ORDER INTERVAL TEST AND ITERATIVE METHOD FOR NONLINEAR SYSTEMS*

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Abstract

An order interval test for existence and uniqueness of solutions to a nonlinear system is given. It combines the interval technique and the monotone iterative technique. It has the main merits of interval iterative methods but need not use interval arithmetic. An order interval Newton method is also given, which is globally convergent. It is a generalization of the results in [3], [4, 13.3].

1. Introduction

Suppose we have a nonlinear system

$$f(x)=0, \tag{1}$$

where $f:D\subset R^n\to R^n$ is continuous on D. Moore and L. Qi introduced interval tests for existence and uniqueness of a solution to a nonlinear system in [1, 2]. However, the interval arithmetic is complicated. In this paper, some n-dimensional order interval iterative methods are presented. They can also be used as interval tests for existence and uniqueness of the solution to (1). Since they use endpoint calculation instead of interval arithmetic, they are simple.

In section 2 a simple interval Newton method and its global convergence is given. In section 3, an order interval Newton method is presented, which is a generalization of the Newton monotone iterative method given in [3], [4, 13.3].

The notation is as follows. Let R^n be the n-dimensional real space and $L(R^n)$ the space of all real $n\times n$ matrices. For vectors $x, y\in R^n$ and matrices $A, B\in L(R^n)$, we denote the usual componentwise partial orderings by $x\leqslant y$ and $A\leqslant B$. If $AB\leqslant I$ $(BA\leqslant I)$, where I is the identity matrix, then A is called a left (right) subinverse of B. If A is both a left and a right subinverse of B, then A is called a subinverse of B.

Let $X=[\underline{x}, \bar{x}]=\{u|\underline{x}\leqslant u\leqslant \bar{x}\}$ be an n-dimensional interval vector; it is an order interval. $W(X)=\bar{x}-\underline{x}$ is called the width of the interval vector $X=[\underline{x}, \bar{x}]$, which is a nonnegative vector. We have the following properties of $W(\cdot)$:

(1) $W(\lambda X)=\lambda W(X)$, $\lambda\in R'$ and $\lambda\geqslant 0$;

(2) $W(x+X)=W(X)$, $x\in R^n$;

(3) $W\left(\sum_{j=1}^m X^j\right)=\sum_{j=1}^m W(X^j)$;

(4) If $X\subset Y$, then $W(X)\leqslant W(Y)$.

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2. Order Interval Test and Simple Interval Newton Method

Let $X=[\underline{x}, \bar{x}]$ be an arbitrary order interval, i. e. $X:=\{u|\underline{x}\leqslant u\leqslant\bar{x}\}$. $P\in L(R^n)$ is a nonsingular matrix.

Define
$$Nu:=u-Pf(u),\ \forall u\in R^n \tag{2}$$
and
$$NX:=[N\underline{x}, N\bar{x}] \tag{3}$$
for any order interval $X=[\underline{x}, \bar{x}]$. Then N is an interval operator.

Lemma 2.1. *Suppose $f:D\subset R^n\to R^n$ is continuous and there is a matrix $A\in L(R^n)$ such that*
$$f(\bar{x})-f(\underline{x})\leqslant A(\bar{x}-\underline{x}),\quad \underline{x}\leqslant\bar{x},\quad \underline{x},\ \bar{x}\in D. \tag{4}$$
If A has nonnegative, nonsingular, left subinverse P, then
$$\{Nu|u\in X\}\subset NX \tag{5}$$
for any $X=[\underline{x}, \bar{x}]\subset D$.

Proof. $\forall u\in X=[\underline{x}, \bar{x}]\subset D$, by (4), we have
$$N\bar{x}-Nu=\bar{x}-u-P(f(\bar{x})-f(u))\geqslant\bar{x}-u-PA(\bar{x}-u)\geqslant 0,$$
$$Nu-N\underline{x}=u-\underline{x}-P(f(u)-f(\underline{x}))\geqslant u-\underline{x}-PA(u-\underline{x})\geqslant 0,$$
i. e. $N\underline{x}\leqslant Nu\leqslant N\bar{x}$, i. e. (5) holds.

