NUMERICAL SOLUTION OF THE REACTION-DIFFUSION EQUATION*1)

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In this paper, we consider the numerical solution for the equation

$$\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}-\frac{\partial}{\partial x}\left(\nu(x, t, U)\frac{\partial U}{\partial x}\right)-F(x, t, U)=0.$$

A finite difference scheme and the basic error equality are given. Then the error estimations are proved for the periodic problem with $\nu(x, t)\geqslant 0$, the first and second boundary value problems with $\nu(x, t)\geqslant\nu_0>0$, and for $\nu(U)\geqslant\nu_0>0$. Under some conditions such estimations imply the stabilities and convergences of the schemes.

§ 1. Introduction

In one-dimensional space, the reaction-diffusion equation is the following

$$\frac{\partial U}{\partial t}+M(x, t, U)\frac{\partial U}{\partial x}-\frac{\partial}{\partial x}\left(\nu(x, t, U)\frac{\partial U}{\partial x}\right)-F(x, t, U)=0.$$

Much work has been done to solve this equation (see [1]). On the other hand some authors worked at error estimations. But there are still some unsolved problems:

(i) In [2], the stability is taken as the boundedness of the solution. But in fact the boundedness of the solution of a non-linear scheme is not uniform with the stability. Besides it is supposed that

$$\nu(x, t, U)\geqslant\frac{K}{2}|M(x, t, U)|, \quad K>0.$$

So the following important case is excluded:

$$M(x, t, U)=U, \quad \nu=\text{positive constant}.$$

(ii) In [3], the author considered the following case:

$$M(x, t, U)=U^p, \quad p\geqslant 1,$$

but only for the periodic problem with $\nu(x, t, U)=$ positive constant.

(iii) Recently the author[4] studied the numerical solution of Burger's equation and used the same technique for the reaction-diffusion equation, but only for some special cases (see [5]).

This paper is concerned with a general problem, i.e.

$$\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}-\frac{\partial}{\partial x}\left(\nu(x, t, U)\frac{\partial U}{\partial x}\right)-F(x, t, U)=0, \quad 0\leqslant x\leqslant 1,\ t>0, \quad (1.1)$$

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1) This work is done on the basis of the proposition of Professor A. R. Mitchell when the author visited Dundee University in May, 1980.

where $\nu(x, t, U)\geqslant 0$.

The technique used is to estimate the index of generalized stability (see [6—8]).

In section 2, we give some notations and lemmas. In section 3, a scheme is constructed and the basic error equality is proved. In section 4, we give a strict error estimation for the periodic problem with $\nu(x, t)\geqslant 0$. In sections 5 and 6, we prove strict error estimations with the first or second boundary value conditions. In section 7, we consider the case $\nu(x, t, U)=\nu(U)>0$.

§ 2. Notations and Lemmas

Let h and τ be the mesh spacing of variables x and t respectively. The mesh point is $(jh, k\tau)$.

$$u_x(jh, k\tau)=\frac{1}{h}[u(jh+h, k\tau)-u(jh, k\tau)],$$

$$u_{\bar{x}}(jh, k\tau)=\frac{1}{h}[u(jh, k\tau)-u(jh-h, k\tau)],$$

$$u_{\hat{x}}(jh, k\tau)=\frac{1}{2h}[u(jh+h, k\tau)-u(jh-h, k\tau)],$$

$$\Delta_h^{\nu(jh,k\tau,u(jh,k\tau))}u(jh, k\tau)=\frac{1}{2}[\nu(jh, k\tau, u(jh, k\tau))u_{\bar{x}}(jh, k\tau)]_x$$
$$+\frac{1}{2}[\nu(jh, k\tau, u(jh, k\tau))u_x(jh, k\tau)]_{\bar{x}},$$

$$u_t(jh, k\tau)=\frac{1}{\tau}[u(jh, k\tau+\tau)-u(jh, k\tau)].$$

We define

$$(u(k\tau), v(k\tau))=h\sum_{j=1}^{N-1}u(jh, k\tau)v(jh, k\tau),$$

$$\|u(k\tau)\|^2=(u(k\tau), u(k\tau)),$$

$$|u(k\tau)|^2_{1,\nu(k\tau,u(k\tau))}=\frac{h}{2}\sum_{j=1}^{N-1}\nu(jh, k\tau, u(jh, k\tau))[u_x^2(jh, k\tau)+u_{\bar{x}}^2(jh, k\tau)].$$

If $\nu(jh, k\tau, u(jh, k\tau))\equiv 1$, then $|u(k\tau)|^2_{1,\nu(k\tau,u(k\tau))}$ is denoted by $|u(k\tau)|_1^2$ for simplicity.

Lemma 1.

$$2(u(k\tau), u_t(k\tau))=\|u(k\tau)\|_t^2-\tau\|u_t(k\tau)\|^2.$$

Lemma 2.

$$(u(k\tau), \Delta_h^{\nu(k\tau,u(k\tau))}u(k\tau))+|u(k\tau)|^2_{1,\nu(k\tau,u(k\tau))}=D_1,$$

where

$$D_1=\frac{1}{2}u_{\bar{x}}(Nh, k\tau)[\nu(Nh, k\tau, u(Nh, k\tau))u(Nh-h, k\tau)$$
$$+\nu(Nh-h, k\tau, u(Nh-h, k\tau))u(Nh, k\tau)]$$
$$-\frac{1}{2}u_x(0, k\tau)[\nu(h, k\tau, u(h, k\tau))u(0, k\tau)$$
$$+\nu(0, k\tau, u(0, k\tau))u(h, k\tau)].$$

Proof. From Abel's formula we have

$$(\eta_x(k\tau), \xi(k\tau))+(\xi_{\bar{x}}(k\tau), \eta(k\tau))$$

$$=\eta(Nh,k\tau)\xi(Nh-h,k\tau)-\eta(h,k\tau)\xi(0,k\tau) \quad (2.1)$$

$$(\xi_x(k\tau),\eta(k\tau))+(\eta_{\bar x}(k\tau),\xi(k\tau))$$
$$=\eta(Nh-h,k\tau)\xi(Nh,k\tau)-\eta(0,k)\xi(h,k\tau). \quad (2.2)$$

