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A NEW UNIFORMLY CONVERGENT ITERATIVE METHOD BY INTERPOLATION, WHERE ERROR DECREASES MONOTONICALLY*

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Introduction

For an arbitrarily refined system of knots, polynomial interpolation does not guarantee the convergency. Hence grew the piecewise interpolation. But the currently widely spread spline interpolation has some shortcomings in practical computation. Adding any new knot, we will have to solve a new linear system of equations. Besides, spline functions always possess some degree of smoothness, and the smooth spline interpolation is not a suitable means for approximation of a less smooth function.

In this paper, we introduce a so called "regenerating kernel" $R_x(y)$, with which to derive a formula of interpolation, and construct a new simple iterative method. Having got some approximation, we put a new knot in each step, interpolate the error function and add the result to the previous approximation. In this way we get another approximation. The formula is very simple and feasible for computer use.

We have proven:

1) With a new knot, the error of approximation decreases in the sence of Sobolev norm

$$\|u\|=\int_a^b u^2\,dx+\int_a^b u'^2\,dx\Big)^{\frac{1}{2}}.$$

2) For an arbitrarily thickened knot system, the iterative process converges uniformly.

Actual computation has verified the theory. Error decreased monotonically. When the knot system was refined, accuracy increased considerably.

For the Lamp function, which has a turning point (derivative discontinuous at the origin), our result is better than that obtained by cubic spline.

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§ 1

Definition.

$$R_x(y)=\frac{1}{2\,\mathrm{sh}(b-a)}[\mathrm{ch}(x+y-a-b)+\mathrm{ch}(|x-y|-b+a)],\quad a\leqslant x\leqslant b,\ a\leqslant y\leqslant b.$$

The following are evident by definition.

1. $R_x(y)=R_y(x)>0.$ (1)

2. $R_x(y)$ satisfies the differential equation

$$-\frac{d^2R_x(y)}{dy^2}+R_x(y)=0,\quad x\neq y. \tag{2}$$

Hence

$$\frac{d^2R_x(y)}{dy^2}>0,\quad x\neq y. \tag{3}$$

3. $$\begin{cases}\dfrac{dR_x(y)}{dy}>0, & a<y<x,\\ \dfrac{dR_x(y)}{dy}<0, & x<y<b.\end{cases} \tag{4}$$

4. $$\left.\frac{dR_x(y)}{dy}\right|_{y=a+0}=\left.\frac{dR_x(y)}{dy}\right|_{y=b-0}=0,\quad a<x<b. \tag{5}$$

5. $$\left.\frac{dR_x(y)}{dy}\right|_{y=x-0}-\left.\frac{dR_x(y)}{dy}\right|_{y=x+0}=1,\quad a<x<b; \tag{6}$$

$$-\left.\frac{dR_a(y)}{dy}\right|_{y=a+0}=\left.\frac{dR_b(y)}{dy}\right|_{y=b-0}=1. \tag{6'}$$

6. Using (3), (4), (6), it is easy to prove

$$\left|\frac{dR_x(y)}{dy}\right|\leqslant 1. \tag{7}$$

7. $$\max R_x(y)=\max R_a(y)=R_a(a)=R_b(b)=\frac{\mathrm{ch}(b-a)}{\mathrm{sh}(b-a)}, \tag{8}$$

$$\min R_x(y)=R_a(b)=R_b(a)=\frac{1}{\mathrm{sh}(b-a)}, \tag{9}$$

$$\min R_y(y)=\frac{1+\mathrm{ch}(b-a)}{2\,\mathrm{sh}(b-a)},\quad y=\frac{a+b}{2}. \tag{10}$$

