CORRECTION PROCEDURE FOR SOLVING PARTIAL DIFFERENTIAL EQUATIONS*

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The correction procedure has been discussed by L. Fox[6] and V. Pereyra[9] for accelerating the convergence of a certain approximate solution. Its theoretical basis is the existence of an asymptotic expansion for the error of discretization proved by Filippov and Rybinskii[10] and Stetter[11] (and Bohmer[2] for the general regions):

$$u-u_h=h^2v+O(h^4),$$

where u is the solution of the original differential equation, u_h the solution of the approximate finite difference equation with parameter h and v the solution of a correction differential equation independent of h. Stetter et al. used the extrapolation procedure to eliminate the auxiliary function v while Pereyra et al. used some special procedure to solve v approximately.

In the present paper we will present a difference procedure for solving v easily.

1. Difference Operator and Truncation Error

Let Δ_h be the 5-point approximation of the Laplace operator Δ in the 2-dimensional case and the 7-point approximation in the 3-dimensional case as usual[5].

Let Δ_h^* be the 5-point approximation defined by

$$\Delta_h^*u(x_1, x_2)=(\Sigma u(x_1\pm h, x_2\pm h)-4u(x_1, x_2))/2h^2,$$

$$\Sigma u(x_1\pm h, x_2\pm h)=u(x_1+h, x_2+h)+u(x_1-h, x_2-h)$$
$$+u(x_1-h, x_2+h)+u(x_1+h, x_2-h)$$

in the 2-dimensional case and the 9-point approximation defined by

$$\Delta_h^*u(x_1, x_2, x_3)=(\Sigma u(x_1\pm h, x_2\pm h, x_3\pm h)-8u(x_1, x_2, x_3))/4h^2$$

in the 3-dimensional case.

Let δ_x, δ_y and δ_x^*, δ_y^* be the 2 and 4-point approximation defined respectively by

$$\delta_x u=(u(x+h, y)-u(x-h, y))/2h,$$
$$\delta_y u=(u(x, y+h)-u(x, y-h))/2h,$$
$$\delta_x^* u=(u(x+h, y-h)-u(x-h, y-h)+u(x+h, y+h)-u(x-h, y+h))/4h,$$
$$\delta_y^* u=(u(x-h, y-h)-u(x-h, y+h)+u(x+h, y-h)-u(x+h, y+h))/4h.$$

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The truncation error will be

$$(\Delta_h-\Delta)u=\frac{h^2}{12}E_1(u)+O(h^4), \tag{1}$$

$$(\Delta_h^*-\Delta)u=\frac{h^2}{12}E_2(u)+O(h^4), \tag{2}$$

$$\left(\delta_x-\frac{\partial}{\partial x}\right)u=\frac{h^2}{6}E_3(u)+O(h^4), \tag{3}$$

$$\left(\delta_y-\frac{\partial}{\partial y}\right)u=\frac{h^2}{6}E_4(u)+O(h^4), \tag{4}$$

$$\left(\delta_x^*-\frac{\partial}{\partial x}\right)u=\frac{h^2}{6}E_5(u)+O(h^4), \tag{5}$$

$$\left(\delta_y^*-\frac{\partial}{\partial y}\right)u=\frac{h^2}{6}E_6(u)+O(h^4), \tag{6}$$

with

$$E_1(u)=\sum\frac{\partial^4}{\partial x_i^4}u,\qquad E_2(u)=\sum\frac{\partial^4}{\partial x_i^4}u+6\sum_{i<j}\frac{\partial^2}{\partial x_i^2}\frac{\partial^2}{\partial x_j^2}u,$$

$$E_3(u)=\frac{\partial}{\partial x}\Delta u-\frac{\partial}{\partial x}\frac{\partial^2}{\partial y^2}u,\qquad E_4(u)=\frac{\partial}{\partial y}\Delta u-\frac{\partial}{\partial y}\frac{\partial^2}{\partial x^2}u$$

$$E_5(u)=\frac{\partial}{\partial x}\Delta u+2\frac{\partial}{\partial x}\frac{\partial^2}{\partial y^2}u,\qquad E_6(u)=\frac{\partial}{\partial y}\Delta u+2\frac{\partial}{\partial y}\frac{\partial^2}{\partial x^2}u.$$

we have

$$\frac{2}{3}E_1(u)+\frac{1}{3}E_2(u)=\Delta^2u, \tag{7}$$

$$\frac{2}{3}E_3(u)+\frac{1}{3}E_5(u)=\frac{\partial}{\partial x}\Delta u, \tag{8}$$

$$\frac{2}{3}E_4(u)+\frac{1}{3}E_6(u)=\frac{\partial}{\partial y}\Delta u. \tag{9}$$

2. Second Order Elliptic Problem

Consider

$$\Delta u=f(x,u)\ \text{in}\ \Omega,\qquad u=g\quad \text{on}\ \partial\Omega \tag{10}$$

in the 1, 2 or 3-dimensional domain Ω with boundary $\partial\Omega$. Suppose that Ω consists of some squares in the 2-dimensional case and of some cubes in the 3-dimensional case, and that the solution u is smooth enough and

$$f_1(u)=f_u'(x,u)\geqslant 0.$$

Consider the finite difference solution u_h defined by

$$\Delta_h u_h=f(x,u_h)\quad \text{in}\ \Omega_h,\qquad u_h=g-\frac{h^2}{12}f(x,g)\quad \text{on}\ \partial\Omega_h \tag{11}$$

and a correction solution φ_h defined by the linearized finite difference equation

$$(\Delta_h^*-f_1(u_h))\varphi_h=f(x,u_h)-f_1(u_h)u_h+\frac{h^2}{4}f_1(u_h)f(x,u_h)\quad \text{in}\ \Omega_h,\qquad \varphi_h=g-\frac{h^2}{12}f(x,g)\quad \text{on}\ \partial\Omega_h \tag{12}$$

with lattice domain Ω_h and the boundary $\partial\Omega_h$ of Ω_h defined as usual[10].