Lemma 2.2. *Suppose the conditions of Lemma 2.1 hold and $X=[\underline{x}, \bar{x}]\subset D$ is an order interval. Then NX contains all solutions of (1) in X. If $NX\subset X$, then there is a solution of (1) in X. If $X\cap NX=\emptyset$, then there is no solution of (1) in X.*

Proof. Suppose x^* is a solution of (1), $x^*\in X$. Then
$$x^*=x^*-Pf(x^*)=Nx^*\in NX.$$
Therefore, NX contains all solutions of (1) in X. This implies the last conclusion directly. Since f is continuous, so is $Nx=x-Pf(x)$. By (4) and Brouwer's fixed point theorem, we know that N has a fixed point in NX if $NX\subset X$. But all the fixed points of N are solutions of (1) and vice versa. This proves the second conclusion.

Lemma 2.3. *Suppose the conditions of Lemma 2.1 hold and $NX\subset X$; then*
$$N(NX)\subset NX. \tag{6}$$

Proof. By (5), we have
$$N(NX)=[N(N\underline{x}), N(N\bar{x})]\subset NX$$
since $N\underline{x}$, $N\bar{x}\in NX\subset X$.

Now we construct simple interval Newton algorithm:

Algorithm 2.1. Let $X^0=[\underline{x}^0, \bar{x}^0]\subset D$. For $k=0, 1, \cdots$, if $X^k\cap NX^k=\emptyset$, then stop; otherwise, let $X^{k+1}=X^k\cap NX^k$.

Theorem 2.1. *Suppose the conditions of Lemma 2.1 hold and $X^0\subset D$ is an order interval, $\{X^k, k=0, 1, \cdots, \}$ is produced by Algorithm 2.1. Then all the solutions of (1) in X^0 are also in X^k for any nonnegative integer k. If $X^k\cap NX^k=\emptyset$ for a certain k, then there is no solution of (1) in X^0. If $NX^k\subset X^k$ for a certain k, then*

(i) *there exists a solution of* (1) *in* X^0.

(ii) $NX^m \subset X^m$ *for* $m=k, k+1, \cdots$.

(iii) $\{\underline{x}^m = N\underline{x}^{m-1}, m=k+1, \cdots\}$ *and* $\{\bar{x}^m = N\bar{x}^{m-1}, m=k+1, \cdots,\}$ *converge,*

$$\lim_{m\to\infty} \underline{x}^m = \underline{x}^*; \quad \lim_{m\to\infty} \bar{x}^m = \bar{x}^*.$$

(iv) $\underline{x}^*$ *and* $\bar{x}^*$ *are solutions of* (1) *in* X^0.

(v) *If* x^* *is a solution of* (1) *in* X^0, *then* $\underline{x}^* \leqslant x^* \leqslant \bar{x}^*$. *If* $\underline{x}^* = \bar{x}^*$, *then* $\underline{x}^* = \bar{x}^*$ *is the unique solution of* (1) *in* X^0.

Proof. The first and the second conclusions and (i), (ii), (iii) are direct consequences from Lemmas 2.1, 2.2 and 2.3. Since

$$\underline{x}^* = \lim_{m\to\infty} \underline{x}^m = \lim_{m\to\infty} N\underline{x}^{m-1} = N\underline{x}^* = \underline{x}^* - Pf(\underline{x}^*),$$

and P is nonsingular, we know that $\underline{x}^*$ is a solution of (1). Similarly for $\bar{x}^*$. Thus we get (iv). If x^* is a solution of (1) in X^0, then $\underline{x}^m \leqslant x^* \leqslant \bar{x}^m$ for $m=k+1, \cdots$, i. e. $\underline{x}^* = \lim\limits_{m\to\infty} \underline{x}^m \leqslant x^* \leqslant \lim\limits_{m\to\infty} \bar{x}^m = \bar{x}^*$. Thus we get (v).

