CARDINALITIES OF RESTRICTED RANGES*

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I

Abstract

Let l and u be upper and lower semicontinuous extended functions on $[a, b]$, respectively, with $l\leqslant u$. Let H be an n-dimensional Haar subspace and $K=\{p\in H: l\leqslant p\leqslant u\}$. This paper gives complete characterizations of K satisfying

$$\operatorname{card} K=0 \text{ or } 1 \text{ or } \infty$$

under certain assumptions, where card K denotes the cardinality of K.

1. Introduction

In approximation by polynomials having restricted ranges[1] and in simultaneous approximation[2] the following problem may be proposed:

Let l and u be upper and lower semicontinuous functions on $X\equiv[a, b]$ (which may take $-\infty$ and $+\infty$, but $l<+\infty$ and $u>-\infty$), respectively, with $l\leqslant u$. Let H be an n-dimensional subspace of $C(X)$ and $K=\{p\in H: l\leqslant p\leqslant u\}$. Characterize K such that

$$\operatorname{card} K=0 \text{ or } 1 \text{ or } \infty,$$

where card K denotes the cardinality of K.

In this paper we give an answer to this problem for H being a Haar subspace. In detail, we give complete characterizations of K satisfying card $K=0$ or 1 or ∞ under certain assumptions.

To begin with let us introduce the following notation.

For $p\in H$ denote

$$X_p^l=\{x\in X: p(x)\leqslant l(x)\},$$
$$X_p^u=\{x\in X: p(x)\geqslant u(x)\},$$
$$X_p=X_p^l\cup X_p^u,$$
$$\sigma(x)=\begin{cases} 1, & x\in X_p^l \\ -1, & x\in X_p^u. \end{cases}$$

By definition if $p(x)=l(x)=u(x)$, $\sigma(x)$ may take both 1 and -1.

A system of $n+1$ ordered points

$$x_1<x_2<\cdots<x_{n+1} \tag{1}$$

in X_p is said to be an alternation system of p (with respect to (l, u)) if it satisfies

$$\sigma(x_{i+1})=-\sigma(x_i), \quad i=1, \cdots, n. \tag{2}$$

It should be pointed out that the restrictions on l and u being upper and lower

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semicontinuous are trivial, because for any l and u we can assume

$$\bar{l}(x)=\limsup_{y\to x} l(y), \qquad \bar{u}(x)=\liminf_{y\to x} u(y)$$

instead, which are upper and lower semicontinuous, respectively[5], and satisfy that

$$\text{card } K=\text{card}\{p\in H:\bar{l}\leqslant p\leqslant\bar{u}\},$$

To verify the last equality we note that, on the one hand, from $l\leqslant\bar{l}\leqslant\bar{u}\leqslant u$ card $K\geqslant$card$\{p\in H:\bar{l}\leqslant p\leqslant\bar{u}\}$ follows, and on the other hand, $l\leqslant p\leqslant u$ implies

$$\limsup_{y\to x} l(y)\leqslant\limsup_{y\to x} p(y)=\liminf_{y\to x} p(y)\leqslant\liminf_{y\to x} u(y),$$

namely, $\bar{l}(x)\leqslant p(x)\leqslant\bar{u}(x)$, from which card $K\leqslant$card$\{p\in H:\bar{l}\leqslant p\leqslant\bar{u}\}$ follows.

2. Main Theorems

Theorem 1. *Let $l<u$ and let H be an n-dimensional Haar subspace. Then for $p\in K$ the following statements are equivalent each to other:*

(a) $K=\{p\}$, *i.e.*, card $K=1$;

(b) $\max_{x\in X_p}\sigma(x)q(x)\geqslant 0,\ \forall q\in H$;

(c) $\max_{x\in X_p}\sigma(x)q(x)>0,\ \forall q\in H\setminus\{0\}$;

(d) $0\in\mathscr{H}\{\sigma(x)\hat{x}:x\in X_p\}$, *where $\mathscr{H}$ denotes the convex hull* [4, p. 17] *and* $\hat{x}=(\emptyset_1(x),\cdots,\emptyset_n(x))$ *with $\emptyset_1,\cdots,\emptyset_n$ being a basis in H*;

(e) *p possesses an alternation system with respect to (l, u).*

Proof. (a)$\Rightarrow$(b). Suppose not and let q satisfy $\max_{x\in X_p}\sigma(x)q(x)<0$, i. e., $\sigma(x)q(x)<0,\ \forall x\in X_p$. We are to prove that $r_t=p-tq$ satisfies $l<r_t<u$ for some $t>0$. Hence from $r_t\neq p$ a contradiction occurs.