Proposition 1.

$$\frac{2}{3}u_h+\frac{1}{3}\varphi_h+\frac{h^2}{12}f(x,\ u_h)=u+O(h^4)\quad \text{in } \Omega_h. \tag{13}$$

Proof. Since

$$u_h-u=O(h^2)\quad \text{in } \Omega_h,$$

by (11), (10), (1),

$$\begin{aligned}(\Delta_h-f_1(u))(u_h-u)&=f(x,\ u_h)-f_1(u)u_h-\Delta_h u+f_1(u)u\\&=\Delta u-\Delta_h u+f(x,\ u_h)-f(x,\ u)-f_1(u)(u_h-u)\\&=-\frac{h^2}{12}E_1(u)+O(h^4)\quad \text{in } \Omega_h.\end{aligned}$$

Let v_i be the solution of the correction differential equation

$$\begin{aligned}(\Delta-f_1(u))v_i&=E_i(u)\quad \text{in } \Omega,\\ v_i&=f(x,\ g)\quad \text{on } \partial\Omega\end{aligned} \tag{14}$$

for $i=1$ or 2. Assume that v_i is smooth (see [13]). Then

$$\begin{aligned}(\Delta_h-f_1(u))\left(u_h-u+\frac{h^2}{12}v_1\right)&=-\frac{h^2}{12}E_1(u)+\frac{h^2}{12}(\Delta_h-f_1(u))v_1+O(h^4)\\&=-\frac{h^2}{12}E_1(u)+\frac{h^2}{12}(\Delta-f_1(u))v_1+O(h^4)\\&=O(h^4)\quad \text{in } \Omega_h,\end{aligned}$$

$$u_h-u+\frac{h^2}{12}v_1=0\quad \text{on } \partial\Omega_h$$

and, by the maximum principle,

$$u_h-u+\frac{h^2}{12}v_1=O(h^4)\quad \text{in } \Omega_h. \tag{15}$$

Let u_h^* be the finite difference solution defined by

$$\begin{aligned}(\Delta_h^*-f_1(u_h))u_h^*&=f(x,\ u_h)-f_1(u_h)u_h\quad \text{in } \Omega_h,\\ u_h^*&=g-\frac{h^2}{12}f(x,\ g)\quad \text{on } \partial\Omega_h.\end{aligned} \tag{16}$$

Then, by (2), (14),

$$\begin{aligned}(\Delta_h^*-f_1(u_h))(u_h^*-u)&=f(x,\ u_h)-f_1(u_h)u_h-\Delta_h^*u+f_1(u_h)u\\&=\Delta u-\Delta_h^*u+f(x,\ u_h)-f(x,\ u)-f_1(u_h)(u_h-u)\\&=-\frac{h^2}{12}E_2(u)+O(h^4)\quad \text{in } \Omega_h,\end{aligned}$$

$$(\Delta_h^*-f_1(u_h))\left(u_h^*-u+\frac{h^2}{12}v_2\right)=O(h^4)\quad \text{in } \Omega_h,$$

$$u_h^*-u+\frac{h^2}{12}v_2=0\quad \text{on } \partial\Omega_h$$

and, by the maximum principle,

$$u_h^*-u+\frac{h^2}{12}v_2=O(h^4)\quad \text{in } \Omega_h. \tag{17}$$

Combining (17) with (15) we obtain

$$\frac{2}{3}u_h+\frac{1}{3}u_h^*-u+\frac{h^2}{12}w=O(h^4) \quad \text{in } \Omega_h$$

with

$$w=\frac{2}{3}v_1+\frac{1}{3}v_2$$

satisfying, by (7),

$$(\Delta-f_1(u))w=\frac{2}{3}E_1(u)+\frac{1}{3}E_2(u)=\Delta^2 u \quad \text{in } \Omega,$$
$$w=f(x, g) \quad \text{on } \partial\Omega.$$

Set

$$w=\Delta u+z=f(x, u)+z.$$

Then

$$(\Delta-f_1(u))w=(\Delta-f_1(u))\Delta u+(\Delta-f_1(u))z$$
$$=\Delta^2 u-f_1(u)f(x, u)+(\Delta-f_1(u))z=\Delta^2 u \quad \text{in } \Omega,$$
$$w=f(x, g)+z=f(x, g) \quad \text{on } \partial\Omega$$

if z satisfies

$$(\Delta-f_1(u))z=f_1(u)f(x, u) \quad \text{in } \Omega,$$
$$z=0 \quad \text{on } \partial\Omega.$$

We now consider the finite difference solution z_h defined by

$$(\Delta_h^*-f_1(u_h))z_h=f_1(u_h)f(x, u_h) \quad \text{in } \Omega_h, \qquad (18)$$
$$z_h=0 \quad \text{on } \partial\Omega_h.$$

Since

$$(\Delta_h^*-f_1(u_h))z=(\Delta_h^*-\Delta)z+(f_1(u)-f_1(u_h))z+f_1(u)f(x, u)$$
$$=f_1(u)f(x, u)+O(h^2),$$

it is easy to see that

$$z_h-z=O(h^2) \quad \text{in } \Omega_h,$$
$$w-f(x, u_h)-z_h=O(h^2) \quad \text{in } \Omega_h.$$

