LACUNARY SPLINE INTERPOLATION*

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Abstract

In this paper, some general kinds of lacunary spline interpolations are discussed. Existence and uniqueness are proved. Order of convergence and saturation of approximation are also obtained. The results generalize some individual results in references.

1. Introduction and Notation

A kind of quintic lacunary splines was studied in [1], and some further information was given by [2], [3]. Since then, various lacunary splines with different degrees have appeared in this background (c. f. [4], [5], [6]). Among them, a kind of higher degree lacunary splines discussed in [6] can be considered as an extension of Meir and Sharma's quintic spline in [1]. By a general method, the existence, uniqueness and convergence of another kind of lacunary splines were obtained in [7]. In this paper, we continue the work in this respect and discuss a more general kind of lacunary splines. The results here, in special cases, may reduce to the results of [1], [5], [6].

Through this paper, we consider interval [0, 1] with a partition of the form Δ_N: $0=x_0<x_1<\cdots<x_n=1$. Let A, B, C denote the sets of nonnegative integers and $|A|$ be the cardinality of A. P_n denotes the set of algebraic polynomials of degree less than or equal to n. $S_p(n, m, \Delta_N)\equiv\{s(x)\,|\,s(x)\in C^{m-1}[0, 1]$ and $s(x)\in P_n$ in the mesh interval $[x_i, x_{i+1}]$ of $\Delta_N\}$. The norm $\|\cdot\|$ is due to Tchebycheff. If g is a function whose jth derivative exists at all points of [0, 1] except finite number of points $y_1, y_2, \cdots, y_m$, we regard $\|g^j\|$ as $\|g^j\|\equiv\sup\limits_{1<i<m-1}\|g^j\|_{(y_i, y_{i+1})}$. Here and afterwards we always use $g^j(x)$ instead of $g^{(j)}(x)$. By saying there exists a relation $J_1\prec J_2$ between two sets of nonnegative integers $J_1\equiv\{j_1^1, j_2^1, \cdots, j_m^1\}$ and $J_2\equiv\{j_1^2, j_2^2, \cdots, j_m^2\}$, we mean that $j_i^1\leqslant j_i^2$ for all i.

2. Existence and Uniqueness

First we introduce a result on polynomial Hermite-Birkhoff interpolation (c. f. [8]):

Definition 1. *A $2\times m$ matrix $E=(e_{ij})_{i=0}^{1}{}_{j=0}^{m-1}$ consisting of zeros and ones is a two-point poised H-B matrix, if the only polynomial P of degree $m-1$ satisfying $P^j(i)=0$ for all (i, j) such that $e_{ij}=1$ is $P\equiv0$.*

Lemma 1 (c. f. [8]). *A $2\times m$ matrix $E=(e_{ij})_{i=0}^{1}{}_{j=0}^{m-1}$ consisting of zeros and ones is a two-point poised H-B matrix if and only if the Pólya conditions*

* Received April 6, 1983.

$$\sum_{i=0}^{1}\sum_{j=0}^{p} e_{ij} \equiv M_p \geqslant p+1, \quad 0 \leqslant p \leqslant m-1 \tag{1}$$

are satisfied.

For the convenience of the discussion below, we give:

Definition 2. *A set of nonnegative integers $J=\{j_0, j_1, \cdots, j_{m-1}\}$, $j_0<j_1<\cdots<j_{m-1}$, is said to be a symmetric set of order m, if J corresponds to a two-point H–B matrix $E=(e_{ij})_{i=0}^{1}{}_{j=0}^{2m-1}$, with*

$$e_{ij}=\begin{cases}1, & j\in J,\ 0\leqslant i\leqslant 1,\\ 0, & \text{otherwise}.\end{cases}$$

When the corresponding H–B matrix of J is poised, J is also called poised.

In [7], the poised symmetric set is called symmetric Pólya set.