Theorem 2.2. *Suppose the conditions of Theorem* 2.1 *hold and* $NX^0 \subset X^0$. *If there exists an* $n\times n$ *matrix* B *such that*

$$f(\bar{x}) - f(\underline{x}) \geqslant B(\bar{x} - \underline{x}), \quad \forall \underline{x} \leqslant \bar{x}, \ \underline{x}, \bar{x} \in D \tag{7}$$

and if

$$\beta = \max_{1\leqslant i\leqslant n} \frac{\sum_{j=1}^{n} R_{ij} W(X_j^0)}{W(X_i^0)} < 1, \tag{8}$$

where $R = (R_{ij}) := I - PB \geqslant 0$, $W(X_i^0) \neq 0$, $i=1, \cdots, n$, *are the widths of* X_i^0, *then* (1) *has a unique solution* x^* *in* X^0, *and the iterative sequence*

$$z^{k+1} = z^k - Pf(z^k), \quad k=0, 1, \cdots \tag{9}$$

converges to x^* *for any starting point* $z^0 \in X^0$, *with the error estimate*

$$|z^k - x^*| \leqslant \beta^k W(X^0), \quad k=0, 1, \cdots. \tag{10}$$

Proof. By Theorem 2.1 we already know the existence of the solutions of (1) in X^0. To prove the theorem, it suffices to prove that

$$W(X^k) \leqslant \beta^k W(X^0), \quad k=0, 1, \cdots. \tag{11}$$

By induction and

$$W(X^{k+1}) = W(NX^k) = N\bar{x}^k - N\underline{x}^k = \bar{x}^k - \underline{x}^k - P(f(\bar{x}^k) - f(\underline{x}^k))$$
$$\leqslant \bar{x}^k - \underline{x}^k - PB(\bar{x}^k - \underline{x}^k) = (I - PB)(\bar{x}^k - \underline{x}^k) = RW(X^k),$$

$$W(X_i^{k+1}) \leqslant \sum_{j=1}^{n} R_{ij} W(X_j^k) = \sum_{j=1}^{n} R_{ij} W(X_j^0) \frac{W(X_j^k)}{W(X_j^0)} \leqslant \beta W(X_i^0) \max_{1\leqslant j\leqslant n} \frac{W(X_j^k)}{W(X_j^0)}$$

we get $\quad W(X_i^k) \leqslant \beta^k W(X_i^0), \quad i=1, \cdots, n, \ k=0, 1, \cdots,$

i. e. we get (11). Moreover, if $z^0 \in X^0$, then

$$Nz^k \in NX^k = X^{k+1}, \quad k=0, 1, \cdots$$

implies $\lim\limits_{k\to\infty} z^k = x^*$. Since z^k and x^* are in X^k, we get (10).

Remark 2.1. Suppose the conditions of Lemma 2.1 hold and there is $x^0 = [\underline{x}^0, \bar{x}^0] \subset D$ such that

$$f(\underline{x}^0) \leqslant 0 \leqslant f(\bar{x}^0). \tag{12}$$

Then $NX^0 \subset X^0$ and vice versa.

The reader may check this himself.

Remark 2.2. If there is $X^0=[\underline{x}^0, \bar{x}^0]\subset D$ such that

$$f_i(\underline{x}^0)f_i(\bar{x}^0)\leqslant 0, \quad i=1, \cdots, n \tag{13}$$

then there exists the diagonal $J=\operatorname{diag}(J_1, \cdots, J_n)$ such that

$$Jf(\underline{x}^0)\leqslant 0\leqslant Jf(\bar{x}^0), \tag{14}$$

where

$$J_i=\begin{cases} 1, & \text{if } f_i(\underline{x}^0)\leqslant 0, \\ -1, & \text{if } f_i(\underline{x}^0)>0. \end{cases}$$

Proof. By (13), if $f_i(\underline{x}^0)\leqslant 0$, then $f_i(\bar{x}^0)\geqslant 0$, so that $J_i=1$ and $J_if_i(\underline{x}^0)\leqslant 0\leqslant J_if_i(\bar{x}^0)$. If $f_i(\underline{x}^0)>0$, then $f_i(\bar{x}^0)\leqslant 0$, so that $J_i=-1$ and $J_if_i(\underline{x}^0)\leqslant 0\leqslant J_if_i(\bar{x}^0)$. We get (14).