Hence

$$\frac{2}{3}u_h+\frac{1}{3}u_h^*-u+\frac{h^2}{12}(f(x, u_h)+z_h)=O(h^4) \quad \text{in } \Omega_h,$$

and then (13) results from (16), (18), (12) and

$$\frac{1}{3}\varphi_h=\frac{1}{3}u_h^*+\frac{h^2}{12}z_h \quad \text{in } \Omega_h.$$

Remark 1. For the Laplace equation

$$\Delta u=f \quad \text{in } \Omega,$$
$$u=g \quad \text{on } \partial\Omega,$$

i. e. $f(x, u)=f$ in (10), then (11), (12) become

$$\Delta_h u_h=f \quad \text{in } \Omega_h,$$
$$u_h=g-\frac{h^2}{12}f \quad \text{on } \partial\Omega;$$
$$\Delta_h^*\varphi_h=f \quad \text{in } \Omega_h,$$
$$\varphi_h=g-\frac{h^2}{12}f \quad \text{on } \partial\Omega_h$$

respectively, and (13) becomes[1)]

$$\frac{2}{3}u_h+\frac{1}{3}\varphi_h+\frac{h^2}{12}f=u+O(h^4) \quad \text{in } \Omega_h.$$

Note that Δ_h, Δ_h^* are a 7-point and a 9-point approximation respectively in the 3-dimensional case. We should mention that Bramble[8] has proposed a 19-point approximation $\square_h'$ in the 3-dimensional case and a difference solution U_h' defined by

$$\square_h' U_h'=f+\frac{h^2}{12}\Delta f \quad \text{in } \Omega_h,$$

$$U_h'=g \quad \text{on } \partial\Omega_h$$

and proved that

$$U_h'=u+O(h^4) \quad \text{in } \Omega_h.$$

Remark 2. Let

$$\square_h=\frac{2}{3}\Delta_h+\frac{1}{3}\Delta_h^* \tag{19}$$

be the 15-point approximation in the 3-dimensional case and U_h the difference solution defined by

$$\square_h U_h=f \quad \text{in } \Omega_h,$$

$$U_h=g-\frac{h^2}{12}f \quad \text{on } \partial\Omega_h.$$

It is easy to see that

$$U_h+\frac{h^2}{12}f=u+O(h^4) \quad \text{in } \Omega_h.$$

Remark 3. In the 1-dimensional case we have

$$\Delta u=\frac{d^2}{dx^2}u,$$

$$\Delta_h u=\Delta_h^* u=\delta_{xx}u=(u(x-h)-2u(x)+u(x+h))/h^2,$$

and then the correction solution φ_h in (12) splits into

$$\varphi_h=u_h+\frac{h^2}{4}z_h,$$

where z_h is the difference solution of (18):

$$(\delta_{xx}-f_1(u_h))z_h=f_1(u_h)f(x,u_h) \quad \text{in } \Omega_h,$$

$$z_h=0 \quad \text{on } \partial\Omega_h$$

and Proposition 1 becomes

$$u_h+\frac{h^2}{12}(f(x,u_h)+z_h)=u+O(h^4) \quad \text{in } \Omega_h.$$

3. More General Elliptic Problem

For simplicity we consider only the 2-dimensional linear equation

$$Lu=\Delta u-bu_x-cu_y-du=f \quad \text{in } \Omega, \qquad u_h=g \quad \text{on } \partial\Omega_h \tag{20}$$

with $d\geqslant 0$, and the corresponding difference equation

1) See Lin Qun and Lü Tao, The combination of approximate solutions for aclerating the convergence, submitted to *RAIRO Numer. Anal.*

$$\begin{aligned} L_h u_h = \Delta_h u_h - b\delta_x u_h - c\delta_y u_h - d u_h = f \quad \text{in } \Omega_h, \\ u_h = g \quad \text{on } \partial\Omega_h \end{aligned} \tag{21}$$

and a correction difference equation defined by

$$\begin{aligned} L_h^* \varphi_h = \Delta_h^* \varphi_h - b\delta_x^* \varphi_h - c\delta_y^* \varphi_h - d\varphi_h \\ = f + \frac{h^2}{4} F(u_h, \delta_x u_h, \delta_y u_h, \delta_{xx} u_h, \delta_{xy} u_h, \delta_{yy} u_h) \quad \text{in } \Omega_h \\ \varphi_h = g \quad \text{on } \partial\Omega_h \end{aligned} \tag{22}$$

with the right hand side

$$F(u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = \left(\Delta - 2b\frac{\partial}{\partial x} - 2c\frac{\partial}{\partial y}\right)(bu_x + cu_y + du + f), \tag{23}$$

in which there appears no third derivative. For instance, if b, c, d are constants,

$$F = -(b^2 u_{xx} + 2bc u_{xy} + c^2 u_{yy}) + d^2 u + df + \Delta f.$$

Proposition 2.

$$\frac{2}{3} u_h + \frac{1}{3} \varphi_h = u + O(h^4) \quad \text{in } \Omega_h. \tag{24}$$

Proof. Since, by (21), (20), (1), (3), (4),

$$\begin{aligned} L_h(u_h - u) = \Delta u - bu_x - cu_y - \Delta_h u + b\delta_x u + c\delta_y u \\ = -\frac{h^2}{12} E_1(u) + \frac{h^2}{6} bE_3(u) + \frac{h^2}{6} cE_4(u) + O(h^4) \quad \text{in } \Omega_h. \end{aligned}$$

Let v_1 be the auxiliary function defined by

$$\begin{aligned} Lv_1 = E_1(u) - 2bE_3(u) - 2cE_4(u) \quad \text{in } \Omega, \\ v_1 = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{25}$$