Let
$$\eta=\nu(x,t,w)v_{\bar x},\ \xi=u,\quad \text{in } (2.1),$$
$$\eta=\nu(x,t,w)v_x,\ \xi=u,\quad \text{in } (2.2).$$

By putting the two results together, we obtain

$$(u(k\tau),\Delta_h^{\nu(k\tau,w(k\tau))}v(k\tau))+\frac{1}{2}(\nu(k\tau,w(k\tau)),u_x(k\tau)v_x(k\tau)+u_{\bar x}(k\tau)v_{\bar x}(k\tau))$$
$$=\frac{1}{2}v_{\bar x}(Nh,k\tau)[\nu(Nh,k\tau,w(Nh,k\tau))u(Nh-h,k\tau)$$
$$+\nu(Nh-h,k\tau,w(Nh-h,k\tau))u(Nh,k\tau)]$$
$$-\frac{1}{2}v_x(0,k\tau)[\nu(h,k\tau,w(h,k\tau))u(0,k\tau)$$
$$+\nu(0,k\tau,w(0,k\tau))u(h,k\tau)]. \quad (2.3)$$

Taking $u=v=w$, we get the conclusion.

Lemma 3.

$$2(u(k\tau),\Delta_h^{\nu(k\tau,u(k\tau))}u_t(k\tau))+[|u(k\tau)|^2_{1,\nu(k\tau,u(k\tau))}]_t-\tau|u_t(k\tau)|^2_{1,\nu(k\tau,u(k\tau))}$$
$$-\frac{1}{2}([\nu(k\tau,u(k\tau))]_t,u_x^2(k\tau+\tau)+u_{\bar x}^2(k\tau+\tau))=D_2,$$

where

$$D_2=u_{\bar x t}(Nh,k\tau)[\nu(Nh,k\tau,u(Nh,k\tau))u(Nh-h,k\tau)$$
$$+\nu(Nh-h,k\tau,u(Nh-h,k\tau))u(Nh,k\tau)]$$
$$-u_{xt}(0,k\tau)[\nu(h,k\tau,u(h,k\tau))u(0,k\tau)$$
$$+\nu(0,k\tau,u(0,k\tau))u(h,k\tau)]. \quad (2.4)$$

Proof. By putting $v=u_t$, $w=u$ in (2.3), we get

$$2(u(k\tau),\Delta_h^{\nu(k\tau,u(k\tau))}u_t(k\tau))+(\nu(k\tau,u(k\tau)),u_x(k\tau)u_{xt}(k\tau)+u_{\bar x}(k\tau)u_{\bar xt}(k\tau))=D_2.$$

Because

$$(\nu(k\tau,u(k\tau)),u_x(k\tau)u_{xt}(k\tau))=\frac{1}{2}[(\nu(k\tau,u(k\tau)),u_x^2(k\tau)]_t$$
$$-\frac{\tau}{2}(\nu(k\tau,u(k\tau)),u_{xt}^2(k\tau))-\frac{1}{2}[(\nu(k\tau,u(k\tau))]_t,u_x^2(k\tau+\tau)),\ \text{etc.},$$

the conclusion follows.

Lemma 4.

$$2(u_t(k\tau),\Delta_h^{\nu(k\tau,u(k\tau))}u(k\tau))+[|u(k\tau)|^2_{1,\nu(k\tau,u(k\tau))}]_t-\tau|u_t(k\tau)|^2_{1,\nu(k\tau,u(k\tau))}$$
$$-\frac{1}{2}([\nu(k\tau,u(k\tau))]_t,u_x^2(k\tau+\tau)+u_{\bar x}^2(k\tau+\tau))=D_3,$$

where

$$D_3=u_{\bar x}(Nh,k\tau)[\nu(Nh,k\tau,u(Nh,k\tau))u_t(Nh-h,k\tau)$$
$$+\nu(Nh-h,k\tau,u(Nh-h,k\tau))u_t(Nh,k\tau)]$$
$$-u_x(0,k\tau)[\nu(h,k\tau,u(h,k\tau))u_t(0,k\tau)+\nu(0,k\tau,u(0,k\tau))u_t(h,k\tau)].$$

Lemma 5.

$$\|u(k\tau)v(k\tau)\|^2 \leqslant h^{-1}\|u(k\tau)\|^2\|v(k\tau)\|^2,$$

$$\sum_{j=1}^{N-1} h|u(jh, k\tau)|^r \leqslant h^{1-\frac{r}{2}}\|u(k\tau)\|^r, \quad r \geqslant 2.$$

Lemma 6[5]. *If $\varepsilon>0$ and h is suitably small, then there exists a positive constant α_0 such that*

$$u^2(0, k\tau)+u^2(h, k\tau)+u^2(Nh-h, k\tau)+u^2(Nh, k\tau)$$

$$\leqslant \varepsilon|u(k\tau)|_1^2+\alpha_0\left(1+\frac{1}{\varepsilon}\right)\|u(k\tau)\|^2.$$

Lemma 7. *If the following conditions are satisfied*

(i) ρ, A *and* β_i *are nonnegative constants,* $1 \leqslant i \leqslant I$;

(ii) a, b_i, c *and* d *are constants,* $1 \leqslant i \leqslant I$;