We notice that by this definition,

$$J^* = \{0, 2, 4, \cdots, 2m-2\} \tag{2}$$

corresponds to a H–B matrix

$$E^* = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0\cdots \\ 1 & 0 & 1 & 0 & 1 & 0\cdots \end{pmatrix}.$$

Obviously it satisfies the Pólya conditions (1), and none of the columns consisting of ones in E^* can move from left to right to maintain Pólya conditions. So we get Lemma 2 by Lemma 1 at once:

Lemma 2. *The symmetric set $J=\{j_0, j_1, \cdots, j_{m-1}\}$ of order m is a symmetric Pólya set if and only if*

$$J \prec J^*. \tag{3}$$

When J is a symmetric Pólya set of order $n+1$, there exists a unique $p\in P_{2n+1}$ such that $p^j(i)=a_{ij}$, $j\in J$, $i=0, 1$, for given arbitrary real data $\{a_{ij}: j\in J, i=0, 1\}$.

Obviously it follows that $0\in J$ by Lemma 2 and there are unique polynomials $g_j, h_j\in P_{2n+1}$ satisfying

$$g_j^i(0)=h_j^i(1)=\delta_{ij}, \quad g_j^i(1)=h_j^i(0)=0, \qquad i, j\in J, \tag{4}$$

with J a symmetric Pólya set, and any $p\in P_{2n+1}$ has the unique expansion

$$p(x)=\sum_{j\in J}\{p^j(0)g_j(x)+p^j(1)h_j(x)\}.$$

In general, for any interval $[x_i, x_{i+1}]$, with $h_{i+1}=x_{i+1}-x_i>0$, we have

$$p^r(x)=\sum_{j\in J} h_{i+1}^{j-r}\left\{ p^j(x_i)g_j^r\left(\frac{x-x_i}{h_{i+1}}\right)+p^j(x_{i+1})h_j^r\left(\frac{x-x_i}{h_{i+1}}\right)\right\}. \tag{5}$$

The expressions of $g_j(x)$ and $h_j(x)$ can be obtained from (4); for some special cases, this has been done (c.f. [9] for $J=J^*$).

Now, let Δ_N be a given partition of $[a, b]$, and A, B, C are the sets of nonnegative integers satisfying conditions:

(i) A, B and C are disjoint from each other

$$|A|=r_0+2, \ |B|=n-r_0-1, \ |C|=2|B|,$$

(ii) $A\cup B$ is a symmetric Pólya set of order $n+1$,

(iii) $A\cup C=\{0, 1, 2, \cdots, 2n-r_0-1\}$, $B\subset\{2n-r_0, 2n-r_0+1, \cdots, 2n\}$.

We consider the interpolating lacunary spline $s\equiv S(f, A, B, C, \Delta_N)$ satisfying:

$$
\begin{aligned}
&\text{(i)} && s\in S_p(2n+1,\ 2n-r_0,\ \Delta_N),\\
&\text{(ii)} && s^j(x_i)=f^j(x_i),\ j\in A,\quad 0\leqslant i\leqslant N,\\
&\text{(iii)} && s^j(0)=f^j(0),\quad j\in C,
\end{aligned}
\tag{6}
$$

where f is a given smooth function. We also denote $s(x)\equiv s(f;\ x)$ to emphasize that $s(x)$ is an interpolation function of $f(x)$.

But r_0 and n cannot be arbitrary nonnegative integers. As $A\cup B$ is a symmetric Pólya set, by Lemma 2 we have

$$A\cup B\prec J^*. \tag{7}$$

Because $A\cap B=\emptyset$, (7) is equivalent to

$$A\prec J_1^*=\{0,\ 2,\ 4,\ \cdots,\ 2(r_0+1)\},\ B\prec J_2^*=\{2(r_0+2),\ 2(r_0+4),\ \cdots,\ 2n\}$$

and also $\quad B=\{j_1,\ j_2,\ \cdots,\ j_{n-r_0-1}\}\subset\{2n-r_0,\ 2n-r_0+1,\ \cdots,\ 2n\}$.