Let $h=\frac{1}{2}\min_{x\in X}\{u(x)-l(x)\}(>0)$ and $e=\max_{x\in X}|q(x)|$. Denote

$$Y_1=\{x\in X:p(x)-l(x)>h \text{ and } q(x)>0\},$$

$$Y_2=\{x\in X:u(x)-p(x)>h \text{ and } q(x)<0\},$$

$$Y=X\setminus(Y_1\cup Y_2).$$

Taking $t_1=h/e$, we have that for $x\in Y_1$ and $0<t\leqslant t_1$

$$r_t(x)=p(x)-tq(x)>l(x)+h-t_1e=l(x)$$

and

$$r_t(x)=p(x)-tq(x)\leqslant u(x)-tq(x)<u(x),$$

that is,

$$l(x)<r_t(x)<u(x). \tag{3}$$

Similarly, (3) holds for $x\in Y_2$ and $0<t\leqslant t_1$.

On the other hand, it is easy to see that $X_p\subset Y_1\cup Y_2$ and, hence,

$$l(x)<p(x)<u(x),\quad \forall x\in Y.$$

Since Y is compact, we can find a number $t_2>0$ so that (3) is also valid for all $x\in Y$ and $0<t\leqslant t_2$.

Thus $l<r_t<u$ is valid for $t=\min\{t_1, t_2\}$.

(b)$\Rightarrow$(d). (b) implies that the linear inequalities

$$\max_{x\in X_p} \sigma(x)q(x)<0$$

or

$$\sigma(x)q(x)<0, \quad x\in X_p \tag{4}$$

is inconsistent in H. (4) may be rewritten as

$$\langle c, \sigma(x)\hat{x}\rangle<0, \quad x\in X_p,$$

where $q(x)=\sum_{i=1}^{n} c_i\phi_i(x)$ and $c=(c_1, \cdots, c_n)$. Noting that X_p is compact, by Theorem on Linear Inequalities [4, p. 19] $0\in\mathscr{H}\{\sigma(x)\hat{x}:x\in X_p\}$.

(d)⇒(e). This is a standard statement and, for example, it may be found in [4, p. 75].

(e)⇒(c). Suppose on the contrary that $\max_{x\in X_p} \sigma(x)q(x)\leqslant 0$ or

$$\sigma(x)q(x)\leqslant 0, \quad \forall x\in X_p \tag{5}$$

for some $q\in H\setminus\{0\}$. Let p have an alternation system (1) with respect to (l, u). From (2) and (5) it follows that

$$(-1)^i\sigma(x_1)q(x_i)\geqslant 0, \quad i=1, 2, \cdots, n+1.$$

Then $q=0$ [3, Lemma 3], a contradiction.

(c)⇒(a). If possible, let $r\in K\setminus\{p\}$. From $l\leqslant r\leqslant u$ it follows that

$$p(x)-r(x)\leqslant l(x)-r(x)\leqslant 0, \quad \forall x\in X_p^l,$$
$$p(x)-r(x)\geqslant u(x)-r(x)\geqslant 0, \quad \forall x\in X_p^u.$$

This means that

$$\sigma(x)(p(x)-r(x))\leqslant 0, \quad \forall x\in X_p$$

or

$$\max_{x\in X_p} \sigma(x)(p(x)-r(x))\leqslant 0.$$

But $q=p-r\neq 0$ and this is a contradiction.

The proof of the theorem is completed.

Remark 1. If $p\in K$ does not have an alternation system, by Theorem 1 we may conclude that $\max_{x\in X_p} \sigma(x)q(x)<0$ will be valid for some $q\in H$. According to the proof of (a)⇒(b) the function $r_t=p-tq$ will satisfy $l<r_t<u$ for some number $t>0$. Whence $l+s\leqslant r_t\leqslant u-s$ will hold for some number $s>0$.

Remark 2. The assumption of $l<u$ could not be deleted. It may be supported by the following.

Example 1. Let $X=[-1, 1]$, $l=-x^2$, $u=x^2$ and $H=\text{span}\{1, x\}$. Clearly $K=\{0\}$ but the function 0 does not have an alternation system.

We turn now to characterizing K for which card $K=0$. To the end we establish next a preliminary result.

