.2).

The following corollaries contain and generalize all of the results established by Bruck[1], Dotson[2] and You[3]. Especially, they allow to weaken the Lipschitz continuity in [3] into general continuity.

Corollary 1. If H is a finite-dimensional space and A is a continuous strongly monotone operator, equation (1.1) is solvable for every $b\in H$, and, for an arbitrary initial value x_0 and for t_n chosen as in Theorem 1 for $g(x)=\|Ax\|$, the sequence x_n produced by (1.2) converges to the unique solution.

Proof. A is maximal and coercive by its continuity and strong monotonicity. Hence, the solvability of equation (1.1) follows from the standarded existence theorem (cf. [5], Th. IV, 2.11). Obviously, $g(x)=\|Ax\|$ is now such a functional as to upper control A. From Theorem 1 the conclusion follows. Q. E. D.

Corollary 2. If equation (1.1) has an inner point x^* as its unique solution, there exist a (spherical) neighborhood $N(x^*, c)$ of x^* and a positive integer M such that, for any initial value $x_0\in N(x^*,c)$ and t_n defined as in Theorem 1 with $M(x_0)=M$ and $\delta<c$, the sequence x_n produced by (1.2) converges to the unique solution x^*.

Proof. Since x^* is an inner point, the local boundedness of a monotone operator (cf. [5], Th. IV, 2.2) implies that there exist a (spherical) neighborhood $N(x^*, 2c)$ of x^* and a positive integer M such that

$$\|Ax\|\leqslant M, \quad \forall x\in N(x^*, 2c). \tag{2.9}$$

Now we have $x_n\in N(x^*, \sqrt{2}c)\subset N(x^*, 2c)$ for each integer n. In fact, this is trivial for $n=0$. If we assume $x_p\in N(x^*, \sqrt{2}c)$ to satisfy $\|x_p-x^*\|\leqslant\left(c^2+\sum_{k=0}^{p-1}\bar{t}_k^2\right)^{1/2}$, then for

$n=p+1$, by reasoning similar to (2.6)—(2.7), we obtain

$$\|x_{p+1}-x^*\|^2\leqslant\|x_p-x^*\|^2+\bar{t}_n^2\leqslant c^2+\sum_{k=0}^{p}\bar{t}_k^2<c^2+\delta^2<2c^2.$$

Then it follows at once that $x_{p+1}\in N(x^*, \sqrt{2}c)$ satisfies $\|x_{p+1}-x^*\|\leqslant\left(c^2+\sum_{k=0}^{p}\bar{t}_k^2\right)^{1/2}$, too. By induction, the claim is true. By (2.9), it shows that

$$\|Ax_n-Ax^*\|\leqslant 2M,\quad n=0, 1, 2, \cdots.$$

For such a specified M, repetition of the proof on convergence in Theorem 1 will finally lead to the corollary.

Note that Theorem 1 yields a class of iterations with the property of global convergence; the convergence rate depends on the choice of parameter $\bar{t}_n$ and initial value x_0. Also, it is easy to see that the larger $\bar{t}_n$ is and the smaller $M(x_0)$ is, the faster convergence rate the iteration (1.2) has. Therefore, in order to make (1.2) converge faster, it is necessary to choose t_n and initial value x_0 as large and near to the real solution as possible.

We now consider choosing t_n as large as possible to speed the convergence rate of (1.2).

Theorem 2. *Suppose that the strongly monotone operator A has continuity of the polynomial*

$$\varphi(\lambda)=a_1\lambda+a_2\lambda^2+\cdots+a_N\lambda^N,$$

where N is a definite integer and $a_k\in[0, \infty)$ are not all zeros ($k=1, 2, \cdots, N$). Then equation (1.1) has the unique solution x^* and, for any initial value $x_0\in H$, if we choose t so that

$$0<t<t^*,$$

where t^* is the unique positive root of the equation

$$2\alpha-\lambda\left(\sum_{k=1}^{N}a_k\alpha^{k-1}R^{k-1}(x_0)\lambda^{k-1}\right)^2=0,$$

then the sequence x_n produced by (1.2) with $t_n=t$ converges to the unique solution x^*, and the following error estimate

$$\|x_n-x^*\|\leqslant\beta^n/(1-\beta)\alpha tR(x_0) \tag{2.10}$$

is valid ($0<\beta<1$ constant).