(iii) $\eta(k\tau)$, $\tilde{A}(\eta(k\tau))$, $\tilde{c}(\eta(k\tau))$, $V(\eta(k\tau))$ *are such mesh functions that* $\eta(k\tau) \geqslant 0$, *and*

$$\tilde{A}(\eta(k\tau)) \leqslant A, \quad \text{for } \eta(k\tau) \leqslant h^a,$$

$$\tilde{c}(\eta(k\tau)) \leqslant 0, \quad \text{for } \eta(k\tau) \leqslant h^c,$$

$$V(\eta(k\tau)) \geqslant 0, \quad \text{for } \eta(k\tau) \leqslant h^d;$$

(iv) $\eta(0) \leqslant \rho$ *and*

$$\eta(k\tau) \leqslant \rho+\tau\sum_{j=0}^{k-1}\left\{\tilde{A}(\eta(j\tau))\eta(j\tau)\left(1+\sum_{i=1}^{I}h^{-b_i}\eta^{\beta_i}(j\tau)\right)+\tilde{c}(\eta(j\tau))V(\eta(j\tau))\right\};$$

(v) $\rho e^{Ak\tau(I+1)} \leqslant \min(h^a, \min\limits_{1\leqslant i\leqslant I} h^{\frac{b_i}{\beta_i}}, h^c, h^d)$, *then*

$$\eta(k\tau) \leqslant \rho e^{Ak\tau(I+1)}.$$

Lemma 7 is a special case of Lemma 1 in [7].

§ 3. Difference Scheme and Basic Error Equality

We define the following difference operator

$$J(u(jh, k\tau), v(jh, k\tau)) = \frac{1}{3}v(jh, k\tau)u_{\hat{x}}(jh, k\tau) + \frac{1}{3}[v(jh, k\tau)u(jh, k\tau)]_{\hat{x}}. \tag{3.1}$$

We have (see [4])

$$\begin{aligned}
&(J(u(k\tau), v(k\tau)), w(k\tau))+(J(w(k\tau), v(k\tau)), u(k\tau))\\
&=\frac{1}{6}\{u(Nh, k\tau)v(Nh, k\tau)w(Nh-h, k\tau)\\
&\quad+u(Nh-h, k\tau)v(Nh-h, k\tau)w(Nh, k\tau)\\
&\quad+v(Nh, k\tau)w(Nh, k\tau)u(Nh-h, k\tau)\\
&\quad+v(Nh-h, k\tau)w(Nh-h, k\tau)u(Nh, k\tau)\\
&\quad-u(h, k\tau)v(h, k\tau)w(0, k\tau)-u(0, k\tau)v(0, k\tau)w(h, k\tau)\\
&\quad-v(h, k\tau)w(h, k\tau)u(0, k\tau)-v(0, k\tau)w(0, k\tau)u(h, k\tau)\}.
\end{aligned} \tag{3.2}$$

The difference scheme for solving (1.1) is the following

$$L_h(u(jh, k\tau)) = u_t(jh, k\tau) + J(u(jh, k\tau) + \delta\tau u_t(jh, k\tau), u(jh, k\tau))$$
$$-\Delta_h^{\nu(jh, k\tau, u(jh, k\tau))}[u(jh, k\tau) + \sigma\tau u_t(jh, k\tau)] - F(jh, k\tau, u(jh, k\tau)) = 0, \quad (3.3)$$

where $0 \leqslant \delta \leqslant 1$, $0 \leqslant \sigma \leqslant 1$. If $\sigma = \delta = 0$, (3.3) is an explicit scheme. Otherwise it is an implicit scheme.

Assume that $u(jh, k\tau)$ and the right term of the scheme have respectively the errors $\tilde{u}(jh, k\tau)$ and $\tilde{f}(jh, k\tau)$; then

$$\tilde{u}_t(jh, k\tau) + J(\tilde{u}(jh, k\tau) + \delta\tau\tilde{u}_t(jh, k\tau), u(jh, k\tau) + \tilde{u}(jh, k\tau))$$
$$+ J(u(jh, k\tau) + \delta\tau u_t(jh, k\tau), \tilde{u}(jh, k\tau))$$
$$= \Delta_h^{\nu(jh, k\tau, u(jh, k\tau) + \tilde{u}(jh, k\tau))}[\tilde{u}(jh, k\tau) + \sigma\tau\tilde{u}_t(jh, k\tau)]$$
$$+ \Delta_h^{\tilde{\nu}(jh, k\tau)}[u(jh, k\tau) + \sigma\tau u_t(jh, k\tau)]$$
$$+ \tilde{F}(jh, k\tau) + \tilde{f}(jh, k\tau), \quad (3.4)$$

where

$$\tilde{F}(jh, k\tau) = F(jh, k\tau, u(jh, k\tau) + \tilde{u}(jh, k\tau)) - F(jh, k\tau, u(jh, k\tau)),$$
$$\tilde{\nu}(jh, k\tau) = \nu(jh, k\tau, u(jh, k\tau) + \tilde{u}(jh, k\tau)) - \nu(jh, k\tau, u(jh, k\tau)).$$

Taking the scalar product of (3.4) with $2\tilde{u}(jh, k\tau)$, we have from Lemmas 1, 2 and 3

$$\|\tilde{u}(k\tau)\|_t^2 - \tau\|\tilde{u}_t(k\tau)\|^2 + 2(\tilde{u}(k\tau), J(\tilde{u}(k\tau) + \delta\tau\tilde{u}_t(k\tau), u(k\tau) + \tilde{u}(k\tau)))$$
$$+ 2(\tilde{u}(k\tau), J(u(k\tau) + \delta\tau u_t(k\tau), \tilde{u}(k\tau))) + 2|\tilde{u}(k\tau)|^2_{1, \nu(k\tau, u(k\tau) + \tilde{u}(k\tau))}$$
$$+ \sigma\tau[|u(k\tau)|^2_{1, \nu(k\tau, u(k\tau) + \tilde{u}(k\tau))}]_t - \sigma\tau^2|u_t(k\tau)|^2_{1, \nu(k\tau, u(k\tau) + \tilde{u}(k\tau))}$$
$$+ \sigma\tau B_0(k\tau) + B_1(k\tau) + \sigma\tau B_2(k\tau)$$
$$= 2(\tilde{u}(k\tau), \Delta_h^{\tilde{\nu}(k\tau)}[u(k\tau) + \sigma\tau u_t(k_\tau)] + \tilde{F}(k\tau) + \tilde{f}(k\tau)), \quad (3.5)$$