8. Let $W_2^1=\{u|u$ absolutely continuous, $u'\in L^2[a,b]\}$. For $u,v\in W_2^1$, we define the inner product

$$(u,v)=\int_a^b uv\,dx+\int_a^b u'v'\,dx.$$

Now, we verify

$$(R_x(\cdot),u(\cdot))=u(x),\quad a\leqslant x\leqslant b. \tag{11}$$

For $a<x<b$,

$$(R_x(y),u(y))=\left(\int_a^{x-\varepsilon}+\int_{x-\varepsilon}^{x+\varepsilon}+\int_{x+\varepsilon}^b\right)R_x(y)u(y)\,dy+\left(\int_a^{x-\varepsilon}+\int_{x-\varepsilon}^{x+\varepsilon}+\int_{x+\varepsilon}^b\right)R_x'(y)u'(y)dy$$

$$=\int_a^{x-\varepsilon}(-R_x''(y)+R_x(y))u(y)dy+\int_{x+\varepsilon}^b(-R_x''(y)+R_x(y))u(y)dy$$

$$+\int_{x-\varepsilon}^{x-\varepsilon} R_x(y)u(y)dy+\int_{x-\varepsilon}^{x+\varepsilon} R'_x(y)u'(y)dy$$
$$+R'_x(y)u(y)\,|_a^{x-\varepsilon}+R'_x(y)u(y)\,|_{x+\varepsilon}^b$$
$$=I_1+I_2+I_3+I_4+I_5+I_6.$$

$I_1=I_2=0$ by (2). Since $R_x(y)$, $u(y)$ are continuous and $R'_x(y)$, $u'(y)\in L^2$, therefore $I_3\to 0$, $I_4\to 0$, when $\varepsilon\to 0$. Finally from equations (5) and (6)

$$I_5+I_6=R'_x(x-\varepsilon)u(x-\varepsilon)-R'_x(x+\varepsilon)u(x+\varepsilon)$$
$$\xrightarrow[(\varepsilon\to 0)]{}(R'_x(x-0)-R'_x(x+0))u(x)=u(x).$$

Hence

$$R_x(y),\ u(y))=u(x),\quad a<x<b.$$

Similarly, by means of (6'), it is easy to verify

$$(R_a(\cdot),\ u(\cdot))=u(a),\quad (R_b(\cdot),\ u(\cdot))=u(b).$$

Finally, we have

$$(R_x(\cdot),\ u(\cdot))=u(x),\quad a\leqslant x\leqslant b. \tag{11}$$

Because of (11), we call $R_x(y)$ the regenerating kernel or kernel of regeneration.

9. Let y_k and z_k be two fixed points. Throughout we designate

$$\phi_k(\cdot)=R_{y_k}(\cdot),\quad \psi_k(\cdot)=R_{z_k}(\cdot).$$

For any $\delta\in W_2^1$, we have

$$(\delta(\cdot),\ \phi_k(\cdot))=\delta(y_k), \tag{12}$$
$$(\delta(\cdot),\ \psi_k(\cdot))=\delta(z_k). \tag{13}$$

Particularly,

$$(\phi_k(\cdot),\ \phi_k(\cdot))=\phi_k(y_k)=R_{y_k}(y_k), \tag{14}$$
$$(\psi_k(\cdot),\ \psi_k(\cdot))=\psi_k(z_k)=R_{z_k}(z_k), \tag{15}$$
$$(\phi_k(\cdot),\ \psi_k(\cdot))=\phi_k(z_k)=\psi_k(y_k)=R_{y_k}(z_k)=R_{z_k}(y_k). \tag{16}$$

$\phi_k(x)=R_{y_k}(x)$, $\psi_k(x)=R_{z_k}(x)$ are orthonormalized by the Gram Schmidt process as $\phi_k^*(x)$, $\psi_k^*(x)$. So

$$\phi_k^*(x)=\alpha_k\phi_k(x), \tag{17}$$
$$\psi_k^*(x)=\beta_k\phi_k(x)+\gamma_k\psi_k(x). \tag{18}$$