Then,

$$L_h\left(u_h - u + \frac{h^2}{12} v_1\right) = -\frac{h^2}{12}(E_1(u) - 2bE_3(u) - 2cE_4(u)) + \frac{h^2}{12} L_h v_1 = O(h^4) \quad \text{in } \Omega_h,$$

$$u_h - u + \frac{h^2}{12} v_1 = 0 \quad \text{on } \partial\Omega_h$$

and, by the maximum principle,

$$u_h - u + \frac{h^2}{12} v_1 = O(h^4) \quad \text{in } \Omega_h. \tag{26}$$

Consider the difference equation defined by

$$\begin{aligned} L_h^* u_h^* = f \quad \text{in } \Omega_h, \\ u_h^* = g \quad \text{on } \partial\Omega_h \end{aligned} \tag{27}$$

and the auxiliary differential equation

$$\begin{aligned} Lv_2 = E_2(u) - 2bE_5(u) - 2cE_6(u) \quad \text{in } \Omega, \\ v_2 = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{28}$$

Then, by the same reason,

$$u_h^* - u + \frac{h^2}{12} v_2 = O(h^4) \quad \text{in } \Omega_h. \tag{29}$$

Combining (29) with (26) we obtain

$$\frac{2}{3}u_h+\frac{1}{3}u_h^*-u+\frac{h^2}{12}w=O(h^4)\quad \text{in } \Omega_h,$$

with

$$w=\frac{2}{3}v_1+\frac{1}{3}v_2$$

satisfying, by (25), (28), (7), (8), (9), (23),

$$Lw=\Delta^2u-2b\Delta u_x-2c\Delta u_y=F(u,\ u_x,\ u_y,\ u_{xx},\ u_{xy},\ u_{yy})\quad \text{in } \Omega,$$

$$w=0\quad \text{on } \partial\Omega.$$

We now consider the difference solution w_h defined by

$$L_h^*w_h=F(u_h,\ \delta_xu_h,\ \delta_yu_h,\ \delta_{xx}u_h,\ \delta_{xy}u_h,\ \delta_{yy}u_h)\quad \text{in } \Omega_h,$$
$$w_h=0\quad \text{on } \partial\Omega_h. \tag{30}$$

Note that from (26) we have

$$\delta_x(u_h-u)=-\frac{h^2}{12}\delta_xv_1+O(h^3)=O(h^2),$$

$$\delta_{xy}(u_h-u)=-\frac{h^2}{12}\delta_{xy}v_1+O(h^2)=O(h^2)$$

and

$$\delta_y(u_h-u)=O(h^2),$$
$$\delta_{xx}(u_h-u)=O(h^2),$$
$$\delta_{yy}(u_h-u)=O(h^2).$$

Therefore it is easy to prove

$$w_h=w+O(h^2)\quad \text{in } \Omega_h.$$

Hence

$$\frac{2}{3}u_h+\frac{1}{3}u_h^*-u+\frac{h^2}{12}w_h=O(h^4)\quad \text{in } \Omega_h,$$

and (24) results from (27), (30), (22) and

$$\frac{1}{3}\varphi_h=\frac{1}{3}u_h^*+\frac{h^2}{12}w_h.$$

Remark 4. The nonlinear problem

$$Lu=\Delta u-f(x,\ u,\ u_x,\ u_y)=0\quad \text{in } \Omega,$$
$$u=g\quad \text{on } \partial\Omega$$

can also be treated by the same argument.

4. Another Correction Approach

Consider again the elliptic problem (10). We suppose that there exists an approximate solution u_h which possesses an asymptotic expansion

$$u_h=u+h^2v_1+h^4v_2+O(h^6)$$

so that we will have

$$\delta_{x_ix_i}\delta_{x_jx_j}u_h=u_{x_ix_ix_jx_j}+O(h^2),\qquad i\neq j. \tag{31}$$

For instance, u_h may be the solution of the difference equation (11) corresponding to (10).

Starting from u_h we define an iterative correction solution $\bar{u}_h$ which satisfies the linearized difference equation

$$(\Delta_h - f_1(u))\bar{u}_h = f(x, u_h) - f_1(u_h)u_h + \frac{h^2}{12} f_1(u_h) f(x, u_h)$$

$$- \frac{h^2}{6} \sum_{i<j} \delta_{x_i x_i} \delta_{x_j x_j} u_h \quad \text{in } \Omega_h, \tag{32}$$

$$\bar{u}_h = g - \frac{h^2}{12} f(x, g) \quad \text{on } \partial\Omega_h.$$

We remark that in the 1-dimensional case the assumption (31) will be satisfied automatically and the difference equation (32) will be reduced to

$$(\delta_{xx} - f_1(u_h))\bar{u}_h = f(x, u_h) - f_1(u_h)u_h + \frac{h^2}{12} f_1(u_h) f(x, u_h) \quad \text{in } \Omega_h,$$

$$\bar{u}_h = g - \frac{h^2}{12} f(x, g) \quad \text{on } \partial\Omega_h,$$

where u_h is any approximation with second order accuracy.