where

$$B_0(k\tau) = -\frac{1}{2}([\nu(k\tau, u(k\tau) + \tilde{u}(k\tau))]_t, \tilde{u}_x^2(k\tau + \tau) + \tilde{u}_{\bar{x}}^2(k\tau + \tau)),$$
$$B_1(k\tau) = -\tilde{u}_{\bar{x}}(Nh, k\tau)[\nu(Nh, k\tau, u(Nh, k\tau) + \tilde{u}(Nh, k\tau))\tilde{u}(Nh - h, k\tau)$$
$$+ \nu(Nh - h, k\tau, u(Nh - h, k\tau) + \tilde{u}(Nh - h, k\tau))\tilde{u}(Nh, k\tau)]$$
$$+ \tilde{u}_x(0, k\tau)[\nu(h, k\tau, u(h, k\tau) + \tilde{u}(h, k\tau))\tilde{u}(0, k\tau)$$
$$+ \nu(0, k\tau, u(0, k\tau) + \tilde{u}(0, k\tau))\tilde{u}(h, k\tau)],$$
$$B_2(k\tau) = -\tilde{u}_{\bar{x}t}(Nh, k\tau)[\nu(Nh, k\tau, u(Nh, k\tau) + \tilde{u}(Nh, k\tau))\tilde{u}(Nh - h, k\tau)$$
$$+ \nu(Nh - h, k\tau, u(Nh - h, k\tau) + \tilde{u}(Nh - h, k\tau))\tilde{u}(Nh, k\tau)]$$
$$+ \tilde{u}_{xt}(0, k\tau)[\nu(h, k\tau, u(h, k\tau) + \tilde{u}(h, k\tau))\tilde{u}(0, k\tau)$$
$$+ \nu(0, k\tau, u(0, k\tau) + \tilde{u}(0, k\tau))\tilde{u}(h, k\tau)].$$

We take the scalar product of (3.4) with $m\tau\tilde{u}_t(jh, k\tau)$, where m is a positive number chosen below, then

$$m\tau\|\tilde{u}_t(h\tau)\|^2 + m\tau(\tilde{u}_t(k\tau), J(\tilde{u}(k\tau) + \delta\tau\tilde{u}_t(k\tau), u(k\tau) + \tilde{u}(k\tau)))$$
$$+ m\tau(\tilde{u}_t(k\tau), J(u(k\tau) + \delta\tau u_t(k\tau), \tilde{u}(k\tau))) + m\sigma\tau^2|\tilde{u}_t(k\tau)|^2_{1, \nu(k\tau, u(k\tau) + \tilde{u}(k\tau))}$$

$$+\frac{m\tau}{2}[|\tilde{u}(k\tau)|^2_{1,\nu(k\tau,u(k\tau)+\tilde{u}(k\tau))}]_t-\frac{m\tau^2}{2}|\tilde{u}_t(k\tau)|^2_{1,\nu(k\tau,u(k\tau)+\tilde{u}(k\tau))}$$

$$+\frac{m\tau}{2}(B_0(k\tau)+B_3(k\tau)+\sigma\tau B_4(k\tau))$$

$$=m\tau(\tilde{u}_t(k\tau),\ \Delta_h^{\tilde{\nu}(k\tau)}[u(k\tau)+\sigma\tau u_t(k\tau)]+\bar{F}(k\tau)+\bar{f}(k\tau)), \tag{3.6}$$

where

$$\begin{aligned}B_3(k\tau)=&-\tilde{u}_x(Nh,\ k\tau)[\nu(Nh,\ k\tau,\ u(Nh,\ k\tau)+\tilde{u}(Nh,\ k\tau))\tilde{u}_t(Nh-h,\ k\tau)\\&+\nu(Nh-h,\ k\tau,\ u(Nh-h,\ k\tau)+\tilde{u}(Nh-h,\ k\tau))\tilde{u}_t(Nh,\ k\tau)]\\&+\tilde{u}_x(0,\ k\tau)[\nu(h,\ k\tau,\ u(h,\ k\tau)+\tilde{u}(h,\ k\tau))\tilde{u}_t(0,\ k\tau)\\&+\nu(0,\ k\tau,\ u(0,\ k\tau)+\tilde{u}(0,\ k\tau))\tilde{u}_t(h,\ k\tau)],\\B_4(k\tau)=&-\tilde{u}_{xt}(Nh,\ k\tau)[\nu(Nh,\ k\tau,\ u(Nh,\ k\tau)+\tilde{u}(Nh,\ k\tau))\tilde{u}_t(Nh-h,\ k\tau)\\&+\nu(Nh-h,\ k\tau,\ u(Nh-h,\ k\tau)+\tilde{u}(Nh-h,\ k\tau))\tilde{u}_t(Nh,k\tau)]\\&+\tilde{u}_{xt}(0,\ k\tau)[\nu(h,\ k\tau,\ u(h,\ k\tau)+\tilde{u}(h,\ k\tau))\tilde{u}_t(0,\ k\tau)\\&+\nu(0,\ k\tau,\ u(0,\ k\tau)+\tilde{u}(0,\ k\tau))\tilde{u}_t(h,\ k\tau)].\end{aligned}$$

