A FAMILY OF PARALLEL AND INTERVAL ITERATIONS FOR FINDING ALL ROOTS OF A POLYNOMIAL SIMULTANEOUSLY WITH RAPID CONVERGENCE*

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Abstract

This paper suggests a family of parallel iterations with parameter p ($p=1, 2, \cdots$) for finding all roots of a polynomial simultaneously. The convergence of the methods is of order $p+2$. The methods may also be applyed to interval iterations.

Let $x\in\mathbb{C}$,

$$\Delta_0(x)=\Delta_0[f;x]=1,$$

$$\Delta_p(x)=\Delta_p[f;x]=\begin{vmatrix}\sigma_1 & 1 & & 0\\ \sigma_2 & \sigma_1 & 1 & \\ \vdots & \ddots & \ddots & \ddots\\ \sigma_{p-1} & \cdots & \sigma_2 & \sigma_1 & 1\\ \sigma_p & \cdots & \sigma_3 & \sigma_2 & \sigma_1\end{vmatrix},$$

where

$$\sigma_\nu=\sigma_\nu(x)=\frac{f^{(\nu)}(x)}{\nu!f(x)}.$$

Expanding the determinant in the first column, we see that $\Delta_p(x)$ satisfies the following recursion relations:

$$\Delta_p(x)=\sum_{\nu=1}^{p}(-1)^{\nu+1}\sigma_\nu(x)\Delta_{p-\nu}(x),$$

which are called first recursion relations. To find a zero of equation

$$f(x)=0,$$

a family of iteration method

$$x^{(n+1)}=x^{(n)}-\frac{\Delta_{p-1}[f;x^{(n)}]}{\Delta_p[f;x^{(n)}]},\quad n\in\mathbb{N}_0$$

has been discussed in [1], where p is any positive integer. The method is of order $p+1$ if the zero is simple. The special cases of the method contain the well-known Newton iteration ($p=1$) and Halley iteration ($p=2$). From the first recursion relations we see that the method is connected with Bernoulli's method for finding a root of a polynomial. On the other hand, if we write $\sigma_\nu=\dfrac{f^{(\nu+1)}(x)}{\nu!f'(x)}$, the method becomes immediately an optimum seeking method for finding a minimizer of $f(x)$, which is the same as the method of Hua and Xia[2] in evaluation and rates of convergence. Obviously, the method may be applied to the operator in the Banach

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space provided a suitable arrangement for the computed pattern is made.

In this paper we will transform the family of the methods into another family, which can help to find simultaneously all zeros of a polynomial, while its order of convergence is raised by 1. Moreover, we show that the family is suitable especially for parallel and interval operations.

First, it is clear that the generating function of $\Delta_p(x)$ is $\frac{f(x)}{f(x-z)}$, that is,

$$\Delta_p(x)=\frac{\partial^p}{p!\partial z^p}\frac{f(x)}{f(x-z)}\bigg|_{z=0}.$$

Using Leibniz' formula for

$$\frac{\partial}{\partial z}\frac{f(x)}{f(x-z)}=\frac{f(x)}{f(x-z)}\cdot\frac{f'(x-z)}{f(x-z)},$$

we obtain

$$\frac{\partial^p}{p!\partial z^p}\frac{f(x)}{f(x-z)}$$
$$=\frac{1}{p}\sum_{\nu=1}^{p}\left(\frac{\partial^{\nu-1}}{(\nu-1)!\partial z^{\nu-1}}\frac{f'(x-z)}{f(x-z)}\right)\left(\frac{\partial^{p-\nu}}{(p-\nu)!\,\partial z^{p-\nu}}\frac{f(x)}{f(x-z)}\right).$$

Let $z=0$. We have the recursion relations

$$\Delta_p(x)=\frac{1}{p}\sum_{\nu=1}^{p}s_\nu(x)\Delta_{p-\nu}(x),$$

where $$s_\nu(x)=\frac{\partial^{\nu-1}}{(\nu-1)!\partial z^{\nu-1}}\frac{f'(x-z)}{f(x-z)}\bigg|_{z=0}=\frac{(-1)^{\nu-1}}{(\nu-1)!}\left(\frac{f'(x)}{f(x)}\right)^{(\nu-1)},$$

which are called the second recursion relations.