We now verify that

$$H_k(x)=\alpha_k\delta_k(y_k)\phi_k^*(x)+[\beta_k\delta_k(y_k)+\gamma_k\delta_k(z_k)]\psi_k^*(x) \tag{19}$$

satisfies the condition of interpolation

$$H_k(y_k)=\delta_k(y_k),\quad H_k(z_k)=\delta_k(z_k). \tag{20}$$

That means $H(x)$ is the interpolating function of $\delta_k(x)$ with respect to knots y_k, z_k,

$$H_k(y_k)=(H_k(\cdot),\ \phi_k(\cdot))$$
$$=(\alpha_k\delta_k(y_k)\phi_k^*(\cdot)+[\beta_k\delta_k(y_k)+\gamma_k\delta_k(z_k)]\psi_k^*(\cdot),\ \phi_k(\cdot))$$
$$=\alpha_k\delta_k(y_k)(\phi_k^*(\cdot),\ \phi_k(\cdot))=\delta_k(y_k)(\phi_k^*(\cdot)\ \phi_k^*(\cdot))=\delta_k(y_k);$$

$$\begin{aligned}H_k(z_k)&=(H_k(\cdot),\ \psi_k(\cdot))\\&=(\alpha_k\delta_k(y_k)\phi_k^*(\cdot)+[\beta_k\delta_k(y_k)+\gamma_k\delta_k(z_k)]\psi_k^*(\cdot),\ \psi_k(\cdot))\\&=\Big(\alpha_k\delta_k(y_k)\phi_k^*(\cdot)+[\beta_k\delta_k(y_k)+\gamma_k\delta_k(z_k)]\psi_k^*(\cdot),\ \frac{1}{\gamma_k}(\psi_k^*(\cdot)-\beta_k\phi_k(\cdot))\Big)\\&=\frac{1}{\gamma_k}[\beta_k\delta_k(y_k)+\gamma_k\delta_k(z_k)](\psi_k^*(\cdot),\ \psi_k^*(\cdot))-\frac{\alpha_k\beta_k}{\gamma_k}\delta_k(y_k)(\phi_k^*(\cdot),\ \phi_k(\cdot))\\&=\frac{1}{\gamma_k}[\beta_k\delta_k(y_k)+\gamma_k\delta_k(z_k)]-\frac{\beta_k}{\gamma_k}\delta_k(y_k)=\delta_k(z_k).\end{aligned}$$

10. From inequality (7), we have

$$|R_x(y_1)-R_x(y_2)|\leqslant|y_1-y_2| \tag{21}$$

i.e. $R_x(y)$ satisfies the Lipschitz condition.

11. The α_k, β_k, γ_k in (17), (18) are all bounded.

1° $$1=(\phi_k^*,\ \phi_k^*)=\alpha_k^2(\phi_k,\ \phi_k)=\alpha_k^2\phi_k(y_k)=\alpha_k^2R_{y_k}(y_k).$$

By (8), (10),

$$\frac{\operatorname{sh}(b-a)}{\operatorname{ch}(b-a)}\leqslant\alpha_k^2=\frac{1}{R_{y_k}(y_k)}\leqslant\frac{2\operatorname{sh}(b-a)}{1+\operatorname{ch}(b-a)}. \tag{22}$$

2° As $(\phi_k,\ \psi_k^*)=0$, $(\psi_k^*,\ \psi_k^*)=1$; from (18), we have

$$\beta_k(\phi_k,\ \phi_k)+\gamma_k(\psi_k,\ \phi_k)=0,$$
$$(\beta_k\phi_k+\gamma_k\psi_k,\ \beta_k\phi_k+\gamma_k\psi_k)=1.$$

That is

$$\beta_k\phi_k(y_k)+\gamma_k\phi_k(z_k)=0,$$
$$\beta_k^2\phi_k(y_k)+2\beta_k\gamma_k\phi_k(z_k)+\gamma_k^2\psi_k(z_k)=1.$$

So

$$\gamma_k^2=\frac{\phi_k(y_k)}{\phi_k(y_k)\psi_k(z_k)-\phi_k^2(z_k)}. \tag{23}$$