So, to maintain (7), we must have $2n-r_0\leqslant j_1\leqslant 2(r_0+2)$, that is

$$n\leqslant\frac{3}{2}r_0+2. \tag{8}$$

Furthermore, when $|B|=0,\ |C|=0$, we have $n=r_0+1,\ s\in S_p(2n+1,\ n+1,\ \Delta_N)$, and the spline (6) is exactly Hermite interpolation function. This is well known (cf. [10]). So we consider only $|B|>0,\ |C|>0$. This means

$$n>r_0+1. \tag{9}$$

To sum up (8), (9), we get the relation between r_0 and n:

$$r_0+1<n\leqslant\frac{3}{2}r_0+2. \tag{10}$$

Let $J=A\cup B$ in (5). From (6) we have

$$
\begin{aligned}
s^r(x)=&\sum_{j\in A}h_{i+1}^{j-r}\Big\{f^j(x_i)g_j^r\Big(\frac{x-x_i}{h_{i+1}}\Big)+f^j(x_{i+1})h_j^r\Big(\frac{x-x_i}{h_{i+1}}\Big)\Big\}\\
&+\sum_{j\in B}h_{i+1}^{j-r}\Big\{s^j(x_i+)g_j^r\Big(\frac{x-x_i}{h_{i+1}}\Big)+s^j(x_{i+1}-)h_j^r\Big(\frac{x-x_i}{h_{i+1}}\Big)\Big\}.
\end{aligned}
\tag{11}
$$

So we need only determine $s^j(x_i\pm)$, $j\in B$, $0\leqslant i\leqslant N$, to get $s(x)$ uniquely, and this can be done according to continuous properties and boundary conditions. In the following, we concentrate our attention on the existence and uniqueness of this kind of splines. Assume that Δ_N is a uniform partition with $x_{i+1}-x_i\equiv h$.

By virtue of (4), we have

$$
\begin{aligned}
&g_j(x)=(-1)^j h_j(1-x).\\
&g_j^r(0)=(-1)^{j+r}h_j^r(1),\quad g_j^r(1)=(-1)^{j+r}h_j^r(0),
\end{aligned}
\tag{12}
$$

and then from (11), $s^r(x_i+)=s^r(x_i-)$, $1\leqslant i\leqslant N-1$, $r\in C$, yields the following equation:

$$
\begin{aligned}
&\sum_{j\in B}h^{j-r}\{[(-1)^{r+j}s^j(x_{i+1}-)-s^j(x_{i-1}+)]g_j^r(1)+[(-1)^{j+r}s^j(x_i+)-s^j(x_i-)]h_j^r(1)\}\\
&\quad=\sum_{j\in A}h^{j-r}\{f^j(x_{i-1})g_j^r(1)+[1-(-1)^{j+r}]f^j(x_i)h_j^r(1)-(-1)^{j+r}f^j(x_{i+1})g_j^r(1)\}.
\end{aligned}
\tag{13}
$$

Now let $r\in C$ be always even or always odd (since under condition (10), $|C|=2n-2r_0-2\leqslant r_0+2=|A|$, there is no question about this assumption according as $A\cup C=\{0,\ 1,\ 2,\ \cdots,\ 2n-r_0-1\}$). For each i, (13) can be regarded as a system about $[(-1)^{s+j}s^j(x_{i+1}-)-s^j(x_{i-1}+)]$ and $[(-1)^{s+j}s^j(x_i+)-s^j(x_i-)]$, $j\in B$, where $s=0$ or 1. Let $B=(l_1, l_2, \cdots, l_k)$, $C=(r_1, r_2, \cdots, r_k)$. The coefficient matrix of (13) is

$$M=\begin{bmatrix} g_{l_1}^{r_1}(1) & h_{l_1}^{r_1}(1) & g_{l_2}^{r_1}(1) & h_{l_2}^{r_1}(1) & \cdots & g_{l_k}^{r_1}(1) & h_{l_k}^{r_1}(1) \\ g_{l_1}^{r_2}(1) & h_{l_1}^{r_2}(1) & g_{l_2}^{r_2}(1) & h_{l_2}^{r_2}(1) & \cdots & g_{l_k}^{r_2}(1) & h_{l_k}^{r_2}(1) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ g_{l_1}^{r_{2k}}(1) & h_{l_1}^{r_{2k}}(1) & g_{l_2}^{r_{2k}}(1) & h_{l_2}^{r_{2k}}(1) & \cdots & g_{l_k}^{r_{2k}}(1) & h_{l_k}^{r_{2k}}(1) \end{bmatrix}. \tag{14}$$