If $f(x)$ is a polynomial of degree N, we have

$$\frac{f'(x)}{f(x)}=\sum_{i=1}^{N}\frac{1}{x-\xi_i},$$

where $\xi_1, \cdots, \xi_N$ are all zeros of $f(x)$. From above, it is seen that $s_\nu(x)$ is the sum of the terms of power ν of $\frac{1}{x-\xi_i}$:

$$s_\nu=\sum_{i=1}^{N}\frac{1}{(x-\xi_i)^\nu}.$$

Also from $$\frac{f(x)}{f'(x-z)}=\prod_{i=1}^{N}\frac{1}{1-\frac{z}{x-\xi_i}},$$

it is seen that $\Delta_p(x)$ is the sum of all homogeneous products of degree p of $\frac{1}{x-\xi_i}$. We denote it by Bell's polynomials $Y_p(z_1, \cdots, z_p)$ in terms of $s_\nu=s_\nu(x)$ as follows (see, e. g., [3], 74—84):

$$\Delta_p(x)=\frac{1}{p!}Y_p(s_1, s_2, 2!s_3, \cdots, (p-1)!s_p)\overset{\text{df}}{=}B_p(s_1, s_2, \cdots, s_p).$$

Using the second recursion relations of $\Delta_p(x)$ or Bell's polynomials, the expressions of

$$B_p=B_p(s_1, s_2, \cdots, s_p)$$

for $p=1, 2, \cdots, 8$ are as follows:

$$B_1=s_1,$$

$$B_2=\frac{1}{2}s_2+\frac{1}{2}s_1^2,$$

$$B_3=\frac{1}{3}\,s_3+\frac{1}{2}\,s_2s_1+\frac{1}{6}\,s_1^3,$$

$$B_4=\frac{1}{4}\,s_4+\frac{1}{3}\,s_3s_1+\frac{1}{8}\,s_2^2+\frac{1}{4}s_2s_1^2+\frac{1}{24}\,s_1^4,$$

$$B_5=\frac{1}{5}\,s_5+\frac{1}{4}\,s_4s_1+\frac{1}{6}\,s_3s_2+\frac{1}{6}\,s_3s_1^2+\frac{1}{8}\,s_2^2s_1+\frac{1}{12}\,s_2s_1^3+\frac{1}{120}\,s_1^5,$$

$$B_6=\frac{1}{6}\,s_6+\frac{1}{5}\,s_5s_1+\frac{1}{8}\,s_4s_2+\frac{1}{8}\,s_4s_1^2+\frac{1}{18}\,s_3^2+\frac{1}{6}\,s_3s_2s_1+\frac{1}{18}\,s_3s_1^3$$
$$+\frac{1}{48}\,s_2^3+\frac{1}{16}\,s_2^2s_1^2+\frac{1}{48}\,s_2s_1^4+\frac{1}{720}\,s_1^6,$$

$$s_7=\frac{1}{7}\,s_7+\frac{1}{6}\,s_6s_1+\frac{1}{10}\,s_5s_2+\frac{1}{12}\,s_4s_3+\frac{1}{8}\,s_4s_2s_1+\frac{1}{40}\,s_4s_1^2$$
$$+\frac{1}{24}\,s_4s_1^3+\frac{1}{18}\,s_3^2s_1+\frac{1}{24}\,s_3s_2^2+\frac{1}{12}\,s_3s_2s_1^2+\frac{1}{72}\,s_3s_1^4+\frac{1}{48}\,s_2^3s_1$$
$$+\frac{1}{48}\,s_2^2s_1^3+\frac{1}{21}\,s_2s_1^5+\frac{1}{5040}\,s_1^7,$$

$$s_8=\frac{1}{8}\,s_8+\frac{1}{7}\,s_7s_1+\frac{1}{12}\,s_6s_2+\frac{1}{12}\,s_6s_1^2+\frac{1}{15}\,s_5s_3+\frac{1}{10}\,s_5s_2s_1$$
$$+\frac{1}{30}\,s_5s_1^3+\frac{1}{128}\,s_4^2+\frac{1}{12}\,s_4s_3s_1+\frac{1}{32}\,s_4s_2^2+\frac{1}{16}\,s_4s_2s_1^2$$
$$+\frac{1}{96}\,s_4s_1^4+\frac{1}{36}\,s_3^2s_2+\frac{1}{36}\,s_3^2s_1^2+\frac{1}{24}\,s_3s_2^2s_1+\frac{1}{36}\,s_3s_2s_1^3+\frac{1}{360}\,s_3s_1^4$$
$$+\frac{1}{384}\,s_2^4+\frac{1}{96}\,s_2^3s_1^2+\frac{1}{192}\,s_2^2s_1^4+\frac{1}{1440}\,s_2s_1^6+\frac{1}{40320}\,s_1^8.$$