Since both the numerator and denominator of (23) are positive, so

$$\gamma_k^2>0.$$

The denominator in (23)

$$\begin{aligned}&\phi_k(y_k)\psi_k(z_k)-\phi_k^2(z_k)\\&\quad=\Big[\frac{1}{2\operatorname{sh}(b-a)}\Big]^2\{\operatorname{ch}(2y_k-a-b)\operatorname{ch}(2z_k-a-b)+\operatorname{ch}^2(b-a)\\&\qquad+[\operatorname{ch}(2y_k-a-b)+\operatorname{ch}(2z_k-a-b)]\operatorname{ch}(b-a)\\&\qquad-\operatorname{ch}^2(y_k+z_k-a-b)-\operatorname{ch}^2(b-a-|y_k-z_k|)\\&\qquad-2\operatorname{ch}(y_k+z_k-a-b)\operatorname{ch}(b-a-|y_k-z_k|)\}.\end{aligned} \tag{24}$$

For any A, B,

$$\operatorname{ch}\Big(\frac{A+B}{2}\Big)\leqslant(\operatorname{ch}A\cdot\operatorname{ch}B)^{\frac{1}{2}}\leqslant\frac{\operatorname{ch}A+\operatorname{ch}B}{2}.$$

Using this in (24), we get

$$\phi_k(y_k)\psi_k(z_k)-\phi_k^2(z_k)\geqslant\Big[\frac{1}{2\operatorname{sh}(b-a)}\Big]^2[\operatorname{ch}^2(b-a)-\operatorname{ch}^2(b-a-|y_k-z_k|)].$$

If we can make

$$|y_k-z_k|\geqslant D,\quad 0<D\leqslant(b-a)/2, \tag{25}$$

then

$$\gamma_k^2 \leqslant \frac{4\mathrm{sh}^2(b-a)\,\mathrm{ch}(b-a)}{\mathrm{sh}(b-a)\,[\mathrm{ch}^2(b-a)-\mathrm{ch}^2(b-a-D)]}$$
$$=\frac{4\mathrm{sh}(b-a)\,\mathrm{ch}(b-a)}{\mathrm{ch}^2(b-a)-\mathrm{ch}^2(b-a-D)}=\text{const.}$$

3°.
$$\beta_k = -\frac{\gamma_k \phi_k(z_k)}{\phi_k(y_k)}. \tag{26}$$

But $0<\phi_k(z_k)<\phi_k(y_k)$. Therefore

$$|\beta_k|<|\gamma_k|.$$

§ 2

Let the value of function $u(x)$ at $n+1$ points $\{x_i\}_0^n$ be given as $\{u_i\}_0^n$, and $v(x)\in W_2^1$ be the initial guess of $u(x)$ (may be arbitrarily chosen). Set

$$\delta_1(x)=u(x)-v(x).$$

Let y_1 be either of the two knots x_0, x_1, where $|\delta_1(x)|$ takes the maximum value, and let z_1 be the other knot point. Let $\phi_1(x)=R_{y_1}(x)$, $\psi_1(x)=R_{z_1}(x)$, $\delta_2(x)=\delta_1(x)-\{\alpha_1\delta_1(y_1)\phi_1^*(x)+[\beta_1\delta_1(y_1)+\gamma_1\delta_1(z_1)]\psi_1^*(x)\}$.

In general, let y_k be one of the first $k+1$ knots $x_0, x_1, \cdots, x_k$, where $|\delta_k(x)|$ takes the maximum value, and let z_k be one of the rest k knots, which is the farthest from y_k. Let

$$\phi_k(x)=R_{y_k}(x),\quad \psi_k(x)=R_{z_k}(x),$$
$$\delta_{k+1}(x)=\delta_k(x)-\{\alpha_k\delta_k(y_k)\phi_k^*(x)+[\beta_k\delta_k(y_k)+\gamma_k\delta_k(z_k)]\psi_k^*(x)\}.$$

The last one can be rewritten as

$$\delta_{k+1}(x)=\delta_1(x)-\sum_{j=1}^{k}\{\alpha_j\delta_j(y_j)\phi_j^*(x)+[\beta_j\delta_j(y_j)+\gamma_j\delta_j(z_j)]\psi_j^*(x)\},\quad k=1,2,\cdots,n-1.$$