Combining (3.5) with (3.6), we get

$$\begin{aligned}&\|u(k\tau)\|_t^2+\tau(m-1)\|\tilde{u}_t(k\tau)\|^2+2|\tilde{u}(k\tau)|^2_{1,\nu(k\tau,u(k\tau)+\tilde{u}(k\tau))}\\&+\tau\Big(\sigma+\frac{m}{2}\Big)[|\tilde{u}(k\tau)|^2_{1,\nu(k\tau,u(k\tau)+\tilde{u}(k\tau))}]_t+\tau^2\Big(m\sigma-\frac{m}{2}-\sigma\Big)|\tilde{u}_t(k\tau)|^2_{1,\nu(k\tau,u(k\tau)+\tilde{u}(k\tau))}\\&+2(\tilde{u}(k\tau),\ J(\tilde{u}(k\tau),\ u(k\tau)+\tilde{u}(k\tau)))+m\delta\tau^2(\tilde{u}_t(k\tau),\ J(\tilde{u}_t(k\tau),\ u(k\tau)+\tilde{u}(k\tau)))\\&+m\tau(\tilde{u}_t(k\tau),\ J(\tilde{u}(k\tau),\ u(k\tau)+\tilde{u}(k\tau)))\\&+2\delta\tau(\tilde{u}(k\tau),\ J(\tilde{u}_t(k\tau),\ u(k\tau)+\tilde{u}(k\tau)))\\&+\tau\Big(\sigma+\frac{m}{2}\Big)B_0(k\tau)+B_1(k\tau)+\sigma\tau B_2(k\tau)+\frac{m\tau}{2}B_3(k\tau)+\frac{m\sigma\tau^2}{2}B_4(k\tau)\\&=\sum_{l=0}^{5}A_l(k\tau),\end{aligned} \tag{3.7}$$

where

$$\begin{aligned}A_0(k\tau)&=2(\tilde{u}(k\tau),\ \bar{F}(k\tau)),\\A_1(k\tau)&=m\tau(\tilde{u}_t(k\tau),\ \bar{F}(k\tau)),\\A_2(k\tau)&=(2\tilde{u}(k\tau)+m\tau\tilde{u}_t(k\tau),\ \bar{f}(k\tau)),\\A_3(k\tau)&=(2\tilde{u}(k\tau)+m\tau\tilde{u}_t(k\tau),\ \Delta_h^{\tilde{\nu}(k\tau)}[u(k\tau)+\sigma\tau u_t(k\tau)]),\\A_4(k\tau)&=-m\tau(\tilde{u}_t(k\tau),\ J(u(k\tau)+\delta\tau u_t(k\tau),\ \tilde{u}(k\tau))),\\A_5(k\tau)&=-2(\tilde{u}(k\tau),\ J(u(k\tau)+\delta\tau u_t(k\tau),\ \tilde{u}(k\tau))).\end{aligned}$$

According to (3.2), (3.7) identifies the following equality

$$\begin{aligned}&\|\tilde{u}(k\tau)\|_t^2+\tau(m-1)\|\tilde{u}_t(k\tau)\|^2+2|\tilde{u}(k\tau)|^2_{1,\nu(k\tau,u(k\tau)+\tilde{u}(k\tau))}\\&+\tau\Big(\sigma+\frac{m}{2}\Big)[|\tilde{u}(k\tau)|^2_{1,\nu(k\tau,u(k\tau)+u(k\tau))}]_t+\tau^2\Big(m\sigma-\frac{m}{2}-\sigma\Big)|\tilde{u}_t(k\tau)|^2_{1,\nu(k\tau,u(k\tau)+\tilde{u}(k\tau))}\\&+\tau\Big(\frac{m}{2}+\sigma\Big)B_0(k\tau)+B_1(k\tau)+\sigma\tau B_2(k\tau)+\frac{m\tau}{2}B_3(k\tau)+\frac{m\sigma\tau^2}{2}B_4(k\tau)\\&+\sum_{l=0}^{3}C_l(k\tau)=\sum_{l=0}^{5}A_l(k\tau),\end{aligned} \tag{3.8}$$

where

$$C_0(k\tau)=\tau(m-2\delta)(\tilde{u}_t(k\tau),\ J(\tilde{u}(k\tau),\ u(k\tau)+\tilde{u}(k\tau))),$$

$$C_1(k\tau)=\frac{1}{3}\tilde{u}(Nh,\ k\tau)\tilde{u}(Nh-h,\ k\tau)(u(Nh,\ k\tau)+u(Nh-h,\ k\tau)$$
$$+\tilde{u}(Nh,\ k\tau)+\tilde{u}(Nh-h,\ k\tau))-\frac{1}{3}\tilde{u}(0,\ k\tau)\tilde{u}(h,\ k\tau)(u(0,\ k\tau)$$
$$+u(h,\ k\tau)+\tilde{u}(0,\ k\tau)+\tilde{u}(h,\ k\tau)),$$

$$C_2(k\tau)=\frac{m\delta\tau^2}{6}\tilde{u}_t(Nh,\ k\tau)\tilde{u}_t(Nh-h,\ k\tau)(u(Nh,\ k\tau)+u(Nh-h,\ k\tau)$$
$$+\tilde{u}(Nh,\ k\tau)+\tilde{u}(Nh-h,\ k\tau))-\frac{m\delta\tau^2}{6}\tilde{u}_t(h,\ k\tau)\tilde{u}_t(0,\ k\tau)(u(h,\ k\tau)$$
$$+u(0,\ k\tau)+\tilde{u}(h,\ k\tau)+\tilde{u}(0,\ k\tau)),$$

$$C_3(k\tau)=\frac{m\tau}{6}(\tilde{u}_t(Nh,\ k\tau)\tilde{u}(Nh-h,\ k\tau)+\tilde{u}_t(Nh-h,\ k\tau)\tilde{u}(Nh,\ k\tau))(u(Nh,\ k\tau)$$
$$+u(Nh-h,\ k\tau)+\tilde{u}(Nh,\ k\tau)+\tilde{u}(Nh-h,\ k\tau))$$
$$-\frac{m\tau}{6}(\tilde{u}_t(h,\ k\tau)\tilde{u}(0,\ k\tau)+\tilde{u}_t(0,\ k\tau)\tilde{u}(h,\ k\tau))(u(h,\ k\tau)$$
$$+u(0,\ k\tau)+\tilde{u}(h,\ k\tau)+\tilde{u}(0,\ k\tau)).$$

§ 4. Error Estimation for Periodic Problem

In this section, we consider the periodic problem with $\nu(x,\ t,\ u)=\nu(x,\ t)\geqslant 0$.