Therefore, we may use (13) instead of $NX^0\subset X^0$ in Theorems 2.1 and 2.2.

3. Order Interval Newton Method

The simple interval Newton method (Algorithm 2.1) is linearly convergent. To improve the convergence rate, we give the order interval Newton method. We use order convexity. Recall [4] that $f:D\subset R^n\to R^n$ is order convex on a convex set D, if

$$f(\lambda x+(1-\lambda)y)\leqslant\lambda f(x)+(1-\lambda)f(y) \tag{15}$$

whenever $x, y\in D$, $x\leqslant y$ or $y\leqslant x$ and $\lambda\in(0, 1)$.

Also recall that if f is G-differentiable. Then f is order convex in D if and only if

$$f(y)-f(x)\leqslant f'(y)(y-x), \quad x\leqslant y,\ x, y\in D. \tag{16}$$

By (15) and (16), we have

$$f(y)-f(x)\leqslant f'(x)(y-x), \quad x\leqslant y,\ x, y\in D. \tag{17}$$

Therefore, we get the following Lemma.

Lemma 3.1. *Suppose $f:D\subset R^n\to R^n$ is G-differentiable and there is a nonsingular matrix $P\in L(R^n)$, such that $F=Pf$ is order convex in $X^0=[\underline{x}^0, \bar{x}^0]\subset D$.* Then

$$F'(\underline{x})(\bar{x}-\underline{x})\leqslant F(\bar{x})-F(\underline{x})\leqslant F'(\bar{x})(\bar{x}-\underline{x}) \tag{18}$$

for any $\underline{x}\leqslant\bar{x}$, $\underline{x}, \bar{x}\in X^0=[\underline{x}^0, \bar{x}^0]$.

Now we define the order interval Newton method as follows:

Algorithm 3.1. Let $X^0=[\underline{x}^0, \bar{x}^0]\subset D$. For $k=0, 1, \cdots$, define

$$X^{k+1}=\bar{N}_kX^k=[\bar{N}_k\underline{x}^k, \bar{N}_k\bar{x}^k], \tag{19}$$

where

$$X^k=[\underline{x}^k, \bar{x}^k],\ \bar{N}_kx=x-f'(\bar{x}^k)^{-1}f(x), \quad \forall x\in X^k.$$

Theorem 3.1. *Suppose $f:D\subset R^n\to R^n$ is G-differentiable and $f'(x)$ is nonsingular in D. Suppose there are $X^0=[\underline{x}^0, \bar{x}^0]\subset D$ and a nonsingular matrix $P\in L(R^n)$ such that*

$$Pf(\underline{x}^0)\leqslant 0\leqslant Pf(\bar{x}^0). \tag{20}$$

Let $F=Pf$ be order convex in X^0 and $F'(x)^{-1}\geqslant 0$, $\forall x\in X^0$. Then (1) has a unique solution x^* in X^0 and the conclusions (i)—(v) of Theorem 2.1 hold. Moreover, if

$$\|F'(x)-F'(y)\|\leqslant r\|x-y\|, \quad x, y\in X^0 \tag{21}$$

then the interval vector sequence $\{X^k,\ k=0,1,\cdots\}$ *defined by* (19) *converges to* x^* *quadratically.*