Proposition 3.

$$\bar{u}_h + \frac{h^2}{12} f(x, u_h) = u + O(h^4) \quad \text{in } \Omega_h. \tag{33}$$

Proof. Since, by (10), (1), (31), (32),

$$(\Delta_h - f_1(u_h))u = \Delta_h u - \Delta u + f(x, u) - f_1(u_h)u$$

$$= \frac{h^2}{12}(\Delta^2 u - 2\sum_{i<j} u_{x_i x_i x_j x_j}) + f(x, u) - f_1(u_h)u + O(h^4)$$

$$= \frac{h^2}{12}(\Delta_h - f_1(u_h))\Delta u + \frac{h^2}{12} f_1(u_h)\Delta u$$

$$- \frac{h^2}{6} \sum_{i<j} u_{x_i x_i x_j x_j} + f(x, u) - f_1(u_h)u + O(h^4)$$

$$= \frac{h^2}{12}(\Delta_h - f_1(u_h))f(x, u) + \frac{h^2}{12} f_1(u_h) f(x, u_h)$$

$$- \frac{h^2}{6} \sum_{i<j} \delta_{x_i x_i} \delta_{x_j x_j} u_h + f(x, u_h) - f_1(u_h)u_h$$

$$+ (f(x, u) - f_1(u_h)u - f(x, u_h) + f_1(u_h)u_h) + O(h^4)$$

$$= \frac{h^2}{12}(\Delta_h - f_1(u_h))f(x, u) + (\Delta_h - f_1(u_h))\bar{u}_h + O(h^4) \quad \text{in } \Omega_h$$

$$u - \frac{h^2}{12} f(x, u) - \bar{u}_h = 0 \quad \text{on } \partial\Omega_h,$$

we have, by the maximum principle,

$$u - \frac{h^2}{12} f(x, u) - \bar{u}_h = O(h^4) \quad \text{in } \Omega_h$$

and (33) is proved.

For the general problem (20) we can define the iterative correction solution $\bar{u}_h$ by

$$L_h \bar{u}_h = f + \frac{h^2}{6}(b\delta_x \delta_{yy} u_h + c\delta_y \delta_{xx} u_h - \delta_{xx}\delta_{yy} u_h)$$

$$+ \frac{h^2}{12} F(u_h, \delta_x u_h, \delta_y u_h, \delta_{xx} u_h, \delta_{xy} u_h, \delta_{yy} u_h) \quad \text{in } \Omega_h,$$

$$\bar{u}_h = g \quad \text{on } \partial\Omega_h,$$

where F has been defined in (23).

Proposition 4.

$$\bar{u}_h = u + O(h^4) \quad \text{in } \Omega_h.$$

Proof. Since, by (21), (20), (23), (31),

$$\begin{aligned} L_h u &= \Delta_h u - \Delta u - b(\delta_x u - u_x) - c(\delta_y u - u_y) + f \\ &= \frac{h^2}{12}(\Delta^2 u - 2u_{xxyy}) - \frac{h^2}{6} b(\Delta u_x - u_{xyy}) - \frac{h^2}{6} c(\Delta u_y - u_{xxy}) + f + O(h^4) \\ &= \frac{h^2}{12} F(u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) + \frac{h^2}{6}(bu_{xyy} + cu_{xxy} - u_{xxyy}) + f + O(h^4) \\ &= L_h \bar{u}_h + O(h^4) \quad \text{in } \Omega_h \end{aligned}$$

$$u - \bar{u}_h = 0 \quad \text{on } \partial\Omega_h$$

the proposition follows by the maximum principle.

We remark that in the special case

$$b = c = 0 \quad \text{on } \partial\Omega$$

the definition of $\bar{u}_h$ can be simplified by

$$L_h \bar{u}_h = f + \frac{h^2}{6}(b\delta_x \delta_{yy} u_h + c\delta_y \delta_{xx} u_h - \delta_{xx} \delta_{yy} u_h)$$

$$- \frac{h^2}{12} G(u_h, \delta_x u_h, \delta_y u_h, \delta_{xx} u_h, \delta_{xy} u_h, \delta_{yy} u_h) \quad \text{in } \Omega_h$$

$$\bar{u}_h = g - \frac{h^2}{12}(dg + f) \quad \text{on } \partial\Omega_h$$

with $G(u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = \left(b\frac{\partial}{\partial x} + c\frac{\partial}{\partial y} - d\right)(bu_x + cu_y + du + f)$

and Proposition 4 will become

$$u_h + \frac{h^2}{12}(b\delta_x u_h + c\delta_y u_h + du_h + f) = u + O(h^4) \quad \text{in } \Omega_h.$$

In fact we have

$$\begin{aligned} L_h u &= \frac{h^2}{12}\left(\Delta - b\frac{\partial}{\partial x} - c\frac{\partial}{\partial y} - d\right)\Delta u - \frac{h^2}{12}\left(b\frac{\partial}{\partial x} + c\frac{\partial}{\partial y} - d\right)\Delta u \\ &\quad + \frac{h^2}{6}(bu_{xyy} + cu_{xxy} - u_{xxyy}) + f + O(h^4) \\ &= \frac{h^2}{12} L_h \Delta u + L_h \bar{u}_h + O(h^4) \quad \text{in } \Omega_h, \end{aligned}$$

$$u - \frac{h^2}{12}\Delta u - \bar{u}_h = 0 \quad \text{on } \partial\Omega_h.$$

5. Eigenvalue Problem

Consider

$$\Delta u + (\lambda - p)u = 0 \quad \text{in } \Omega, \qquad (34)$$
$$u = 0 \quad \text{on } \partial\Omega$$

with $p \geq 0$ and the smallest eigenvalue λ and the corresponding eigenfunction u normalized by

$$(u, u)=\int_\Omega u^2=1,$$

in the 2 and 3-dimensional cases.

Consider for simplicity the 9 or 15-point approximation $\Box_h$ defined in (19) and the corresponding difference eigenvalue problem

$$\begin{aligned}\Box_h U_h+(\Lambda_h-p)U_h=0 \quad &\text{in } \Omega_h,\\ U_h=0 \quad &\text{on } \partial\Omega_h\end{aligned} \tag{35}$$

with the smallest eigenvalue Λ_h and the corresponding eigenvector U_h normalized by

$$(U_h, U_h)_h=h^\alpha\sum_{Q\in\Omega_h} U_h(Q)^2=1,$$

where $\alpha=2$ and 3 in the 2 and 3-dimensional cases respectively.