Lemma 3. *Let A, B, C, g_j, h_j be defined as above. Then M is a nonsingular matrix whose elements do not depend on h.*

Proof. Let $\{\phi_s\}_{s\in C}$ denote the row vectors of M. It suffices to show that $\{\phi_s\}_{s\in C}$ are linearly independent. Assume $\sum_{s\in C} e_s\phi_s=0$. This can be written as

$$\sum_{s\in C} e_s g_l^s(1)=0,\quad \sum_{s\in C} e_s h_l^s(1)=0,\quad l\in B,$$

and this is equivalent to

$$\sum e_s p^s(1)=0,$$

where $P(x)=\sum_{l\in B}[\alpha_l g_l(x)+\beta_l h_l(x)]$ and $\{\alpha_l\}_{l\in B}$, $\{\beta_l\}_{l\in B}$ are arbitrary real data. So it suffices to prove that there exists a polynomial $P_m(x)$ of degree $2n+1$ such that $P_m^k(1)=\delta_{km}$, $k\in C$, for each $m\in C$. But P_m relates to $\{g_l, h_l\}$ only. By (4) and $A\cap B=\phi$, one can easily show that

$$P_m^i(1)=0,\ P_m^i(0)=0,\quad i\in A.$$

So the $H-B$ matrix $E=(e_{ij})_{i=0}^{1}{}_{j=0}^{2n+1}$ satisfies

$$e_{0j}=\begin{cases}1, & j\in A\\ 0, & \text{otherwise,}\end{cases}\qquad e_{1j}=\begin{cases}1, & j\in A\cup C,\\ 0, & \text{otherwise.}\end{cases}$$

From the restrictions put on A, B and C, we can easily verify that Pólya conditions (1) hold for $0\leqslant p\leqslant 2n+1$. Therefore, there is a P_m having the desired properties by Lemma 1. The proof is completed. Q. E. D.

By virtue of Lemma 3, we can solve from (13) the unknowns

$$\begin{aligned}&[(-1)^{s+j}s^j(x_{i+1}-)-s^j(x_{i-1}+)],\\ &[(-1)^{s+j}s^j(x_i+)-s^j(x_i-)],\quad i=1, 2, \cdots, N-1,\ j\in B,\end{aligned} \tag{15}$$

and then put boundary conditions (6iii) into (11). It follows that

$$\begin{aligned}&\sum_{j\in B} h^{j-r}\{s^j(x_0+)g_j^r(0)+s^j(x_1-)h_j^r(0)\}\\ &\quad =f^r(x_0)-\sum_{j\in A} h^{j-r}\{f^j(x_0)g_j^r(0)+f^j(x_1)h_j^r(0)\},\ r\in C.\end{aligned} \tag{16}$$

Noticing (12), by virtue of Lemma 3, we obtain from (16)

$$s^j(x_0+)\text{ and } s^j(x_1-),\quad j\in B.$$

Then for each $j\in B$, we can easily solve every $s^j(x_i+)$ and $s^j(x_i-)$, $0<i<N$, from (15). So we have determined spline (11) uniquely, and got the information about the algorithm. Thus, we have proved

Theorem 1. *Let $r_0+1\leqslant n\leqslant\frac{3}{2}r_0+2$, and let Δ_N be a uniform partition of $[a, b]$. Let A, B and C be disjoint sets such that $|A|=r_0+2$, $|B|=n-r_0-1$, $|C|=2|B|$, $A\cup C=\{0, 1, 2, \cdots, 2n-r_0-1\}$, $B\subset\{2n-r_0, 2n-r_0+1, \cdots, 2n\}$. Let $A\cup B$*

be a symmetric Pólya set and assume that there are only even or odd numbers in C. Then, given arbitrary real data $\{f_{ij}: j\in A,\ 0\le i\le N\}$, $\{g_j: j\in C\}$, there is a unique element $s\in S_p(2n+1,\ 2n-r_0,\ \Delta_N)$ satisfying

$$s^j(x_i)=f_{ij},\quad j\in A,\ 0\le i\le N,$$
$$s^j(x_0)=g_j,\quad j\in C.$$

3. Order of Convergence and Saturation of Approximation

Let $S_{n1}(x)\equiv S_{n1}(f;x)$ denote the Hermite interpolation function of degree $2n+1$ on Δ_N, which satisfies