Proof. Define function ψ: $[0, \infty)\to[0, \infty)$ as

$$\psi(\lambda)=\left[1-2\alpha t+\left(\sum_{k=1}^{N}a_k t\lambda^{k-1}\right)^2\right]^{1/2},$$

which is obviously continuous and increasing. We have from (1.2) and strong monotonicity that

$$\begin{aligned}\|x_{n+1}-x_n\|^2&=\|x_n-x_{n-1}\|^2-2t(Ax_n-Ax_{n-1}, x_n-x_{n-1})+t^2\|Ax_n-Ax_{n-1}\|^2\\&\leqslant(1-2\alpha t)\|x_n-x_{n-1}\|^2+t^2\left(\sum_{k=1}^{N}a_k\|x_n-x_{n-1}\|^k\right)^2\\&\leqslant\|x_n-x_{n-1}\|^2\psi^2(\|x_n-x_{n-1}\|)\end{aligned}$$

(also using the hypothesis about A), i.e.,

$$\|x_{n+1}-x_n\|\leqslant\|x_n-x_{n-1}\|\psi(\|x_n-x_{n-1}\|). \tag{2.11}$$

We now prove that for any integer n the following

$$\|x_{n+1}-x_n\| \leqslant \beta^n t\alpha R(x_0) \tag{2.12}$$

holds, where $\beta=\psi(\|x_1-x_0\|)$ and, from (1.2), (2.1) and the choice of t,

$$\beta^2=\psi^2(\|x_1-x_0\|)=\psi^2(\alpha t R(x_0))$$
$$=1-2\alpha t+t^2\Big(\sum_{k=1}^{N} a_k \alpha^{k-1} R^{k-1}(x_0) t^{k-1}\Big)^2<1.$$

For $n=1$, (2.12) is obvious because (1.2) and (2.11) imply the inequality:

$$\|x_2-x_1\| \leqslant \|x_1-x_0\| \psi(\|x_1-x_0\|) \leqslant t\alpha R(x_0)\beta.$$

If we assume (2.12) to be true for $n=p-1$, using the increase of ψ, we have

$$\|x_{p+1}-x_p\| \leqslant \|x_p-x_{p-1}\| \psi(\|x_p-x_{p-1}\|)$$
$$\leqslant \beta^{p-1}\alpha t R(x_0)\psi(\beta^{p-1} t\alpha R(x_0)) \leqslant \beta^p t\alpha R(x_0),$$

which indicates the trueness of (2.12) for $n=p$. By induction, (2.12) is true for any integer n.

Now, for any positive integer m, it follows from (2.12) that

$$\|x_{n+m}-x_n\| \leqslant \sum_{k=n}^{n+m-1} \|x_{k+1}-x_k\| \leqslant \sum_{k=n}^{n+m-1} \alpha t R(x_0)\beta^k < \beta^n/(1-\beta)(\alpha t R(x_0)). \tag{2.13}$$

Since $\beta<1$, $\|x_{n+m}-x_n\| \to 0$ $(n, m\to\infty)$ and hence $\{x_n\}$ is a Cauchy sequence. Let x^* be such that $x_n \to x^*(n\to\infty)$. Then (1.2) and the continuity and strong monotonicity of A together imply that x^* is the unique solution of (1.1). Finally, taking the limit as $m\to\infty$ in (2.13), the estimate (2.10) follows and hence the proof is completed.

Remark. The hypothesis of Theorem 2 is clearly sharper than that of Theorem 1, yet it is still weaker than that of [2] and [3]. Particularly, the present hypothesis guarantees that (1.2) converges at a rate of a geometric series.

3. Application

We now apply the result to finding the finite element approximate solution of a class of nonlinear elliptic boundary value problem.