Suppose now that

$$f(x)=(x-\xi_i)f_i(x),$$
$$\Delta_{p,i}(x)=\Delta_p[f_i;x],\quad i=1,\ 2,\ \cdots,\ N.$$

Then

$$\frac{f_i(x)}{f_i(x-z)}=\frac{f(x)}{f(x-z)}\Big(1-\frac{z}{x-\xi_i}\Big),$$
$$\Delta_{p,i}(x)=\frac{\partial^p}{p!\,\partial z^p}\,\frac{f_i(x)}{f_i(x-z)}\Big|_{z=0}.$$

By Leibniz' formula, we have

$$\Delta_{p,i}(x)=\Delta_p(x)-\frac{\Delta_{p-1}(x)}{x-\xi_i}.$$

Therefore,

$$\xi_i=x-\frac{\Delta_{p-1}(x)}{\Delta_p(x)-\Delta_{p,i}(x)},\quad i=1,\ 2,\ \cdots,\ N.$$

$\Delta_{p,i}(x)$ may also be computed by the first and second recursion relations and by polynomials B_p. Especially, the expressions of $\Delta_{p,i}(x)$ by polynomials B_p are

$$\Delta_{p,i}(x)=B_p(s_{1,i},\ s_{2,i},\ \cdots,\ s_{p,i}),$$

where

$$s_{\nu,i}=s_{\nu,i}(x)=\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{1}{(x-\xi_j)^\nu},$$

whence we obtain

$$\xi_i=x-\frac{\Delta_{p-1}(x)}{\Delta_p(x)-B_p\Big(\sum\limits_{\substack{j=1\\ j\neq i}}^{N}\frac{1}{x-\xi_j},\ \cdots,\ \sum\limits_{\substack{j=1\\ j\neq i}}^{N}\frac{1}{(x-\xi_j)^p}\Big)},\quad i=1,\ 2,\ \cdots,\ N.$$

Thus, we have a family of the following iteration method for finding all zeros of $f(x)$ simultaneously:

$$x_i^{(n+1)}=x_i^{(n)}-\frac{\Delta_{p-1}(x_i^{(n)})}{\Delta_p(x_i^{(n)})-B_p\left(\sum\limits_{\substack{j=1\\j\neq i}}^{N}\frac{1}{x_i^{(n)}-x_j^{(n)}},\ \cdots,\ \sum\limits_{\substack{j=1\\j\neq i}}^{N}\frac{1}{(x_i^{(n)}-x_j^{(n)})^p}\right)},$$

$$i=1,\ 2,\ \cdots,\ N;\ n\in\mathbb{N}_0.$$

In Appendix A we show that the order of convergence is $p+2$ if the zeros are simple.

Specifically, letting $p=1$, the iteration method of Ehrlich[4] is obtained:

$$x_i^{(n+1)}=x_i^{(n)}-\frac{1}{\dfrac{f'(x_i^{(n)})}{f(x_i^{(n)})}-\sum\limits_{\substack{j=1\\j\neq i}}^{N}\dfrac{1}{x_i^{(n)}-x_j^{(n)}}},\quad i=1,\ 2,\ \cdots,\ N;\ n\in\mathbb{N}_0,$$

which is of order 3 if the zeros are simple. Letting $p=2$, we have a new method

$$x_i^{(n+1)}=x_i^{(n)}-\frac{2\dfrac{f'(x_i^{(n)})}{f(x_i^{(n)})}}{\dfrac{2f'(x_i^{(n)})^2-f(x_i^{(n)})f''(x_i^{(n)})}{f(x_i^{(n)})^2}-\left(\sum\limits_{\substack{j=1\\j\neq i}}^{N}\dfrac{1}{x_i^{(n)}-x_j^{(n)}}\right)^2-\sum\limits_{\substack{j=1\\j\neq i}}^{N}\dfrac{1}{(x_i^{(n)}-x_j^{(n)})^2}},$$

$$i=1,\ 2,\ \cdots,\ N;\quad n\in\mathbb{N}_0,$$

which is of order 4 if the zeros are simple.