Let φ_h be a correction solution defined by the linear difference equation

$$\begin{aligned}(\Lambda_h+\Lambda_h-p)\varphi_h=(\Lambda_h-p)^2U_h-((\Lambda_h-p)^2U_h,\ U_h)_hU_h \quad &\text{in } \Omega_h,\\ \varphi_h=0 \quad &\text{on } \Omega_h,\\ (\varphi_h, U_h)_h=\Lambda_h-(pU_h, U_h)_h.&\end{aligned} \tag{36}$$

Proposition 5.

$$\Lambda_h+\frac{h^2}{12}((\Lambda_h-p)^2U_h,\ U_h)_h=\lambda+O(h^4), \tag{37}$$

$$U_h+\frac{h^2}{12}((p-\Lambda_h)U_h+\varphi_h)=\pm u+O(h^4). \tag{38}$$

Proof. Set

$$\overline{U}_h=U_h/(u, U_h)_h. \tag{39}$$

Then $\overline{U}_h$ is an eigenvector of (35) with a convenience normalization condition:

$$(u, \overline{U}_h)_h=1. \tag{40}$$

Consider, by (5), (2),

$$(\Box_h+\Lambda_h-p)u=(\Box_h-\Delta)u+(\Lambda_h-\lambda)u=\frac{h^2}{12}\Delta^2u+(\Lambda_h-\lambda)u+O(h^4) \quad \text{in } \Omega_h. \tag{41}$$

Computing the inner product of both sides with $\overline{U}_h$ and noting that

$$((\Box_h+\Lambda_h-p)u,\ \overline{U}_h)_h=(u,\ (\Box_h+\Lambda_h-p)\overline{U}_h)_h=0$$

we obtain by (40)

$$\Lambda_h-\lambda+\frac{h^2}{12}(\Delta^2 u,\ \overline{U}_h)_h=O(h^4). \tag{42}$$

By the usual estimate

$$\overline{U}_h=u+O(h^2),\ U_h=\pm u+O(h^2),\ \Lambda_h=\lambda+O(h^2),$$

and noting from (34) that

$$\Delta u=(p-\lambda)u=0 \quad \text{on } \partial\Omega$$

we have

$$\begin{aligned}(\Delta^2u,\ \overline{U}_h)_h&=(\Delta^2u,\ u)_h+O(h^2)=(\Delta^2u,\ u)+O(h^2)\\ &=(\Delta u,\ \Delta u)+O(h^2)=((\lambda-p)^2u,\ u)+O(h^2)\\ &=((\Lambda_h-p)^2U_h,\ U_h)_h+O(h^2).\end{aligned} \tag{43}$$

Then (37) follows from (42).

Since, by (41), (43),

$$(\Box_h+\Lambda_h-p)u=\left(\Lambda_h-\lambda+\frac{h^2}{12}(\Delta^2u,\ u)\right)+\frac{h^2}{12}(\Delta^2u-(\Delta^2u,\ u)u)+O(h^4)$$
$$=\frac{h^2}{12}(\Delta^2u-(\Delta^2u,\ u)u)+O(h^4)$$

and letting v be an auxiliary function defined by

$$(\Delta+\lambda-p)v=\Delta^2u-(\Delta^2u,\ u)u \quad \text{in } \Omega, \qquad (44)$$
$$v=0 \quad \text{on } \partial\Omega \quad \text{and} \quad (v,\ u)=0$$

we have

$$(\Box_h+\Lambda_h-p)\left(u-\frac{h^2}{12}v-\overline{U}_h\right)=(\Box_h+\Lambda_h-p)u$$
$$-\frac{h^2}{12}(\Delta+\lambda-p)v+\frac{h^2}{12}(\Delta+\lambda-\Box_h-\Lambda_h)v=O(h^4) \quad \text{in } \Omega_h.$$

Set

$$s_h=\left(u-\overline{U}_h-\frac{h^2}{12}v,\ \overline{U}_h\right)_h=O(h^2),$$
$$w_h=u-\overline{U}_h-\frac{h^2}{12}v-s_hu.$$

We have

$$(w_h,\ \overline{U}_h)_h=s_h-s_h(u,\ \overline{U}_h)_h=0,$$
$$(\Box_h+\Lambda_h-p)w_h=O(h^4)-s_h(\Box_h+\Lambda_h-p)u=O(h^4) \quad \text{in } \Omega_h,$$
$$w_h=0 \quad \text{on } \partial\Omega_h.$$

Letting

$$S=\{w:(w,\ \overline{U}_h)_h=0\}$$

and noting that $(\Box_h+\Lambda_h-p)^{-1}$ exists in S and is uniformly bounded:

$$\|(\Box_h+\Lambda_h-p)^{-1}|_s\|=\frac{1}{\Lambda_h^2-\Lambda_h^1}\to\frac{1}{\lambda^2-\lambda^1},$$

where $0<\lambda=\lambda^1<\lambda^2\leqslant\cdots$ and $\Lambda_h=\Lambda_h^1<\Lambda_h^2\leqslant\cdots$ are the eigenvalues of (34) and (35) respectively, we have

$$w_h=(1-s_h)u-\overline{U}_h-\frac{h^2}{12}v=O(h^4) \quad \text{in } \Omega_h.$$