Lemma. *Let l and u be bounded with $l\leqslant u$ and H be an n-dimensional Haar subspace. If $K=\emptyset$, then there exists a number $d>0$ such that*

$$\text{card } K_t=\begin{cases} 0, & t<d, \\ 1, & t=d, \\ \infty, & t>d, \end{cases}$$

where $K_t=\{p\in H:l-t\leqslant p\leqslant u+t\}$.

Proof. It is easy to see that $K_t \neq \emptyset$ for t large enough. Set

$$d = \inf\{t \geqslant 0 : K_t \neq \emptyset\}.$$

We claim first that $K_d \neq \emptyset$. In fact, let $p_m \in K_{d+1/m}$, $m = 1, 2, \cdots$. Clearly $\{p_m\}$ is bounded and possesses a convergent subsequence. Suppose without loss of generality that $p_m \rightrightarrows p$, $m \to \infty$. We now show that $p \in K_d$. If not, we can find a point x such that, say, $p(x) > u(x) + d$. Thus $p_m(x) > u(x) + d + 1/m$ is true for m large enough, and hence $p_m \overline{\in} K_{d+1/m}$. This contradiction proves $p \in K_d$ and $K_d \neq \emptyset$.

Now $K_d \neq \emptyset$ imples that $d > 0$. p must possess an alternation system with respect to $(l-d, u+d)$, because, otherwise, according to Remark 1 to Theorem 1 $K_{d-\varepsilon} \neq \emptyset$ for $\varepsilon > 0$ small enough and it will contradict the definition of d. By Theorem 1 $K_d = \{p\}$ and card $K_d = 1$. By the definition of d we obtain that card $K_t = 0$ for $t < d$ and by an observation we conclude that card $K_t = \infty$ for $t > d$.

It is to ask whether or not the assumption of boundness of l and u may be deleted? Unfortunately, the answer is negative. Let us give an example to show it.

Example 2. Let $[a, b] \equiv [-1, 1]$, $H = \text{span}\{1, x\}$,

$$l = \begin{cases} -\infty, & -1 \leqslant x < 0, \\ \sqrt{x}, & 0 \leqslant x \leqslant 1 \end{cases}$$

and

$$u = \begin{cases} \sqrt{-x}, & -1 \leqslant x \leqslant 0, \\ \infty, & 0 < x \leqslant 1. \end{cases}$$

It is easy to verify $K = \emptyset$. In fact, for $p = c_1 + c_2 x$ from $l(0) = u(0) = 0$ it follows that $c_1 = 0$. But for any c_2 the inequality $c_2 x \geqslant \sqrt{x}$ with $0 \leqslant x \leqslant 1$ could not be valid.

On the other hand, for any $t > 0$ card $K_t = \text{card}\{p \in H : l - t \leqslant p \leqslant u + t\} = \infty$.

This shows that there does not exist such a number d.

Theorem 2. *Let l and u be bounded with $l \leqslant u$ and let H be an n-dimensional Haar subspace. Then* card $K = 0$ *if and only if there exists a $p \in H \backslash K$ which possesses an alternation system with respect to (l, u).*

Proof. Necessity. By lemma there exists a number $d > 0$ such that card $K_d = 1$, where $K_d = \{p \in H : l - d \leqslant p \leqslant u + d\}$.

Let $p \in K_d$. Then $p \overline{\in} K$. Moreover, by Theorem 1 p has an alternation system with respect to $(l-d, u+d)$. This alternation system is, of course, one of p with respect to (l, u).

Sufficiency. If the conclusion is false, suppose $q \in K$. Let (1) be an alternation system of p with respect to (l, u). Then we have that for $\sigma(x_1) = 1$

$$p(x_{2j-1}) - q(x_{2j-1}) \leqslant l(x_{2j-1}) - q(x_{2j-1}) \leqslant 0, \quad j = 1, \cdots, \left[\frac{n+2}{2}\right]$$

and

$$p(x_{2j}) - q(x_{2j}) \geqslant u(x_{2j}) - q(x_{2j}) \geqslant 0, \quad j = 1, \cdots, \left[\frac{n+1}{2}\right].$$

Hence

$$(-1)^i (p(x_i) - q(x_i)) \geqslant 0, \quad i = 1, 2, \cdots, n+1.$$

This implies $p = q$. The same conclusion may be deduced for $\sigma(x_1) = -1$. But it is impossible, because $p \overline{\in} K$ and $q \in K$.