The family may be applied to parallel operation, and is suitable specially for a vector computer. We suggest that the computation for $\Delta_p(x_i)$ be progressed by the first recursion relations; but for

$$\Delta_p[\tilde{f}_i;\ x_i]=B_p\left(\sum_{\substack{j=1\\j\neq i}}^{N}\frac{1}{x_i-x_j},\ \cdots,\ \sum_{\substack{j=1\\j\neq i}}^{N}\frac{1}{(x_i-x_j)^p}\right),$$

where $$\tilde{f}_i(x)=\tilde{f}_i(x;\ \boldsymbol{x})=\prod_{\substack{j=1\\j\neq i}}^{N}(x-x_j),\quad \boldsymbol{x}=(x_1,\ \cdots,\ x_N)^T,$$

by the second recursion relations if p is large.

The family of the method may be easily transformed into the interval iteration:

$$W_i^{(n+1)}=x_i^{(n)}-\frac{\Delta_{p-1}(x_i^{(n)})}{\Delta_p(x_i^{(n)})-B_p\left(\sum\limits_{\substack{j=1\\j\neq i}}^{N}\frac{1}{x_i^{(n)}-W_j^{(n)}},\ \cdots,\ \sum\limits_{\substack{j=1\\j\neq i}}^{N}\frac{1}{(x_i^{(n)}-W_j^{(n)})^p}\right)},$$

$$i=1,\ 2,\ \cdots,\ N;\quad n\in\mathbb{N}_0,$$

where $W_1^{(0)},\ \cdots,\ W_N^{(0)}$ are isolated discs with centers $x_1,\ \cdots,\ x_N$, and contain $\xi_1,\ \cdots,\ \xi_N$, respectively. If $p=1$, this is the method of Gargantini and Henrici[5]. If $p=2$, we have

$$W_i^{(n+1)}=x_i^{(n)}-\frac{2\dfrac{f'(x_i^{(n)})}{f(x_i^{(n)})}}{\dfrac{2f'(x_i^{(n)})^2-f(x_i^{(n)})f''(x_i^{(n)})}{f(x_i^{(n)})^2}-\left(\sum\limits_{\substack{j=1\\j\neq i}}^{N}\dfrac{1}{x_i^{(n)}-W_j^{(n)}}\right)^2-\sum\limits_{\substack{j=1\\j\neq i}}^{N}\dfrac{1}{(x_i^{(n)}-W_j^{(n)})^2}},$$

$$i=1,\ 2,\ \cdots,\ N;\quad n\in\mathbb{N}_0.$$

This method, with the same order of convergence, is simpler than the method of Gargantini[6]. We will discuss the property of the convergence for this method in another paper.

Appendix A. On the Orders of Iteration Functions

The iteration functions, proposed above, for finding all zeros $\xi_1, \cdots, \xi_N$ of polynomial $f(x)$ of degree N are actually N-dimensional vector functions of a vector in N dimensions. For $\boldsymbol{x}=(x_1, \cdots, x_N)^T$, the components of the vector functions $\boldsymbol{\varphi}(\boldsymbol{x})$ are

$$\varphi_i(\boldsymbol{x})=x_i-\frac{\Delta_{p-1}(x_i)}{\Delta_p(x_i)-B_p\left(\sum_{\substack{j=1\\j\neq i}}^{N}\frac{1}{x_i-x_j}, \cdots, \sum_{\substack{j=1\\j\neq i}}^{N}\frac{1}{(x_i-x_j)^p}\right)}, \quad i=1, 2, \cdots, N.$$

For the asymptotic property of $\boldsymbol{\varphi}(\boldsymbol{x})$ at $\boldsymbol{\xi}=(\xi_1, \cdots, \xi_N)^T$, we have the following theorem. According to the definition of the order given by Traub[7], the theorem implies that the order of $\boldsymbol{\varphi}(\boldsymbol{x})$ is $p+2$ if the zeros of $f(x)=0$ are simple.

Theorem. *Suppose that* $\xi_1, \cdots, \xi_N$ *are distinct. Let* $e_i=x_i-\xi_i$ $(i=1, \cdots, N)$. *Then hold the asymptotic formulas*

$$\varphi_i(\boldsymbol{x})-\xi_i\sim e_i^{p+1}\sum_{\substack{j=1\\j\neq i}}^{N}\sum_{\nu=1}^{p}\frac{\Delta_{p-\nu,i}(\xi_i)}{(\xi_i-\xi_j)^{\nu+1}}e_j,$$

for $i=1, 2, \cdots, N$ *when* $\boldsymbol{x}\to\boldsymbol{\xi}$.