Let $H_m^0(\Omega)$ be a $(2, m)$-th Sobolev space on which the inner product and norm are defined respectively by

$$(u, v)=\sum_{|\alpha|\leqslant m}\int_\Omega D^\alpha u \cdot D^\alpha v \, dx,$$
$$\|u\|_m^2=(u, u)_m,$$

where $\alpha=(\alpha_1, \alpha_2, \cdots, \alpha_n)\in Z_+^n$, Z_+^n being the Cartesian product of n spaces of nonnegative integers, $|\alpha|=\alpha_1+\alpha_2+\cdots+\alpha_n$ and

$$D^\alpha u=\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\cdots\partial x_n^{\alpha_n}}.$$

For given a bounded open subset Ω of R^n, we suppose that its boundary $\partial\Omega$ is such that the Sobolev Imbedding Theorem holds, we consider the following $2m$-th order elliptic boundary value problem:

$$\begin{cases} \sum_{|\alpha|\leqslant m}(-1)^{|\alpha|}D^\alpha\{A_\alpha(x, u, \cdots, D^m u)\}=0, & x\in\Omega, \\ D^\beta u(x)=0, & x\in\partial\Omega, \ \forall|\beta|\leqslant m-1. \end{cases} \tag{3.1}$$

For every $u, v\in H_m^0(\Omega)$, define the quasibilinear form as

$$B(u, v)=\sum_{|\alpha|\leq m}\int_\Omega A_\alpha(x, u, \cdots, D^m u)\cdot D^\alpha v\,dx. \tag{3.2}$$

Suppose that (3.1) satisfies the following conditions:

i) The function $A_\alpha(x, u, \cdots, D^m u)$ is measurable in $x\in\Omega$ and continuous in its other argument $u, \cdots, D^m u$ for almost all $x\in\Omega$, and there exists a nonnegative continuous function g on $[0, +\infty)$ such that

$$|A_\alpha(x, u, \cdots, D^m u)|\leq\{g(\sum_{|\beta|<m-\frac{n}{2}}|D^\beta u|)\}$$
$$\cdot\{1+\sum_{|\beta|=m-\frac{n}{2}}|D^\beta u|+\sum_{m-\frac{n}{2}<|\beta|\leq m}|D^\beta u|^{q_\beta/p_\alpha}\},$$

for all $|\alpha|\leq m$, almost all $x\in\Omega$, and all $D^\gamma u$, $|\gamma|\leq m$, where

$$q_\beta=2n/(n-2m+2|\beta|),$$
$$\frac{1}{p_\alpha}=\begin{cases}1, & |\alpha|<m-n/2,\\ \frac{1}{2}+(m-|\alpha|)/n, & |\alpha|\geq m-n/2.\end{cases}$$

ii) Strongly elliptic condition, i.e.,

$$B(u, u-v)-B(v, u-v)=\sum_{|\alpha|\leq m}(A_\alpha(x, D^\gamma u)-A_\alpha(x, D^\gamma v), D^\alpha(u-v))\geq\alpha\|u-v\|_m^2$$
$$\forall u, v\in H_m(\Omega), \quad u-v\in \overset{0}{H}_m. \tag{3.3}$$

Recall that $u\in \overset{0}{H}_m$ is a generalized solution of (3.1), relative to the space $\overset{0}{H}_m$, if and only if

$$B(u, v)=0, \quad \forall v\in \overset{0}{H}_m. \tag{3.4}$$

Let S_h be a finite element subspace. By the finite element method for approximating the generalized solution we mean to find $u_h\in S_h$ so that

$$B(u_h, v)=0, \qquad \forall v\in S_h. \tag{3.5}$$

The following result has been proved in [7].

Lemma 1. *$B(u, v)$ in (3.2) is a bounded linear functional of $v\in \overset{0}{H}_m$ for each $u\in \overset{0}{H}_m$, and there exists a continuous functional $h(u)$ independent of v such that*

$$|B(u, v)|\leq h(u)\|v\|_m, \quad \forall v\in \overset{0}{H}_m, \tag{3.6}$$

where

$$h(u)=\{\|g(\sum_{|\beta|<m-\frac{n}{2}}|D^\beta u|)\|_\infty\}\cdot\{(\text{meas}\,\Omega)^{1/2}+\sum_{|\beta|=m-\frac{n}{2}}\|D^\beta u\|_2+f(u)\}, \tag{3.7}$$

$$f(u)=\begin{cases}k_1\sum_{m-\frac{n}{2}<|\beta|\leq m}(\|D^\beta u\|_{q_\beta})^{q_\beta}, & |\alpha|<m-\frac{1}{2},\\ k_2\sum_{m-\frac{n}{2}<|\beta|\leq m}(\|D^\beta u\|_{q_\beta})^{q_\beta/p_\alpha}, & |\alpha|\geq m-\frac{1}{2}\end{cases}$$

and k_1, k_2 are constants.