Or by the definition of $\delta_1(x)$,

$$u(x)=v(x)+\sum_{j=1}^{n-1}\{\alpha_j\delta_j(y_j)\phi_j^*(x)+[\beta_j\delta_j(y_j)+\gamma_j\delta_j(z_j)]\psi_j^*(x)\}+\delta_n(x).$$

Theorem. *Let $u(x)\in W_2^1$. Then for any initial approximation $v(x)\in W_2^1$, we have $\|\delta_{n+1}\|\leqslant\|\delta_n\|$ and $\max|\delta_n(x)|\to 0$, provided $\sqrt{n}\,h_n\to 0$, where h_n has the following meaning: Sort $x_0, x_1, \cdots, x_n$ to get $x'_0, x'_1, \cdots, x'_n$, such that $a\leqslant x'_0<x'_1<\cdots<x'_n\leqslant b$. Then*

$$h_n=\max\{|x'_0-a|,\ |x'_{i+1}-x'_i|_0^{n-1},\ |b-x'_n|\}.$$

Proof.

$$(\delta_{k+1}(\cdot),\delta_{k+1}(\cdot))=(\delta_k,\delta_k)-2(\delta_k,\alpha_k\delta_k(y_k)\phi_k^*+[\beta_k\delta_k(y_k)+\gamma_k\delta_k(z_k)]\psi_k^*)$$
$$+(\alpha_k\delta_k(y_k)\phi_k^*+[\beta_k\delta_k(y_k)+\gamma_k\delta_k(z_k)]\psi_k^*,\alpha_k\delta_k(y_k)\phi_k^*$$
$$+[\beta_k\delta_k(y_k)+\gamma_k\delta_k(z_k)]\psi_k^*)=(\delta_k,\delta_k)-2I_1+I_2.$$

1°. Calculation of I_1. From

$$(\delta_k,\phi_k^*)=(\delta_k,\alpha_k\phi_k)=\alpha_k\delta_k(y_k)$$

and

$$(\delta_k,\psi_k^*)=(\delta_k,\beta_k\phi_k+\gamma_k\psi_k)=\beta_k(\delta_k,\phi_k)+\gamma_k(\delta_k,\psi_k)=\beta_k\delta_k(y_k)+\gamma_k\delta_k(z_k),$$

we get

$$I_1=\alpha_k^2\delta_k^2(y_k)+[\beta_k\delta_k(y_k)+\gamma_k\delta_k(z_k)]^2. \tag{27}$$

2°. Calculation of I_2. ϕ_k^* and ψ_k^* are orthonormal; so

$$I_2=\alpha_k^2\delta_k^2(y_k)+[\beta_k\delta_k(y_k)+\gamma_k\delta_k(z_k)]^2. \tag{28}$$

Therefore

$$(\delta_{k+1}, \delta_{k+1})=(\delta_k, \delta_k)-\alpha_k^2\delta_k^2(y_k)-[\beta_k\delta_k(y_k)+\gamma_k\delta_k(z_k)]^2,$$

i.e.

$$\|\delta_k\|^2=\|\delta_{k+1}\|^2+\alpha_k^2\delta_k^2(y_k)+[\beta_k\delta_k(y_k)+\gamma_k\delta_k(z_k)]^2. \tag{29}$$

So,

$$\|\delta_{k+1}\|\leqslant\|\delta_k\| \text{ (the equality holds only when } \delta_k(y_k)=0). \tag{30}$$

That means error δ_k decreases monotonically in the sence of Sobolev norm.