Proof. Let

$$F_{p,i}(x, \boldsymbol{y})=B_p\left(\sum_{\substack{j=1\\j\neq i}}^{N}\frac{1}{x-y_j}, \cdots, \sum_{\substack{j=1\\j\neq i}}^{N}\frac{1}{(x-y_j)^p}\right)$$

for $x\in\mathbb{C}$, $\boldsymbol{y}\in\mathbb{C}^N$. We have

$$\varphi_i(\boldsymbol{x})=x_i-\frac{\Delta_{p-1}(x_i)}{\Delta_p(x_i)-F_{p,i}(x_i, \boldsymbol{x})}$$

and

$$\xi_i=x_i-\frac{\Delta_{p-1}(x_i)}{\Delta_p(x_i)-F_{p,i}(x_i, \boldsymbol{\xi})}.$$

Thus

$$\varphi_i(\boldsymbol{x})-\xi_i=\frac{\Delta_{p-1}(x_i)}{\Delta_p(x_i)-F_{p,i}(x_i, \boldsymbol{\xi})}\cdot\frac{F_{p,i}(x_i, \boldsymbol{x})-F_{p,i}(x_i, \boldsymbol{\xi})}{\Delta_p(x_i)-F_{p,i}(x_i, \boldsymbol{x})}.$$

The first factor of the right member above is clearly e_i. Now we consider the numerator and denominator of the second factor successively.

Firstly, we consider the numerator. It is known that Bell's polynomials $Y_p(z_1, \cdots, z_p)$ have the differential property:

$$\frac{\partial Y_p(z_1, \cdots, z_p)}{\partial z_\nu}=\binom{p}{\nu}Y_{p-\nu}(z_1, \cdots, z_{p-\nu}), \quad 1\leqslant\nu\leqslant p,$$

which becomes

$$B_{p,\nu}(s_1, \cdots, s_p)\overset{\text{df}}{=}\frac{\partial B_p(s_1, \cdots, s_p)}{\partial s_\nu}=\frac{1}{\nu}B_{p-\nu}(s_1, \cdots, s_{p-\nu}), \quad 1\leqslant\nu\leqslant p$$

for the polynomials

$$B_p(s_1, \cdots, s_p)=\frac{1}{p!}Y_p(s_1, s_2, 2!s_3, \cdots, (p-1)!s_p).$$

Therefore

$$\frac{\partial F_{p,i}(x, \boldsymbol{y})}{\partial y_j}=\sum_{\nu=1}^{p}B_{p,\nu}\left(\sum_{\substack{j=1\\j\neq i}}^{N}\frac{1}{x-y_j}, \cdots, \sum_{\substack{j=1\\j\neq i}}^{N}\frac{1}{(x-y_j)^p}\right)\cdot\frac{\nu}{(x-y_j)^{\nu+1}}$$
$$=\sum_{\nu=1}^{p}F_{p-\nu,i}(x, \boldsymbol{y})\frac{1}{(x-y_j)^{\nu+1}}, \quad j\neq i.$$

Expanding the functions $F_{p,i}(x_i, \boldsymbol{x})$ of several variables in Taylor series, we have

$$F_{p,i}(x_i, \boldsymbol{x})-F_{p,i}(x_i, \boldsymbol{\xi})\sim\sum_{\substack{j=1\\j\neq i}}^{N}\sum_{\nu=1}^{p}F_{p-\nu,i}(x_i, \boldsymbol{\xi})\frac{e_j}{(x_i-\xi_j)^{\nu+1}}$$

if $\boldsymbol{x}\to\boldsymbol{\xi}$. Clearly,

$$F_{p-\nu,i}(x_i, \boldsymbol{\xi})\to F_{p-\nu,i}(\xi_i, \boldsymbol{\xi})=\Delta_{p-\nu,i}(\xi_i).$$

Hence

$$F_{p,i}(x_i, \boldsymbol{x})-F_{p,i}(x_i, \boldsymbol{\xi})\sim\sum_{\substack{j=1\\j\neq i}}^{N}\sum_{\nu=1}^{p}\frac{\Delta_{p-\nu,i}(\xi_i)}{(\xi_i-\xi_j)^{\nu+1}}e_j.$$