(i) $S_{n1}(x)$ is a polynomial of degree not exceeding $2n+1$ when restricted on $[x_i, x_{i+1}]$,

(ii) $S_{n1}^p(x_i)=f^p(x_i)$, $p=0, 1, 2, \cdots, n$.

Theorem 2[10]. *Let $f(x)\in C^s[0, 1]$ $(n\le s\le 2n+1)$. Then for $q\le s$,*

$$\|S_{n1}^q(f;x)-f^q(x)\|=O(h^{s-q}\omega(f^s;h)).$$

If $f(x)\in C^{2n+2}[0, 1]$, then

$$f^q(x)-S_{n1}^q(f;x)=h^{2n+2-q}f^{2n+2}(x_i)R^q\Big(\frac{x-x_i}{h}\Big)+O(h^{2n+2-q}\omega(f^{2n+2};h)), \tag{17}$$

where $R(x)=x^{n+1}(x-1)^{n+1}/(2n+2)!$.

From this theorem, we can derive a result about the H–B interpolation spline $H(f;x)$, with

$$H(f;x)\equiv\sum_{j\in A\cup B}h^j\Big[f^j(x_i)g_j\Big(\frac{x-x_i}{h}\Big)+f^j(x_{i+1})h_j\Big(\frac{x-x_i}{h}\Big)\Big], \tag{18}$$

$x_i\le x\le x_{i+1}$, $0\le i\le N-1$, $g_j(x)$ and $h_j(x)$ as above. The symbol $\max B$ denotes the maximum element in B.

Lemma 4. *Let $f(x)\in C^s[0, 1]$, $\max B\le s\le 2n+1$. Then for $q\le s$,*

$$\|H^q(f;x)-f^q(x)\|=O(h^{s-q}\omega(f^s;h)) \tag{19}$$

when $s=2n+2$,

$$f^q(x)-H^q(f;x)=h^{2n+2-q}Q^q\Big(\frac{x-x_i}{h}\Big)f^{2n+2}(x_i)+O(h^{2n+2-q}\omega(f^{2n+2};h)), \tag{20}$$

where
$$Q(x)=R(x)-\sum_{j\in A\cup B}[R^j(0)g_j(x)+R^j(1)h_j(x)]h^q.$$

Proof. From the obvious equality

$$H(f;x)=S_{n1}(f;x)+\sum_{j\in A\cup B}h^j\Big\{[f^j(x_i)-S_{n1}^j(x_i+)]g_j\Big(\frac{x-x_i}{h}\Big)+[f^j(x_{i+1})-S_{n1}^j(x_{i+1}-)]h_j\Big(\frac{x-x_i}{h}\Big)\Big\}, \tag{21}$$

using Theorem 2, we get

$$\|H^q(f;x)-f^q(x)\|=\|S_{n1}^q(f;x)-f^q(x)\|+O(h^{s-q}\omega(f^s;h))=O(h^{s-q}\omega(f^s;h))$$

for $\max B\le s\le 2n+1$. When $s=2n+2$, it is only necessary to put (17) into (21) to prove (20). Q. E. D.

Theorem 3. *Let A, B, C and $s(x)\equiv s(f;x)$ be the same as in Theorem 1. If $f(x)\in C^s[0, 1]$, $\max B\le s<2n+1$, then*

$$\|s^q(f;x)-f^q(x)\|=O(h^{s-1-q}\omega(f^s;h)),\ q\leqslant s. \tag{22}$$

If $f(x)\in C^{2n+2}[0,1]$, then there exists a polynomial $C(x)$ of degree $2n+1$ such that

$$f^q(x)=s^q(f;x)+C^q\left(\frac{x-x_i}{h}\right)[f^{2n+1}(x_i)-f^{2n+1}(x_0)]h^{2n+1-q}$$
$$+O(h^{2n+1-q}\omega(f^{2n+2};h)) \tag{23}$$

for $x\in[x_i, x_{i+1}]$.