Let L, M and N denote nonnegative constants throughout which may depend on $u(jh,\ k\tau)$. Suppose that

(1) the boundary condition is periodic, i.e.

$$Nh=1+h,\quad u(jh,\ k\tau)=u(jh+1,\ k\tau),\tag{4.1}$$

(2) $\nu\leqslant\nu_1,\quad \left|\frac{\partial\nu}{\partial t}\right|\leqslant\alpha_1,$ (4.2)

(3) $\lambda=\tau h^{-2}<\infty,$ (4.3)

(4) $v[F(x,\ t,\ u+v)-F(x,\ t,\ u)]\leqslant M\sum_{l=0}^{p}\bar{M}_l|v|^{l+2},\quad \bar{M}_l=0$ or 1, (4.4)

(5) $|F(x,\ t,\ u+v)-F(x,\ t,\ u)|\leqslant M\sum_{l=0}^{q}\bar{N}_l|v|^{l+1},\quad \bar{N}_l=0$ or 1. (4.5)

From (4.1), we have $A_3(k\tau)=0$ and

$$B_l(k\tau)=0,\quad 1\leqslant l\leqslant 4,\tag{4.6}$$

$$C_l(k\tau)=0,\quad 1\leqslant l\leqslant 3.\tag{4.7}$$

From (4.2) and (4.3), we have

$$|\tau B_0(k\tau)|\leqslant M\|\tilde{u}(k\tau+\tau)\|^2.\tag{4.8}$$

From Lemma 5

$$|C_0(k\tau)|\leqslant\begin{cases}\varepsilon\tau\|\tilde{u}_t(k\tau)\|^2+\dfrac{M}{\varepsilon}(m-2\delta)^2(\|\tilde{u}(k\tau)\|^2+h\|\tilde{u}(k\tau)\|^2|\tilde{u}(k\tau)|_1^2), & \text{for } \nu_0>0,\\ \varepsilon\tau\|\tilde{u}_t(k\tau)\|^2+\dfrac{m}{\varepsilon}(m-2\delta)^2(\|\tilde{u}(k\tau)\|^2+h^{-1}\|u(k\tau)\|^4), & \text{for } \nu_0=0.\end{cases}\tag{4.9}$$

From (4.4), (4.5) and Lemma 5, we get

$$A_0(k\tau)\leqslant M\sum_{l=0}^{p}\widetilde{M}_l h^{-\frac{l}{2}}\|\widetilde{u}(k\tau)\|^{l+2}, \tag{4.10}$$

$$|A_1(k\tau)|\leqslant\varepsilon\tau\|\widetilde{u}_t(k\tau)\|^2+\frac{M}{\varepsilon}\sum_{l=0}^{q}\widetilde{N}_l h^{2-l}\|\widetilde{u}(k\tau)\|^{2l+2}, \tag{4.11}$$

$$|A_2(k\tau)|\leqslant\varepsilon\tau\|\widetilde{u}_t(k\tau)\|^2+\frac{M}{\varepsilon}(\|\widetilde{u}(k\tau)\|^2+\|\widetilde{f}(k\tau)\|^2), \tag{4.12}$$

$$|A_4(k\tau)|\leqslant\varepsilon\tau\|\widetilde{u}_t(k\tau)\|^2+\frac{M}{\varepsilon}\|\widetilde{u}(k\tau)\|^2. \tag{4.13}$$

Now we are going to estimate $|A_5(k\tau)|$. From (3.1), we get

$$-A_5(k\tau)=\frac{2}{3}(\widetilde{u}^2(k\tau),\ u_{\hat{x}}(k\tau)+\delta\tau u_{\hat{x}t}(k\tau))$$
$$-\frac{2}{3}(\widetilde{u}(k\tau)\widetilde{u}_{\hat{x}}(k\tau),\ u(k\tau)+\delta\tau u_t(k\tau))+\widetilde{A}_5(k\tau),$$

where

$$\widetilde{A}_5(k\tau)=\frac{1}{3}\{\widetilde{u}(Nh,\ k\tau)\widetilde{u}(Nh-h,\ k\tau)(u(Nh,\ k\tau)+\delta\tau u_t(Nh,\ k\tau)$$
$$+u(Nh-h,\ k\tau)+\delta\tau u_t(Nh-h,\ k\tau))-\widetilde{u}(h,\ k\tau)\widetilde{u}(0,\ k\tau)(u(h,\ k\tau)$$
$$+\delta\tau u_t(h,\ k\tau)+u(0,\ k\tau)+\delta\tau u_t(0,\ k\tau))\}.$$

On the other hand we have

$$(\eta\xi)_{\hat{x}}=\eta_{\hat{x}}\xi+\eta\xi_{\hat{x}}+\frac{h}{2}\eta_x\xi_x-\frac{h}{2}\eta_{\bar{x}}\xi_{\bar{x}}.$$

So

$$-A_5(k\tau)=\frac{4}{3}(\widetilde{u}^2(k\tau),\ u_{\hat{x}}(k\tau)+\delta\tau u_{\hat{x}t}(k\tau))$$
$$+\frac{2}{3}(\widetilde{u}(k\tau)\widetilde{u}_{\hat{x}}(k\tau),\ u(k\tau)+\delta\tau u_t(k\tau))$$
$$+\frac{h}{3}(\widetilde{u}(k\tau)\widetilde{u}_x(k\tau),\ u_x(k\tau)+\delta\tau u_{tx}(k\tau))$$
$$-\frac{h}{3}(\widetilde{u}(k\tau)\widetilde{u}_{\bar{x}}(k\tau),\ u_{\bar{x}}(k\tau)+\delta\tau u_{t\bar{x}}(k\tau)).$$