Next, we consider the denominator. We have

$$\frac{f'(x_i)}{f(x_i)}\sim e_i^{-1},$$

$$\Delta_p(x_i)\sim e_i^{-p},$$

$$F_{p,i}(x_i, \boldsymbol{x})=O(1),$$

if $\boldsymbol{x}\to\boldsymbol{\xi}$. Therefore

$$\Delta_p(x_i)-F_{p,i}(x_i, \boldsymbol{x})\sim e_i^{-p}.$$

Thus, we obtain

$$\varphi_i(\boldsymbol{x})-\xi_i=e_i\frac{\left(\sum_{\substack{j=1\\j\neq i}}^{N}\sum_{\nu=1}^{p}\frac{\Delta_{p-\nu,i}(\xi_i)}{(\xi_i-\xi_j)^{\nu+1}}e_j\right)(1+o(1))}{e_i^{-p}(1+o(1))}$$

$$=e_i^{p+1}\sum_{\substack{j=1\\j\neq i}}^{N}\sum_{\nu=1}^{p}\frac{\Delta_{p-\nu,i}(\xi_i)}{(\xi_i-\xi_j)^{\nu+1}}e_j(1+o(1)),\quad \boldsymbol{x}\to\boldsymbol{\xi}.$$

Appendix B. On the Algorithm of Iteration Functions

In Appendix A we see that $\Delta_p(x_i)\sim(x_i-\xi_i)^{-p}$, where ξ_i is a zero of polynomial $f(x)$. Hence overflow happens frequently. Therefore, it is advantageous to introduce a factor $u(x_i)^{-1}$, which is the same as $f(x_i)^{-1}$ in magnitude. For example, let

$$u(x_i)=\frac{f(x_i)}{f'(x_i)},$$

$$\sigma_\nu^*(x_i)=u(x_i)\sigma_\nu(x_i)=\frac{f^{(\nu)}(x_i)}{\nu!f'(x_i)},$$

$$\Delta_\nu^*(x_i)=\begin{vmatrix}1 & u(x_i) & & 0\\ \sigma_2^*(x_i) & 1 & u(x_i) & \\ \vdots & \ddots & \ddots & \ddots\\ \sigma_{\nu-1}^*(x_i)\cdots\sigma_2^*(x_i) & & 1 & u(x_i)\\ \sigma_\nu^*(x_i)\cdots\sigma_3^*(x_i) & & \sigma_2^*(x_i) & 1\end{vmatrix}$$

and let

$$s_{\nu,i}(\boldsymbol{x})=\sum_{\substack{j=1\\j\neq i}}^{N}\frac{1}{(x_i-x_j)^\nu},$$

$$B_{\nu,i}(\boldsymbol{x})=B_\nu(s_{1,i}(\boldsymbol{x}), \cdots, s_{\nu,i}(\boldsymbol{x})).$$

Then the computation of the iteration functions

$$\varphi_i(\boldsymbol{x})=x_i-u(x_i)\cdot\frac{\Delta_{p-1}^*(x_i)}{\Delta_p^*(x_i)-u(x_i)^p\cdot B_{p,i}(\boldsymbol{x})}$$

is stable. For the concrete algorithm, we suggest the following:

1) $\frac{1}{\nu!}f^{(\nu)}(x_i)$ in $u(x_i)$ and $\sigma_\nu^*(x_i)(\nu=2, \cdots, p)$ be computed by the fast

algorithm of Shaw and Traub[8].

2) $\Delta^*_{p-1}(x_i)$ and $\Delta^*_p(x_i)$ be computed by recursion relations

$$\Delta^*_\nu(x_i) = \sum_{\mu=1}^{\nu} (-u(x_i))^{\mu-1}\sigma^*_\mu(x_i)\Delta^*_{\nu-\mu}(x_i), \quad \nu=2, 3, \cdots, p,$$
$$\Delta^*_0(x_i) = \Delta^*_1(x_i) = \sigma^*_1(x_i) = 1.$$

3) The triangular matrix $\left(\frac{1}{x_i - x_j}\right)_{1<i,j<N}$ be computed and stored for further computation of $s_{\nu,i}(x)$ $(\nu=1, 2, \cdots, p)$.

4) $B_{p,i}(x)$ be computed by recursion relations

$$B_{\nu,i}(x) = \frac{1}{\nu}\sum_{\mu=1}^{\nu} s_{\mu,i}(x)B_{\nu-\mu,i}(x), \quad \nu=1, 2, \cdots, p, \quad B_{0,i}(x)=1.$$

5) Because of the above recursion relations, it is not necessary to fix p in every step of iteration. Furthermore, if the initial approximation is chosen satisfactorily, that is, the inequality

$$|x_i^{(0)} - \xi_i| < \min_{j\neq i} |x_i^{(0)} - \xi_j|, \quad i=1, 2, \cdots, N$$

is satisfied, then the computation may be completed perhaps without further iteration provided p is extended sufficiently. The algorithm without further iteration may be regarded as a modification of Bernoulli's method.

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