Proof. By virtue of (13), it follows that

$$\sum_{j\in B}h^j\{[(-1)^{r+j}(s^j(x_{i+1}-)-f^j(x_{i+1}))-(s^j(x_{i-1}+)-f^j(x_{i-1}))]g_j^r(1)$$
$$+[(-1)^{r+j}(s^j(x_i+)-f^j(x_i))-(s^j(x_i-)-f^j(x_i))]h_j^r(1)\}$$
$$=h^r[H^r(f;x_i-)-H^r(f;x_i+)],\quad r\in C,\ i=1,2,3,\cdots,N-1. \tag{24}$$

Under the restriction on r, we can solve

$$\begin{cases}(-1)^{s+j}[s^j(x_{i+1}-)-f^j(x_{i+1})]-[s^j(x_{i-1}+)-f^j(x_{i-1})]=O(h^{s-j}\omega(f^s;h)),\\(-1)^{s+j}[s^j(x_i+)-f^j(x_i)]-[s^j(x_i-)-f^j(x_i)]=O(h^{s-j}\omega(f^s;h)),\\ \qquad j\in B,\ i=1,2,\cdots,N-1\end{cases} \tag{25}$$

from (13) by Lemma 3 as above. And it follows from (16) that

$$\sum_{j\in B}h^j\{[s^j(x_0+)-f^j(x_0)]g_j^r(0)+[s^j(x_1-)-f^j(x_1)]h_j^r(0)\}$$
$$=h^r\{f^r(x_i)-\sum_{j\in A\cup B}h^{j-r}[f^j(x_0)g_j^r(0)+f^j(x_1)h_j^r(0)]\}$$
$$=h^r[f^r(x_0)-H^r(f;x_0+)]=O(h^s\omega(f^s;h)).$$

We can obtain as above

$$\begin{cases}s^j(x_0+)-f^j(x_0)=O(h^{s-j}\omega(f^s;h)),\\ s^j(x_1-)-f^j(x_1)=O(h^{s-j}\omega(f^s;h)).\end{cases} \tag{26}$$

From (25), (26), it is easy to get

$$s^j(x_i\pm)-f^j(x_i)=O(h^{s-j-1}\omega(f^s;h))$$

for all i. Then by

$$\|s^q(f;x)-f^q(x)\|\leqslant\|s^q(f;x)-H^q(f;x)\|+\|H^q(f;x)-f^q(x)\|,$$

we can show that the conclusion holds for $\max B\leqslant s\leqslant 2n+1$ from Lemma 4. When $f(x)\in C^{2n+2}[0,1]$, putting (20) of Lemma 4 into (24) yields

$$\sum_{j\in B}h^j\{[(-1)^{r+j}(s^j(x_{i+1}-)-f^j(x_{i+1}))-(s^j(x_{i-1}+)-f^j(x_{i-1}))]g_j^r(1)$$
$$+[(-1)^{r+j}(s^j(x_i+)-f^j(x_i))-(s^j(x_i-)-f^j(x_i))]h_j^r(1)\}$$
$$=h^{2n+2}f^{2n+2}(x_i)[Q^r(1)-Q^r(0)]+O(h^{2n+2}\omega(f^{2n+2};h)),$$
$$i=1,2,\cdots,N-1,\ r\in C, \tag{27}$$

and from the boundary conditions we have

$$\sum_{j\in B}h^j\{[s^j(x_0+)-f^j(x_0)]g_j^r(0)+[s^j(x_1-)-f^j(x_1)]h_j^r(0)\}$$
$$=h^{2n+2}Q^r(0)f^{2r+2}(x_0)+O(h^{2r+2}\omega(f^{2n+2};h)). \tag{28}$$