As a consequence of Lemma 1 and Riesz's representation theorem, there exists $Tu\in \overset{0}{H}_m(\Omega)$ for each $u\in \overset{0}{H}_m$ such that

$$\begin{aligned}(Tu, v)_m&=B(u, v),\\ \|Tu\|&\leq h(u).\end{aligned} \tag{3.8}$$

It is easy to justify from hypothesis ii) that the operator $T:\overset{0}{H}_m\to \overset{0}{H}_m$ defined by (3.8) is a strongly monotone operator with modulus $\alpha>0$, i.e., it satisfies

$$(Tu-Tv,\ u-v)\geqslant\alpha\|u-v\|^2,\quad \forall u,\ v\in H_m^0. \tag{3.9}$$

Now, let ϕ_i, $1\leqslant i\leqslant n$, be a basis in S_h. Setting a general element $u\in S_h$ as $u=a_1\phi_1+a_2\phi_2+\cdots+a_n\phi_n$, we establish the homeomorphic map between S_h and R^n, denoted by G, as follows

$$u=\sum_{k=1}^{n}a_k\phi_k \underset{G^{-1}}{\overset{G}{\rightleftarrows}} a\triangleq(a_1,\ a_2,\ \cdots,\ a_n)^T,$$

$$\|u\|_m \underset{G^{-1}}{\overset{G}{\rightleftarrows}} \|a\|=\Big(\sum_{k=1}^{n}a_k^2\Big)^{1/2}$$

and assume that the equivalence relation between $\|u\|$ and $\|a\|$ is

$$c_1\|a\|\leqslant\|u\|_m\leqslant c_2\|a\|,\quad \forall u\in S_h. \tag{3.10}$$

By virtue of G, we then define a new operator $F:R^n\to R^n$ by

$$F(a)=(B(G^{-1}a,\ \phi_1),\ B(G^{-1}a,\ \phi_2),\ \cdots,\ B(G^{-1}a,\ \phi_n)). \tag{3.11}$$

With this, after denoting by $h(a)$ the functional $h(G(u))$ transformed from (3.7), we have

Lemma 2. *The operator F defined above is a strongly monotone operator with modulus $c_1^2\alpha>0$ and is upper controlled by the functional*

$$h_1(a)=\Big(\sum_{k=1}^{n}\|\phi_k\|_m^2\Big)^{1/2}\cdot h(a).$$

Proof. Let $a=(a_1,\ a_2,\ \cdots,\ a_n)^T$, $b=(b_1,\ b_2,\ \cdots,\ b_n)^T$ be arbitrary elements in R^n, and $G^{-1}(a)=\sum_{k=1}^{n}a_k\phi_k$, $G^{-1}(b)=\sum_{k=1}^{n}b_k\phi_k\in S_h$, correspondingly. By (3.8), (3.11) and Lemma 1, it is obvious that

$$\begin{aligned}(F(a)-F(b),\ a-b)&=\sum_{k=1}^{n}(B(G^{-1}a,\ \phi_k)-B(G^{-1}b,\ \phi_k))(a_k-b_k)\\&=\sum_{k=1}^{n}[B(G^{-1}a,\ (a_k-b_k)\phi_k)-B(G^{-1}b,\ (a_k-b_k)\phi_k)]\\&=B\Big(G^{-1}a,\ \sum_{k=1}^{n}(a_k-b_k)\phi_k\Big)-B\Big(G^{-1}b,\ \sum_{k=1}^{n}(a_k-b_k)\phi_k\Big)\\&=(TG^{-1}(a)-TG^{-1}(b),\ G^{-1}(a)-G^{-1}(b))\\&\geqslant\alpha\|G^{-1}a-G^{-1}b\|_m^2\geqslant c_1^2\alpha\|a-b\|^2,\end{aligned}$$

which shows the strong monotonicity of T. Again, from (3.6) and (3.11), we have

$$\|F(a)\|^2=\sum_{k=1}^{n}B^2(G^{-1}a,\ \phi_k)\leqslant\sum_{k=1}^{n}\|\phi_k\|_m^2h^2(G^{-1}a)=h_1^2(a),$$

which shows F is upper controlled by h. Since h is continuous, so is h_1. The proof therefore is completed.