Computing the inner product of both sides with u and taking into account ([5], Lemma 24. 9)

$$(u,\ u)_h=(u,\ u)+O(h^4)=1+O(h^4),$$
$$(v,\ u)_h=(v,\ u)+O(h^2)=O(h^2),$$

we obtain

$$1-s_h=1+\frac{h^2}{12}(v,\ u)_h+O(h^4)=1+O(h^4),$$
$$u=\overline{U}_h+\frac{h^2}{12}v+O(h^4) \quad \text{in } \Omega_h.$$

Computing the inner product of both sides with $\overline{U}_h$ and noting that

$$(v,\ \overline{U}_h)_h=(v,\ u+O(h^2))_h=(v,\ u)_h+O(h^2)=O(h^2)$$

we obtain

$$1=(\overline{U}_h,\ \overline{U}_h)_h+O(h^4),$$
$$U_h=\pm\overline{U}_h/(\overline{U}_h,\ \overline{U}_h)_h^{\frac{1}{2}}=\pm\overline{U}_h/(1+O(h^4))^{\frac{1}{2}}=\pm\overline{U}_h+O(h^4).$$

Then, we have, by (45),

$$u=\pm U_h+\frac{h^2}{12}v+O(h^4) \quad \text{in } \Omega_h, \tag{46}$$

where v is the solution of (44). Set

$$v=\Delta u+\varphi \tag{47}$$

with φ being the solution of

$$(\Delta+\lambda-p)\varphi=(\lambda-p)^2u-((\lambda-p)^2u,\ u)u \quad \text{in } \Omega,$$
$$\varphi=0 \quad \text{on } \partial\Omega \quad \text{and} \quad (\varphi,\ u)=\lambda-(pu,\ u).$$

Then we have

$$(\Delta+\lambda-p)(\Delta u+\varphi)=\Delta^2u+(\lambda-p)\Delta u+(\lambda-p)^2u$$
$$-((\lambda-p)^2u,\ u)u=\Delta^2u-(\Delta^2u,\ u)u \quad \text{in } \Omega$$
$$\Delta u+\varphi=0 \quad \text{on } \partial\Omega, \quad (\Delta u+\varphi,\ u)=0.$$

Hence (38) follows from (46), (47) and the usual estimate

$$\varphi_h=\varphi+O(h^2).$$

We remark that in the special case

$$p=0$$

we will have $\varphi_h=\Lambda_h U_h$, and (37) and (38) then reduce to

$$\Lambda_h+\frac{h^2}{12}\Lambda_h^2=\lambda+O(h^4), \tag{48}$$

$$U_h=\pm u+O(h^4) \quad \text{in } \Omega_h. \tag{49}$$

Further, we will have a sixth order approximation:

Proposition 6.

$$\Lambda_h+\frac{h^2}{12}\left(\Lambda_h+\frac{h^2}{12}\Lambda_h^2\right)-\frac{h^4}{360}\Lambda_h^3-\frac{h^4}{180}\Lambda_h(\delta_{xx}\delta_{yy}U_h,\ U_h)_h=\lambda+O(h^6), \tag{50}$$

$$U_h+h^4\varphi_h=u/(u,\ U_h)_h+O(h^6) \quad \text{in } \Omega_h, \tag{51}$$

where φ_h is a correction solution defined by

$$(\Box_h+\Lambda_h)\varphi_h=\frac{1}{180}\Lambda_h((\delta_{xx}\delta_{yy}U_h,\ U_h)_hU_h-\delta_{xx}\delta_{yy}U_h) \quad \text{in } \Omega_h,$$
$$\varphi_h=0 \quad \text{on } \partial\Omega_h \quad \text{and} \quad (\varphi_h,\ U_h)_h=0. \tag{52}$$

Proof. Since

$$(\Box_h+\Lambda_h)u=\Box_hu-\Delta u+(\Lambda_h-\lambda)u$$
$$=\frac{h^2}{12}\Delta^2u+\frac{h^4}{360}(\Delta^3u+2\Delta u_{xxyy})+(\Lambda_h-\lambda)u+O(h^6)$$
$$=\left(\frac{h^2}{12}\lambda^2-\frac{h^4}{360}\lambda^3-\frac{h^4}{180}\lambda\frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial y^2}+\Lambda_h-\lambda\right)u+O(h^6) \quad \text{in } \Omega_h, \tag{53}$$

computing the inner product of both sides with $\overline{U}_h$ defined in (39) we obtain

$$\frac{h^2}{12}\lambda^2-\frac{h^4}{360}\lambda^3-\frac{h^4}{180}\lambda(u_{xxyy},\ \overline{U}_h)_h+\Lambda_h-\lambda=O(h^6). \tag{54}$$

Then (50) follows from (48) and (31).

It follows from (53), (54) and (31) that

$$(\Box_h+\Lambda_h)(u-U_h)=\Big(\frac{h^2}{12}\lambda^3-\frac{h^4}{360}\lambda^3-\frac{h^4}{180}\lambda(u_{xxyy},\overline{U}_h)_h+\Lambda_h-\lambda\Big)u$$

$$-\frac{h^4}{180}\lambda u_{xxyy}+\frac{h^4}{180}\lambda(u_{xxyy},\overline{U}_h)_h u+O(h^6)$$

$$=\frac{h^4}{180}\Lambda_h(\delta_{xx}\delta_{yy}U_h,U_h)_h U_h-\frac{h^4}{180}\Lambda_h\delta_{xx}\delta_{yy}U_h+O(h^6)$$

$$=h^4(\Box_h+\Lambda_h)\varphi_h+O(h^6)\quad\text{in }\Omega_h.$$

Note from (49) and (52) that

$$\varepsilon_h=(u-U_h-h^4\varphi_h,\overline{U}_h)_h=1-\frac{1}{(u,U_h)_h}-h^4(\varphi_h,U_h+O(h^2))_h$$