Therefore

$$-A_5(k\tau)=(\widetilde{u}^2(k\tau),\ u_{\hat{x}}(k\tau)+\delta\tau u_{\hat{x}t}(k\tau))$$
$$+\frac{h}{6}(\widetilde{u}(k\tau)\widetilde{u}_x(k\tau),\ u_x(k\tau)+\delta\tau u_{tx}(k\tau))$$
$$-\frac{h}{6}(\widetilde{u}(k\tau)\widetilde{u}_{\bar{x}}(k\tau),\ u_{\bar{x}}(k\tau)+\delta\tau u_{t\bar{x}}(k\tau))+\frac{\widetilde{A}_5(k\tau)}{2}. \tag{4.14}$$

Because of (4.1), $\widetilde{A}_5(k\tau)=0$; then

$$|A_5(k\tau)|\leqslant M\|u(k\tau)\|^2. \tag{4.15}$$

By substituting (4.6)—(4.15) into (3.8), we obtain

$$\|\widetilde{u}(k\tau)\|_t^2+\tau(m-1-4\varepsilon)\|\widetilde{u}_t(k\tau)\|^2+|\widetilde{u}(k\tau)|_{1,\nu(k\tau)}^2$$
$$+\tau\left(\sigma+\frac{m}{2}\right)[|\widetilde{u}(k\tau)|_{1,\nu(k\tau)}^2]_t+\tau^2\left(m\sigma-\frac{m}{2}-\sigma\right)|\widetilde{u}_t(k\tau)|_{1,\nu(k\tau)}^2$$
$$\leqslant\widetilde{R}(k\tau)+\|\widetilde{f}(k\tau)\|^2, \tag{4.16}$$

where

$$\tilde{R}(k\tau)=M(\|\tilde{u}(k\tau)\|^2+\|\tilde{u}(k\tau+\tau)\|^2+h^{-1}(m-2\delta)^2(1-\text{sign }\nu_0)\|\tilde{u}(k\tau)\|^4)$$
$$+M\sum_{l=1}^{p}\tilde{M}_l h^{-\frac{l}{2}}\|\tilde{u}(k\tau)\|^{l+2}+M\sum_{l=1}^{q}\tilde{N}_l h^{2-l}\|\tilde{u}(k\tau)\|^{2l+2}$$
$$+[Mh(m-2\delta)^2\text{ sign }\nu_0\|\tilde{u}(k\tau)\|^2-1]\,|\tilde{u}(k\tau)|^2_{1,\nu(k\tau)}.$$

Now we are going to choose m for three cases.

Case 1. $\sigma>\frac{1}{2}$. We take

$$m=m_1=\max\left(1+4s+p_0,\ \frac{2\sigma}{2\sigma-1}\right),\quad p_0>0.$$

Then $m\sigma-\frac{m}{2}-\sigma\geqslant 0$, and (4.16) implies

$$\|\tilde{u}(k\tau)\|_t^2+p_0\tau\|\tilde{u}_t(k\tau)\|^2+|\tilde{u}(k\tau)|^2_{1,\nu(k\tau)}+\tau\left(\sigma+\frac{m}{2}\right)[|u(k\tau)|^2_{1,\nu(k\tau)}]_t$$
$$\leqslant\tilde{R}(k\tau)+\|\tilde{f}(k\tau)\|^2. \tag{4.17}$$

We use the following notations

$$Q(\tilde{u}(k\tau),\nu,p_0)=\|\tilde{u}(k\tau)\|^2+2\tau\sum_{j=0}^{k-1}(|u(j\tau)|^2_{1,\nu(j\tau)}+p_0\tau\|\tilde{u}_t(j\tau)\|^2),$$
$$\rho_k(\tilde{u}(0),\tilde{f})=\|\tilde{u}(0)\|^2+\tau\sum_{j=0}^{k-1}\|\tilde{f}(j\tau)\|^2.$$

Then from (4.17)

$$Q(\tilde{u}(k),\nu,p_0)\leqslant M\rho_k(\tilde{u}(0),\tilde{f})+M\tau\sum_{j=0}^{k-1}\tilde{R}(j\tau). \tag{4.18}$$

Finally we use Lemma 7 with

$$\rho=\rho_k(\tilde{u}(0),\tilde{f}),\quad \eta(k\tau)=Q(\tilde{u}(k\tau),\nu,p_0),$$
$$V(\eta(k\tau))=\text{sign }\nu_0|\tilde{u}(k\tau)|^2_{1,\nu(k\tau)},$$

where a and d are arbitrary, $I=p+q+1$,

$$\frac{b_i}{\beta_i}=\begin{cases}1, & \text{for } 1\leqslant i\leqslant p,\ \tilde{M}_i=1,\\ \text{arbitrary}, & \text{for } 1\leqslant i\leqslant p,\ \tilde{M}_i=0,\end{cases}$$

$$\frac{b_{p+i}}{\beta_{p+i}}=\begin{cases}\dfrac{i-2}{i}, & \text{for } 1\leqslant i\leqslant q,\ \tilde{N}_i=1,\\ \text{arbitrary}, & \text{for } 1\leqslant i\leqslant q,\ \tilde{N}_i=0,\end{cases}$$

$$\frac{b_{p+q+1}}{\beta_{p+q+1}}=\begin{cases}\text{sign}\,|m-2\delta|, & \text{for } \nu_0=0,\\ \text{arbitrary}, & \text{for } \nu_0>0,\end{cases}$$

$$C=\begin{cases}\text{arbitrary}, & \text{for } \nu_0=0,\\ -1, & \text{for } \nu_0>0.\end{cases}$$

Therefore there exists a positive constant $T(\rho_k)$ depending on ρ_k such that if $\rho_k(\tilde{u}(0),\tilde{f})\leqslant Nh^{2s}$, $k\tau\leqslant T(\rho_k)$, then

$$Q(\tilde{u}(k\tau),\nu,p_0)\leqslant Me^{Lk\tau}\rho_k(\tilde{u}(0),\tilde{f}), \tag{4.19}$$

where

$$s=\begin{cases}s_1, & \text{if all } \tilde{M}_i=0,\ i\geqslant 1,\\ \dfrac{1}{2}, & \text{otherwise,}\end{cases} \tag{4.20}$$

$$s_1=\max\left(\frac{1}{2}\operatorname{sign}|m-2\delta|(1-2\operatorname{sign}\nu_0),\ \max_{\substack{1\le i<q\\ \tilde N_i\ne 0}}\left(\frac{i-2}{2i}\right)\right).$$

Especially, if $\delta>\frac{m_1}{2}$, we can take $m=2\delta$. Hence if $p=q=0$, then for all ρ_k and k, we have (4.19).