Lemma 3. *If a^0 is the solution of equation $F(a)=0$, then $u_h=G^{-1}a^0$ is a finite element solution of the problem (3.5) on subspace S_h. Conversely, if u_h is a finite element solution of (3.5), Gu_h is the unique solution of equation $F(a)=0$.*

Proof. If a^0 satisfies $F(a^0)=0$, it follows from (3.11) that

$$B(G^{-1}a^0,\ \phi_k)=0,\qquad k=1,\ 2,\ \cdots,\ n.$$

Therefore, $B(G^{-1}a^0,\ v)=0$ for every $v\in S_h$ since v is a linear combination of $\phi_k(k=1,\ 2,\ \cdots,\ n)$, which indicates $G^{-1}a^0$ is the solution of (3.5). Conversely, if u_h is the solution of (3.5), it satisfies

$$B(u_h, v)=0, \quad \forall v \in S_h$$

and, in particular, $B(u_h, \phi_k)=0$ for each k, $1 \leqslant k \leqslant n$. Hence, $F(Gu_h)=0$ and Gu_h is the unique solution of equation $F(a)=0$.

By the lemmas above, we can now safely say that finding the finite element solution of problem (3.1) is essentially equivalent to constructing the solution of equation $F(a)=0$. Note that, under our assumption, there exist the generalized solution and finite element solution to the problem (3.1) (cf. [7]), and to equation $F(a)=0$ as well. Further, if the unique solution of equation $F(a)=0$ is $a^0=(a_1^0, a_2^0, \cdots, a_n^0)$, the finite element solution of the problem (3.1) on subspace S_h is then given by

$$u_h=G^{-1}a^0=\sum_{k=1}^{n} a_k^0 \phi_k.$$

To summarize, by denoting

$$R(x)=(1/c_1^2\alpha)\cdot\|Fx\|,$$
$$U(x, \delta)=\{y\in R^n \mid \|y\|\leqslant (R(x)+\delta)^{1/2}\},$$
$$M(x)=\{h_1(a) \mid \|a\|\leqslant (R(x)+\delta)^{1/2}+R(0)\}$$

and applying Theorem 1 (or Corollary 1), we obtain at last the following

Theorem 3. *For any initial value $x^{(0)}=(x_1^{(0)}, x_2^{(0)}, \cdots, x_n^{(0)})$, choose any sequence $\{\bar{t}_n\}$ of positive reals that satisfies*

1) $0<\bar{t}_n<1$, $\bar{t}_0^2\geqslant R(x_0)(R(x_0)-1)$, $n=1, 2, \cdots$;

2) *series $\sum_{(n)} \bar{t}_n$ diverges but $\sum_{(n)} \bar{t}_n^2$ converges, and set*

$$\delta=\sum_{(n)} \bar{t}_n^2,$$
$$t_n=\min\{1, 1/2\alpha c_1^2, \bar{t}_n/2M(x_0)\}.$$

Then the sequence $a^{(m)}=(a_1^{(m)}, a_2^{(m)}, \cdots, a_n^{(m)})$, and $u_m=\sum_{k=1}^{n} a_k^{(m)}\phi_k$ correspondingly, produced by

$$a^{(m+1)}=a^{(m)}-t_m F(a^{(m)}), \tag{3.12}$$

converges to the unique solution of equation $F(a)=0$ and the finite element solution of the problem (3.1) on subspace S_h respectively.

Remark. Using an iteration method to find the finite element solution of (3.1) was first suggested by Li Kai-tai and Huang Ai-xiang in [8], to avoid solving nonlinear simultaneous equations, which is often very complicated. But, in their iteration scheme, it is required to calculate the Gramian matrix of the basis $\{\phi_i\}$ and to determine the parameter t_n at each iterative step through an integral evaluation. Theorem 3 obviously overcomes all these difficulties. Therefore, the iteration (3.12) is more convenient and efficient in generating the finite element solution of the problem (3.1).

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