$$=1-\frac{1}{(u,U_h)_h}+O(h^6)=1-\frac{1}{1+O(h^4)}+O(h^6)=O(h^4),$$

$$1-\varepsilon_h=\frac{1}{(u,U_h)_h}+O(h^6),$$

$$((1-\varepsilon_h)u-U_h-h^4\varphi_h,\overline{U}_h)_h=0$$

and $$(\Box_h+\Lambda_h)((1-\varepsilon_h)u-U_h-h^4\varphi_h)=O(h^6)-\varepsilon_h(\Box_h+\Lambda_h)u=O(h^6)\quad\text{in }\Omega_h$$

$$(1-\varepsilon_h)u-U_h-h^4\varphi_h=0\quad\text{on }\partial\Omega.$$

Then we have $$(1-\varepsilon_h)u=U_h+h^4\varphi_h+O(h^6)\quad\text{in }\Omega_h$$

and (51) is proved.

We mention that a sixth order approximate eigenvalue has been obtained by Birkhoff and Fix[1] by using the spline finite element method.

6. Fourth Order Elliptic Problem

The correction approach is also effective for the fourth order problem like

$$u_{xxxx}=f(x,u,u_x,u_{xx})\quad\text{in }(0,1),$$

$$u=g_1,\quad u'=g_2\quad\text{on 0 and 1.}$$

Let u_h be an approximation with the property

$$\delta_{xx}u_h=u_{xx}+O(h^2)$$

and $\bar{u}_h$ the correction solution defined by

$$L_h\bar{u}_h=(\delta_{xx}\delta_{xx}-f_1(U_h)-f_2(U_h)\delta_x-f_3(U_h)\delta_{xx})\bar{u}_h$$

$$=f(U_h)-f_1(U_h)u_h-f_2(U_h)\delta_x u_h-f_3(U_h)\delta_{xx}u_h$$

$$+\frac{h^2}{6}f_1(U_h)\delta_{xx}u_h+\frac{h^2}{12}f_3(U_h)f(U_h)\quad\text{in }\Omega_h,$$

$$\bar{u}_h=g_1-\frac{h^2}{6}\delta_{xx}u_h,\quad\delta_x\bar{u}_h=g_2\quad\text{on 0 and 1,}$$

where

$$f(U_h)=f(x,u_h,\delta_x u_h,\delta_{xx}u_h),$$

$$f_1(U_h)=f_u(x,u_h,\delta_x u_h,\delta_{xx}u_h),$$

$$f_2(U_h)=f_{u_x}(x,u_h,\delta_x u_h,\delta_{xx}u_h),$$

$$f_3(U_h)=f_{u_{xx}}(x,u_h,\delta_x u_h,\delta_{xx}u_h).$$

Proposition 7.

$$u-\bar{u}_h-\frac{h^2}{6}\delta_{xx}u_h=O(h^4)\quad \text{in } \Omega_h. \tag{55}$$

Proof. Since

$$\begin{aligned}L_h u&=\delta_{xx}\delta_{xx}u-f_1(U_h)u-f_2(U_h)\delta_x u-f_3(U_h)\delta_{xx}u\\&=\delta_{xx}\delta_{xx}u-u_{xxxx}+f(U)-f(U_h)-f_1(U_h)(u-u_h)\\&\quad-f_2(U_h)(u_x-\delta_x u_h)-f_3(U_h)(u_{xx}-\delta_{xx}u_h)\\&\quad+f(U_h)-f_1(U_h)u_h-f_2(U_h)\delta_x u_h-f_3(U_h)\delta_{xx}u_h\\&\quad-f_2(U_h)(\delta_x u-u_x)-f_3(U_h)(\delta_{xx}u-u_{xx})\\&=\frac{h^2}{6}u_{xxxxxx}-\frac{h^2}{6}f_2(U_h)u_{xxx}-\frac{h^2}{12}f_3(U_h)u_{xxxx}\\&\quad+f(U_h)-f_1(U_h)u_h-f_2(U_h)\delta_x u_h-f_3(U_h)\delta_{xx}u_h+O(h^4)\\&=\frac{h^2}{6}\Big(\frac{d^4}{dx^4}-f_1(U_h)-f_2(U_h)\frac{d}{dx}-f_3(U_h)\frac{d^2}{dx^2}\Big)u_{xx}\\&\quad+\frac{h^2}{6}f_1(U_h)u_{xx}+\frac{h^2}{12}f_3(U_h)f(U)+f(U_h)-f_1(U_h)u_h\\&\quad-f_2(U_h)\delta_x u_h-f_3(U_h)\delta_{xx}u_h+O(h^4)\\&=\frac{h^2}{6}L_h u_{xx}+L_h\bar{u}_h+O(h^4)\quad \text{in } \Omega_h,\end{aligned}$$

$$u-\frac{h^2}{6}u_{xx}-\bar{u}_h=O(h^4),\quad \delta_x\Big(u-\frac{h^2}{6}u_{xx}-\bar{u}_h\Big)=O(h^4)\quad \text{in 0 and 1,}$$

then, if L_h^{-1} is uniformly bounded, we have

$$u-\frac{h^2}{6}u_{xx}-\bar{u}_h=O(h^4)\quad \text{in } \Omega_h$$

and (55) is proved.

We would like to mention that the above analysis is crude and we must treat the curved boundary case to which Bohmer[2], Bramble[3], Marchouk and Shaydourov[8], and Weinberger[12] have already made important contributions.

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