From (29),

$$\|\delta_1\|^2=\|u-v\|^2=\sum_{k=1}^{n}\{\alpha_k^2\delta_k^2(y_k)+[\beta_k\delta_k(y_k)+\gamma_k\delta_k(z_k)]^2\}+\|\delta_{n+1}\|^2. \tag{31}$$

Hence, for any n

$$\sum_{k=1}^{n}\{\alpha_k^2\delta_k^2(y_k)+[\beta_k\delta_k(y_k)+\gamma_k\delta_k(z_k)]^2\}\leqslant\|u-v\|^2. \tag{32}$$

Naturally we get

$$\alpha_k^2\delta_k^2(y_k)\to 0, \quad k\to\infty. \tag{33}$$

But

$$\alpha_k^2=\phi_k(y_k)^{-1}=R_{y_k}(y_k)^{-1}\geqslant\frac{\operatorname{sh}(b-a)}{\operatorname{ch}(b-a)}.$$

This means α_k^2 is positively bounded below; so

$$\delta_k(y_k)\to 0, \quad k\to\infty.$$

Due to the way of selection of y_k,

$$|\delta_k(x_i)|\leqslant|\delta_k(y_k)|, \quad i=0, 1, \cdots, n.$$

Therefore

$$|\delta_k(x)|\to 0, \quad i=0, 1, \cdots, k, \ k\to\infty, \tag{34}$$

i.e. error function $\delta_k(x)$ tends to zero at knots x, when k tends to infinity.

Now we prove that $\delta_k(x)$ converges to zero uniformly on $[a, b]$.

Let y_n^* be the point where $|\delta_n(x)|$ takes the maximum value on $[a, b]$, i.e.

$$|\delta_n(y_n^*)|=\max_{x\in[a,b]}|\delta_n(x)|, \tag{35}$$

and let $x_{i_0}^{(n)}$ be one of the knot points, which is nearest to y_n^*,

$$|\delta_n(y_n^*)|\leqslant|\delta_n(y_n^*)-\delta_n(x_{i_0}^{(n)})|+|\delta_n(x_{i_0}^{(n)})|. \tag{36}$$

But

$$\begin{aligned}|\delta_n(y_n^*)-\delta_n(x_{i_0}^{(n)})|&\leqslant|\delta_1(y_n^*)-\delta_1(x_{i_0}^{(n)})|\\&\quad+\Big|\sum_{k=1}^{n-1}\{\alpha_k\delta_k(y_k)\phi_k^*(y_n^*)+[\beta_k\delta_k(y_k)+\gamma_k\delta_k(z_k)]\psi_k^*(y_n^*)\}\\&\quad-\sum_{k=1}^{n-1}\{\alpha_k\delta_k(y_k)\phi_k^*(x_{i_0}^{(n)})+[\beta_k\delta_k(y_k)+\gamma_k\delta_k(z_k)]\psi_k^*(x_{i_0}^{(n)})\}\Big|\\&=I_1+I_2.\end{aligned}$$

1°. Estimation of I_1. By the uniform continuity of $u(x)$, we have

$$|\delta_1(y_n^*)-\delta_1(x_{i_0}^{(n)})|<\varepsilon,$$

whenever the knot system is sufficiently refined.

2°. Estimation of I_2. In (19) we have proven that $R_y(x)$ satisfies the Lipschitz condition. So

$$|\phi_k(y_n^*) - \phi_k(x_{i_0}^{(n)})| < |y_n^* - x_{i_0}^{(n)}|,$$
$$|\psi_k(y_n^*) - \psi_k(x_{i_0}^{(n)})| < |y_n^* - x_{i_0}^{(n)}|.$$

Recalling the boundedness of α_k, β_k and γ_k, we have

$$|\phi_k^*(y_n^*) - \phi_k^*(x_{i_0}^{(n)})| < L|y_n^* - x_{i_0}^{(n)}|,$$
$$|\psi_k^*(y_n^*) - \psi_k^*(x_{i_0}^{(n)})| < L|y_n^* - x_{i_0}^{(n)}|.$$