Proof. Because P is nonsingular, so

$$F'(\bar{x}^k)^{-1}F(x)=[Pf(\bar{x}^k)]^{-1}Pf(x)=f'(\bar{x}^k)^{-1}f(x).$$

Therefore,

$$\bar{N}_k x=x-f'(\bar{x}^k)^{-1}f(x)=x-F'(\bar{x}^k)^{-1}F(x).$$

Since $F(x)$ is order convex in X^0, it is continuous in X^0 (see [5]) and (18) holds. Since $F'(\bar{x})^{-1}\geqslant 0$, we have

$$F'(\bar{x})^{-1}F'(\underline{x})(\bar{x}-\underline{x})\leqslant F'(\bar{x})^{-1}F'(\bar{x})(\bar{x}-\underline{x})=\bar{x}-\underline{x},\quad \forall\underline{x}\leqslant\bar{x},\quad \underline{x},\bar{x}\in X^0.$$

We get

$$F'(\bar{x})^{-1}F'(\underline{x})\leqslant I$$

and

$$F'(\bar{x})^{-1}\leqslant F'(\underline{x})^{-1},\quad \forall\underline{x}\leqslant\bar{x},\quad \underline{x},\bar{x}\in X^0. \tag{21}$$

Now we prove that

$$N_k X^k\subset X^k \text{ and } F(\underline{x}^k)\leqslant 0\leqslant F'(\bar{x}^k),\quad k=0,1,\cdots. \tag{22}$$

By induction and (18), we have

$$F(\bar{x}^k)\geqslant F(\bar{x}^{k-1})+F'(\bar{x}^{k-1})(\bar{x}^k-\bar{x}^{k-1})=F(\bar{x}^{k-1})-F'(\bar{x}^{k-1})F'(\bar{x}^{k-1})^{-1}F(\bar{x}^{k-1})=0$$

and $\quad F(\underline{x}^k)\leqslant F(\underline{x}^{k-1})+F'(\underline{x}^k)(\underline{x}^k-\underline{x}^{k-1})=F(\underline{x}^{k-1})-F'(\underline{x}^k)F'(\bar{x}^{k-1})^{-1}F(\underline{x}^{k-1}).$

Since $F'(\underline{x}^k)^{-1}\geqslant 0$ and (21), we get

$$F'(\underline{x}^k)^{-1}F(\underline{x}^k)\leqslant F'(\underline{x}^k)^{-1}F(\underline{x}^{k-1})-F'(\underline{x}^k)^{-1}F'(\underline{x}^k)F'(\bar{x}^{k-1})^{-1}F(\underline{x}^{k-1})$$
$$=[F'(\underline{x}^k)^{-1}-F'(\bar{x}^{k-1})^{-1}]F(\underline{x}^{k-1})\leqslant 0,$$

i. e. $F(\underline{x}^k)\leqslant 0$. Therefore, by $F'(\bar{x}^k)^{-1}\geqslant 0$ we have

$$\bar{N}_k\bar{x}^k-\bar{x}^k=-F'(\bar{x}^k)^{-1}F(\bar{x}^k)\leqslant 0,$$
$$\bar{N}_k\underline{x}^k-\underline{x}^k=-F'(\bar{x}^k)^{-1}F(\underline{x}^k)\geqslant 0,$$

i. e. $\bar{N}_k X^k\subset X^k$. Thus we get (22). According to Theorem 2.1, we get conclusions (i)—(v) of Theorem 2.1.

The proof of uniqueness and quadratical convergence may be seen in [4, 13.3].

If $F=Pf$ is order concave on X^0 instead of order convex, we get

$$F'(\bar{x})(\bar{x}-\underline{x})\leqslant F(\bar{x})-F(\underline{x})\leqslant F'(\underline{x})(\bar{x}-\underline{x}),\quad \forall\underline{x},\bar{x}\in D,\ \underline{x}\leqslant\bar{x}.$$

Then we define the order interval Newton method as follows:

Algorithm 3.2. Let $X^0=[\underline{x}^0,\bar{x}^0]\subset D$. For $k=0,1,\cdots$, define

$$X^{k+1}=\underline{N}_k X^k=[\underline{N}_k\underline{x}^k,\ \underline{N}_k\bar{x}^k], \tag{23}$$

where

$$X^k=[\underline{x}^k,\bar{x}^k],\ \underline{N}_k x=x-f(\underline{x}^k)^{-1}f(x),\ \forall x\in X^k.$$

Similarly, the conclusions of Theorem 3.1 hold.