We can find from (27), (28) as above that there exist two constants C_{ij}^0, C_{ij}^1 for every i, such that

$$\begin{cases}[(-1)^{s+j}(s^j(x_{i+1}-)-f^j(x_{i+1}))-(s^j(x_{i-1}+)-f^j(x_{i-1}))]\\ \qquad =C^0_{ij}f^{2n+2}(x_i)h^{2n+2-j}+O(h^{2n+2-j}\omega(f^{2n+2};\ h)),\\ [(-1)^{s+j}(s^j(x_i+)-f^j(x_i))-(s^j(x_i-)-f^j(x_i))]\\ \qquad =C^1_{ij}f^{2n+2}(x_i)h^{2n+2-j}+O(h^{2n+2-j}\omega(f^{2n+2};\ h));\end{cases}$$

$$\begin{cases}s^j(x_0+)-f^j(x_0)=C^0_{0j}f^{2n+2}(x_0)+O(h^{2n+2-j}\omega(f^{2n+2};\ h)),\\ s^j(x_1-)-f^j(x_1)=C^1_{1j}f^{2n+2}(x_1)+O(h^{2n+2-j}\omega(f^{2n+2};\ h)).\end{cases}$$

Subtracting a certain polynomial of degree $2n+1$, we can suppose that $f^{2n+2}(0)=0$. So there are proper constants C_{j0}, C_{j1} such that

$$\begin{aligned}s^j(x_i+)-f^j(x_i)&=C_{j0}h^{2n+2-j}\sum_{v=0}^{i}f^{2n+2}(x_v)+O(h^{2n+1-j}\omega(f^{2n+2};\ h))\\ &=C_{j0}h^{2n+1-j}\int_0^{i}f^{2n+2}(x)dx+O(h^{2n+1-j}\omega(f^{2n+2};\ h))\\ &=C_{j0}h^{2n+1-j}[f^{2n+1}(x_i)-f^{2n+1}(x_0)]+O(h^{2n+1-j}\omega(f^{2n+2};\ h)),\end{aligned}$$

$$s^j(x_i-)-f^j(x_i)=C_{j1}h^{2n+1-j}[f^{2n+1}(x_i)-f^{2n+1}(x_0)]+O(h^{2n+1-j}\omega(f^{2n+2};\ h)).$$

Noticing that

$$\begin{aligned}f^q(x)-s^q(f;\ x)&=f^q(x)-H^q(f;\ x)+H^q(f;\ x)-s^q(f;\ x)\\ &=h^{2n+1-q}f^{2n+2}(x_i)Q^q\left(\frac{x-x_i}{h}\right)+h^{2n+1-q}[f^{2n+1}(x_i)-f^{2n+1}(x_0)]\\ &\quad\times\left(\sum_{j\in A\cup B}C_{j1}g^q_j\left(\frac{x-x_i}{h}\right)\right)+h^{2n+1-q}[f^{2n+1}(x_{i+1})-f^{2n+1}(x_0)]\\ &\quad\times\left(\sum_{j\in A\cup B}C_{j2}h^q_j\left(\frac{x-x_i}{h}\right)\right)+O(h^{2n+1-q}\omega(f^{2n+2};\ h)),\end{aligned}$$

using Lemma 4 once more we find that there exists a polynomial $C(x)$ of degree $2n+1$ such that (23) holds. Q. E. D.

Remark. The general method in [6] is used in the proof of Theorem 3.

Corollary. Let $f(x)\in C^{2n+2}[0,\ 1]$. If

$$\|f^q(x)-s^q(f;\ x)\|=o(h^{2n+1-q})$$

holds for some q, then $f(x)$ is a polynomial of degree not exceeding $2n+1$.

The results of Theorem 1 can be extended to interpolation of less smooth functions with the method used in [2]. Firstly we introduce a conclusion of [11].

Theorem 4[11]. *Given $f(x)\in C^s[0,\ 1]$, $0\leqslant s\leqslant 2n+2$, let Δ_N be a uniform partition on $[a,\ b]$. Then there exists a unique element $g(x)\in S_p(2(2n+2)+1,\ 2n+2,\ \Delta_N)$ satisfying*

(i) $g^j(x_i)=f^j(x_i)$, $0\leqslant j\leqslant s$, $0\leqslant i\leqslant N$,

(ii) $g^j(x_i)=0$, $s<j\leqslant 2n+2$, $0\leqslant i\leqslant N$.