Combining Theorem 1 and Theorem 2 immediately gives

Theorem 3. *Let l and u be bounded with $l < u$ and let H be an n-dimensional Haar subspace. Then* card $K \leqslant 1$ *if and only if there exists a $p \in H$ which possesses an alternation system with respect to (l, u).*

As an equivalent proposition to Theorem 3 a characterization for card $K=\infty$ easily yields.

Theorem 4. *Under the assumptions of Theorem* 3 card $K=\infty$ *if and only if there does not exist a* $p\in H$ *which possesses an alternation system with respect to* (l, u).

Finally we present another characterization for card $K=\infty$ as a corollary to Theorem 1.

Theorem 5. *Under the assumptions of Theorem* 1 card $K=\infty$ *if and only if there exists a* $p\in K$ *which does not have an alternation system with respect to* (l, u).

References

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II*

Abstract

Let l and $-u$ be upper semicontinuous bounded functions on $[a, b]$ with $l\leqslant u$. For H being an n-dimensional subspace of $C[a, b]$ let $K=\{p\in H: l\leqslant p\leqslant u\}$. This paper characterizes K for which

card $K=0$ or 1 or ∞,

where card K denotes the cardinality of K.

In (I) we have given an answer to the following problem in the case when H is a Haar subspace:

Let l and u be upper and lower semicontinuous functions on $X\equiv[a, b]$ (which may take $-\infty$ and $+\infty$, but $l<+\infty$ and $u>-\infty$), respectively, with $l\leqslant u$. Let H be an n-dimensional subspace of $C(X)$ and $K=\{p\in H: l\leqslant p\leqslant u\}$. Characterize K such that

card $K=0$ or 1 or ∞,

where card K denotes the cardinality of K.

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In this paper we continue to investigate this problem in a general setting, where H is not necessarily Haar.

The method used in this paper is to reduce this problem to a certain one of minimization. To this end, assume that l and u are bounded through this paper and consider a function with variables x and y

$$F(x, y)=|y-v(x)|+\frac{l(x)-u(x)}{2}+d, \tag{1}$$

where $v=\frac{1}{2}(l+u)$ and $d=\frac{\|u-l\|}{2}$ with the "sup" norm.

Some important properties of F are as follows.

Lemma. *F is nonnegative and convex with respect to y. Furthermore, $F(x, q) \equiv F(x, q(x))$ is upper semicontinuous with respect to x for any $q\in H$.*

Proof. The proof of the first two claims is trivial. The last claim follows from another equivalent representation of F

$$F(x, q)=\max\{l(x)-q(x), q(x)-u(x)\}+d, \tag{2}$$

in which both $l(x)-q(x)$ and $q(x)-u(x)$ are upper semicontinuous of x.

An element $p\in H$ is said to be a minimum to F from H if it satisfies that

$$\|F(\cdot, p)\|=e\equiv\inf_{q\in H}\|F(\cdot, q)\|. \tag{3}$$

Such a minimum must exist, because we have

Theorem 1. *There exists an element $p\in H$ satisfying* (3).

Proof. Since $F(x, y)\to\infty$ as $|y|\to\infty$ and $F(x, y)$ is continuous with respect to y for each $x\in X$, applying Theorem 1 in [1] proves our conclusion.

We turn now to relations between our previous problem and the one of minimization which are described in the following theorem.

Theorem 2.

(a) card $K=0$ *if and only if* $e>d$;

(b) card $K=1$ *if and only if* $e=d$ *and there exists a unique minimum.*

Proof.

(a) Let card $K=0$. Suppose on the contrary that $e\leqslant d$. Taking $p\in H$ by Theorem 1 so that it satisfies (3), we have $\|F(\cdot, p)\|\leqslant d$ or $F(x, p)\leqslant d$. According to (2), we obtain $\max\{l(x)-p(x), p(x)-u(x)\}\leqslant 0$, which implies that $l(x)\leqslant p(x)\leqslant u(x)$, namely $p\in K$. This is a contradiction. Conversely, since $p\in K$ means successively $\max\{l(x)-p(x), p(x)-u(x)\}\leqslant 0$ and $\|F(\cdot,p)\|\leqslant d$, it follows that $e\leqslant d$. Whence $e>d$ implies card $K=0$.

(b) For the necessity by part (a) of this theorem we can first assert that card $K=1$ implies $e\leqslant d$. On the other hand, if $e<d$, letting $p\in K$ such that (3) is satisfied, for a fixed $q\in H$ with $\|q\|=1$ and $|t|\leqslant d-e$ we would have $p+tq\in K$ and a contradiction. So $e=d$. Furthermore, it is easy to see that there must exist a unique minimum.