Remark 3.1. Let $P=I$; the endpoint sequence $\{\bar{x}^{k+1}=\bar{N}_k\bar{x}^k,\ k=0,1,\cdots\}$ of (19) is the same as the sequence defined by the monotone Newton method (see [4, 13.3, 4]). Therefore, the order interval Newton methods (19) and (23) are generalizations of the monotone Newton method.

Remark 3.2. If $A=F'(x)$ is an M-matrix, then $A^{-1}=F'(x)^{-1}\geqslant 0$.

Example. Let

$$f(x)=\begin{pmatrix}-x_1^3+5x_1^2-x_1+2x_2-3\\ x_2^3+x_2^2-14x_2-x_1-19\end{pmatrix},\quad X^0=\left[\begin{pmatrix}3\\3\end{pmatrix},\begin{pmatrix}6\\5\end{pmatrix}\right].$$

Since
$$f(\underline{x}^0)=\begin{pmatrix}18\\-28\end{pmatrix},\quad f(\bar{x}^0)=\begin{pmatrix}-3.5\\55\end{pmatrix}$$
we have $P=\begin{pmatrix}-1&0\\0&1\end{pmatrix}$. Let
$$F(x)=Pf(x)=\begin{pmatrix}-f_1\\f_2\end{pmatrix},\quad F'(x)=\begin{pmatrix}3x_1^2-10x_1+1&-2\\-1&3x_2^2+2x_2-14\end{pmatrix},$$
where $F'(x)$ is an M-matrix in X^0. We get
$$F'(\underline{x})(\bar{x}-\underline{x})\leqslant F(\bar{x})-F(\underline{x})\leqslant F'(\bar{x})(\bar{x}-\underline{x}),\quad \forall \underline{x}\leqslant\bar{x},\ \underline{x},\ \bar{x}\in X^0,$$
and
$$F'(\underline{x})^{-1}\geqslant F'(\bar{x}^0)^{-1}=\begin{pmatrix}0.02042&0.000572\\0.0002876&0.01409\end{pmatrix}\geqslant 0,\ \forall x\in X^0.$$
Therefore, the conditions of Theorem 3.1 are satisfied, and $f(x)=0$ has a unique solution x^* in X^0. The results from use of Algorithm 3.1 are given in Table 1.

The exact solution is $x^*=\begin{pmatrix}5\\4\end{pmatrix}$, and we get
$$\bar{x}^4=\begin{pmatrix}5.0000004\\4.0000047\end{pmatrix}=x^*.$$

Table 1

k	$x^k=[\underline{x}^k,\bar{x}^k]$	k	$x^k=[\underline{x}^k,\bar{x}^k]$
0	$\left[\begin{pmatrix}3\\3\end{pmatrix},\ \begin{pmatrix}6\\5\end{pmatrix}\right]$	3	$\left[\begin{pmatrix}4.65040\\3.97972\end{pmatrix},\ \begin{pmatrix}5.00020\\4.00006\end{pmatrix}\right]$
1	$\left[\begin{pmatrix}3.38358\\3.39978\end{pmatrix},\ \begin{pmatrix}5.25384\\4.21482\end{pmatrix}\right]$	4	$\left[\begin{pmatrix}4.9544496\\3.998790\end{pmatrix},\ \begin{pmatrix}5.0000004\\4.0000047\end{pmatrix}\right]$
2	$\left[\begin{pmatrix}4.01412\\3.81375\end{pmatrix},\ \begin{pmatrix}5.02251\\4.01346\end{pmatrix}\right]$		

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