Furthermore, there exists a constant K independent of f and N such that

$$Kh^{s-j}\omega(f^s;\ h)\geqslant\begin{cases}\|f^j-g^j\| & 0\leqslant j\leqslant s,\\ \|g^j\| & s<j\leqslant 2n+2.\end{cases}$$

Theorem 5. *Let $f(x)\in C^s[0,\ 1]$, $2n-r_0\leqslant s\leqslant 2n+1$, and let $s(x)$ be the same as in Theorem 1. Then*

$$\|f^q-s^q\|=O(h^{s-1-q}\omega(f^s;\ h)),\quad 0\leqslant q\leqslant s.$$

Proof. Let g be the same as in Theorem 4. By the triangle inequality

$$\|f^q-s^q\|\leqslant\|f^q-g^q\|+\|g^q-s^q\|,$$

the order of the first term on the right side of the above inequality is $O(h^{s-q}\omega(f^s;h))$ by Theorem 4. The order of the second term on the right can be given from Theorem 3 as

$$\|g^q-s^q\|=O(h^{2n-q}\omega(g^{2n+1};h)).$$

Again with the help of Theorem 4, we have

$$\omega(g^{2n+1};h)=O(h^{s-2n-1}\omega(f^s;h)).$$

So

$$\|s^q-g^q\|=O(h^{s-1-q}\omega(f^s;h)). \qquad \text{Q. E. D.}$$

To sum up, we have completely solved the problem of existence, uniqueness, order of convergence and saturation of approximation of this kind of lacunary splines.

4. Special Cases

For special cases of the lacunary splines just discussed, first let $r_0=0$. We get $1<n\leqslant 2$ from (10). So $n=2$ and $s\in S_p(5,4,\Delta_N)$ and also $|A|=2$, $|B|=1$, $A\cup C=\{0,1,2,3\}$, $B=\{4\}$. But by Lemma 2 and the assumption that C consists of only even or only odd numbers, we have

$$A=\{0,2\},\ C=\{1,3\},\ B=\{4\}.$$

Therefore it is the quintic lacunary spline discussed by A. Meir and A. Sharma in [1].

Next, let $n=\frac{3}{2}(r_0+1)$ and $A\cup B=J^*$. We find that we have the lacunary splines in [5], [6]; but neither of the works got the existence and uniqueness for the general case.

Let $r_0=n-k$. Then $k>1$, $n\geqslant 3k-4$ from (10). We list some lacunary splines discussed for smaller n in Table 1, and use symbol (u, v) instead of $S_p(u, v, \Delta_N)$.

Table 1

k \ n	2	3	4	5	6	7	8	9 …
2	(5, 4)	(7, 5)	(9, 6)	(11, 7)	(13, 8)	(15, 9)	(17, 10)	(19, 11) …
3				(11, 8)	(13, 9)	(15, 10)	(17, 11)	(19, 12) …
4							(17, 12)	(19, 13) …
⋮								

It should be pointed out that there is not only one spline in one kind of lacunary spline listed above. For example, for $k=3$, $n\geqslant 5$, let $n=7$. We have $s\in S_p(15,10,\Delta_N)$, and $|A|=6$, $|B|=2$, $|C|=4$, $A\cup C=\{0,1,2,3,4,5,6,7,8,9\}$, $B\subset\{10,11,12,13,14\}$. Then, if C consists of any four elements of $\{1,3,5,7,9\}$ or $C=\{2,4,6,8\}$, and if $B=\{b_1,b_2\}\prec\{12,14\}$, then all conditions in Theorem 1 are satisfied. Thus we have a unique spline and the order of convergence, which are given by Theorems 3, 5.

Remark. We have proved corresponding results about splines of even order, which we will disouss in another paper.

Acknowledgement. The author wishes to thank Professor Xiong Zhen-xiang for her support and kind advice.

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