For the sufficiency let $e=d$ and let a unique minimum exist. By part (a) of this theorem, card $K>0$. Suppose $p, q\in K$. Then $\|F(\cdot, p)\|\leqslant d$ and $\|F(\cdot, q)\|\leqslant d$. By the definition of e it follows that $\|F(\cdot,p)\|\geqslant e$ and $\|F(\cdot,q)\|\geqslant e$. Thus, in fact, $\|F(\cdot,p)\|=\|F(\cdot, q)\|=e$. By uniqueness $p=q$, i. e., card $K=1$.

In order to state theorems of characterization and uniqueness of a minimum to F we assume the notation in [1] and denote for $p\in H$

$$X_p=\{x\in X:F(x,\ p)=\|F(\cdot,\ p)\|\},$$
$$Y_p=\{x\in X_p:p(x)=v(x)\},$$
$$\sigma(x)=\begin{cases}1, & p(x)<v(x),\\ -1, & p(x)>v(x).\end{cases}$$

The characterization for a minimum of F is as follows.

Theorem 3. *A necessary and sufficient condition that $p\in H$ be a minimum to F is that*

$$Y_p\neq\emptyset \quad or \quad \max_{x\in X_p}\sigma(x)q(x)\geqslant 0,\ \ \forall q\in H. \tag{4}$$

Proof. By the previous lemma, F satisfies the assumptions of Theorem 1 in [2]. According to this theorem p is a minimum to F if and only if

$$\sup_{x\in X_p}F'(x,\ p;\ q,\ p)\geqslant 0,\ \ \forall q\in H, \tag{5}$$

where
$$F'(x,\ p;\ q,\ p)=\lim_{t\to 0+}\frac{F(x,\ p+t(q-p))-F(x,\ p)}{t}.$$

A simple calculation yields

$$F'(x,\ p;\ q,\ p)=\begin{cases}q(x)-p(x), & p(x)>v(x),\\ -(q(x)-p(x)), & p(x)<v(x),\\ |q(x)-p(x)|, & p(x)=v(x).\end{cases}$$

In the case $p(x)\neq v(x)$ it is rewritten as

$$F'(x,\ p;\ q,\ p)=(q(x)-p(x))\operatorname{sgn}(p(x)-v(x))=\sigma(x)(p(x)-q(x)).$$

Thus (5) becomes

$$Y_p\neq\emptyset \quad \text{or} \quad \max_{x\in X_p}\sigma(x)(p(x)-q(x))\geqslant 0,\ \ \forall q\in H.$$

Since H is a linear subspace, the above is equivalent to (4).

For the uniqueness of a minimum we have the following

Theorem 4. *In order that $p\in H$ be the unique minimum to F it is necessary and sufficient that*

$$Y_q=\emptyset \quad and \quad \max_{x\in X_q}\sigma_q(x)(q(x)-p(x))<0,\ \ \forall q\in H\setminus\{p\}, \tag{6}$$

where $\sigma_q(x)$ is the σ's function with respect to q.

Proof. By Theorem 3 in [2], $p\in H$ is the unique minimum to F if and only if

$$\sup_{x\in X_q}F'(x,\ q;\ p,\ q)<0,\ \ \forall q\in H\setminus\{p\}, \tag{7}$$

where
$$F'(x,\ q;\ p,\ q)=\begin{cases}\sigma_q(x)(q(x)-p(x)), & q(x)\neq v(x),\\ |p(x)-q(x)|, & q(x)=v(x).\end{cases}$$

Thus, in order that (7) be valid, it is necessary and sufficient that (6) be valid.

With Theorem 2, 3 and 4 we can now deduce the main results of this paper.

Theorem 5. *The following statements are equivalent each to other*:

(a) card $K=0$;

(b) $e>d$;

(c) *There exists an element $p\in H\setminus K$ satisfying* (4).

Proof. Theorem 2 points out the equivalence of (a) and (b). And the equivalence of (b) and (c) follows dire ctly from Theorem 3.

Similarly, from Theorem 2 and Theorem 4 it follows that

Theorem 6. *The following statements are equivalent each to other:*

(a) card $K=1$;

(b) $e=d$ *and there exists a unique minimum;*

(c) $e=d$ *and there exists an element* $p\in H$ *satisfying* (6).

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