Hence, from (36)

$$\begin{aligned}|\delta_n(y_n^*) - \delta_n(x_{i_0}^{(n)})| &< \varepsilon + L\sum_{k=1}^{n-1}|\alpha_k\delta_k(y_k)|\,|y_n^* - x_{i_0}^{(n)}| \\ &\quad + L\sum_{k=1}^{n-1}|\beta_k\delta_k(y_k) + \gamma_k\delta_k(z_k)|\,|y_n^* - x_{i_0}^{(n)}| \\ &< \varepsilon + L\sqrt{2(n-1)}\left\{\sum_{k=1}^{n-1}\alpha_k^2\delta_k^2(y_k) + [\beta_k\delta_k(y_k) + \gamma_k\delta_k(z_k)]^2\right\}^{\frac{1}{2}}\cdot h_n \\ &\leqslant \varepsilon + L\sqrt{2(n-1)}h_n\|u - v\|,\end{aligned}$$

where $h_n = \max_i |x'_{i+1} - x'_i|$. Therefore

$$|\delta_n(y_n^*) - \delta_n(x_{i_0}^{(n)})| < 2\varepsilon, \tag{37}$$

when $\sqrt{n}\,h_n$ is sufficiently small. Moreover, by (34), we also have

$$|\delta_n(x_{i_0}^{(n)})| < \varepsilon.$$

Finally, from (36)

$$\max_{x\in[a,b]}|\delta_n(x)| = |\delta_n(y_n^*)| < 3\varepsilon, \tag{38}$$

when the knot system is sufficiently refined.

Now we have proven that $\delta_n(x)$ converges to zero uniformly, when the knot system is sufficiently thickened.

Remark. During the selection of the first two knots, if we make

$$|y_1 - z_1| = |x_1 - x_0| \geqslant 2D,$$

then (25) ($|y_k - z_k| \geqslant D$) will be valid for all k.

§ 3

Algorithm.

(1) Give the table of interpolation x_i, u_i, $i=0, 1, \cdots, n$, and a, b.

(2) Select the initial guess $v(x)$ of $u(x)$;

$$\delta_1(x) \overset{\text{def.}}{=\!=} u(x) - v(x).$$

Compute $\quad \delta_1(x_i) = u_i - v_i \Rightarrow D, \quad i=0, 1, \cdots, n.$

For $k=1$ to n,

(1) For $i=0, 1, \cdots, k$, select the knot point, where $|\delta_k(x)|$ takes the maximum value.

Let

$$|\delta_k(x_{i_0})| = \max_{i=0,1,\cdots,n}|\delta_k(x_{i_0})|,$$

$$x_{i_0} \Rightarrow y_k, \quad \delta_k(x_{i_0}) \Rightarrow DY_k.$$

(2) For $i=0, 1, \cdots, k$, select a point from $k+1$ knots which is the farthest from y_k, i.e. $|y_k - x_i|$ takes the maximum value.

Let

$$|y_k - x_{i_0}| = \max_{i=0,1,\cdots,k} |y_k - x_i|,$$

$$x_{i_0} \Rightarrow z_k, \quad \delta_k(z_k) \Rightarrow Dz_k.$$

(3) Compute

$$\alpha_k = \{[\text{ch}(2y_k - a - b) + \text{ch}(b-a)]/2\text{sh}(b-a)\}^{-\frac{1}{2}},$$

$$\gamma_k = \left[\frac{\phi_k(y_k)}{\phi_k(y_k)\psi_k(z_k) - \phi_k^2(z_k)}\right]^{\frac{1}{2}},$$

$$\beta_k = -\frac{\gamma_k \phi_k(z_k)}{\phi_k(y_k)}.$$

(4) Compute $\delta_{k+1}(x_i)$, $i=0, 1, \cdots, n$;

$$\delta_{k+1}(x_i) = D_i - \{\alpha_k DY_k \phi_k^*(x_i) + [\beta_k DY_k + Y_k Dz_k]\psi_k^*(x_i)\} \Rightarrow D_i,$$

$$\phi_k^*(x) = \alpha_k \phi_k(x) = \alpha_k R_{y_k}(x),$$

$$\psi_k^*(x) = \beta_k \phi_k(x) + \gamma_k \psi_k(x) = \beta_k R_{y_k}(x) + \gamma_k R_{z_k}(x),$$

$$v(x) \Leftarrow v(x) + \alpha_k DY_k \phi_k^*(x) + [\beta_k DY_k + \gamma_k Dz_k]\psi_k^*(x).$$