Case 2. $\sigma=\frac{1}{2}$. We take

$$m=m_2=1+4s+p_0+2\lambda\nu_1.$$

Since

$$\tau|\tilde u_t(k\tau)|^2_{1,\nu(k\tau)}\leqslant 4\lambda\nu_1\|u_t(k\tau)\|^2,$$

we have

$$\tau(m-1-4s)\|\tilde u_t(k\tau)\|^2+\tau^2\left(m\sigma-\frac{m}{2}-\sigma\right)|u_t(k\tau)|^2_{1,\nu(k\tau)}\geqslant p_0\tau\|\tilde u_t(k\tau)\|^2. \quad (4.21)$$

Then we have (4.17)—(4.20) also. Especially, if $\delta>\frac{m_2}{2}$, $p=q=0$, then for all ρ_k and k, (4.19) holds.

Case 3. $\sigma<\frac{1}{2}$, $\lambda<\frac{1}{2\nu_1(1-2\sigma)}$. We take

$$m=m_3=\frac{1+4s+p_0+4\lambda\nu_1\sigma}{1+4\lambda\nu_1\sigma-2\lambda\nu_1}.$$

Then we have (4.21) too, and (4.19) follows. Especially, if $\delta>\frac{m_3}{2}$, $p=q=0$, then for all ρ_k and k, we have (4.19).

Theorem 1. *If the following conditions are satisfied*

(1) *conditions* (4.1)—(4.5) *hold,*

(2) $\sigma\geqslant\frac{1}{2}$ *or* $\lambda<\frac{1}{2\nu_1(1-2\sigma)}$,

(3) $\rho_k(\tilde u(0),\tilde f)\leqslant Nh^{2s}$, $k\tau\leqslant T(\rho_k)$,

then

$$Q(\tilde u(k\tau),\nu,p_0)\leqslant Me^{Lk\tau}\rho_k(\tilde u(0),\tilde f).$$

Especially, if $p=q=0$, *and*

$$\delta>\begin{cases}\frac{m_1}{2}, & \text{for } \sigma>\frac{1}{2},\\ \frac{m_2}{2}, & \text{for } \sigma=\frac{1}{2},\\ \frac{m_3}{2}, & \text{for } \sigma<\frac{1}{2},\end{cases}$$

then for all ρ_k *and* k, *estimation* (4.19) *holds.*

Theorem 2. *If the conditions of Theorem* 1 *are satisfied and scheme* (3.3) *is consistent with order* s_0, $s_0>s$, *then the scheme is convergent.*

Proof. Let $U(jh,k\tau)$ and $u(jh,k\tau)$ be the solutions of (1.1) and (3.3) respectively,

$$u(jh,k\tau)=U(jh,k\tau)+\tilde{\tilde u}(jh,k\tau).$$

Then

$$\begin{cases}L_h[U(jh,k\tau)+\tilde{\tilde u}(jh,k\tau)]=0,\\ L_h[U(jh,k\tau)]=\tilde{\tilde f}(jh,k\tau),\end{cases}$$

where
$$\|\tilde{\tilde{f}}(k\tau)\|^2=O(h^{2S_0}).$$
By the same proof as in Theorem 1, we get
$$Q(\tilde{\tilde{u}}(k\tau),\ \nu,\ p_0)\leqslant \overline{M}e^{\overline{L}k\tau}\rho_k(\tilde{u}(0),\ \tilde{\tilde{f}}),$$
where $\overline{L}$ and $\overline{M}$ only depend on U. So
$$Q(\tilde{\tilde{u}}(k\tau),\ \nu,\ p_0)=O(h^{2S_0})\to 0.\quad \text{as } h\to 0.$$

Remark 1. For the convergence, we require $S_0\leqslant\frac{1}{2}$ at most. For the special values of p, q and ν_0, we can get better results.

§ 5. The First Boundary Value Problem

This section deals with the first boundary value problem. We suppose that

(i) $Nh=1$, (5.1)

(ii) $\tilde{u}(1,\ k\tau)=\tilde{g}_1(k\tau),\ \tilde{u}(0,\ k\tau)=\tilde{g}_0(k\tau)$, (5.2)

(iii) $\nu=$ positive constant, for simplicity. (5.3)

Clearly (3.8) holds still with $B_0(k\tau)=A_3(k\tau)=0$. (4.9)—(4.12) hold also. We have
$$|A_4(k\tau)|\leqslant\varepsilon\tau\|\tilde{u}_t(k\tau)\|^2+M(\|\tilde{u}(k\tau)\|^2+\tilde{g}_1^2+\tilde{g}_0^2). \tag{5.4}$$
From (4.14), we have
$$|A_5(k\tau)|\leqslant M\|\tilde{u}(k\tau)\|^2+\frac{\varepsilon\nu}{h}[\tilde{u}^2(1-h,\ k\tau)+\tilde{u}^2(h,\ k\tau)]+Mh(\tilde{g}_1^2+\tilde{g}_0^2). \tag{5.5}$$
We find that
$$B_1(k\tau)=\frac{\nu}{h}[\tilde{u}^2(1-h,\ k\tau)+\tilde{u}^2(h,\ k\tau)-\tilde{g}_1^2(k\tau)-\tilde{g}_0^2(k\tau)], \tag{5.6}$$
$$\begin{aligned}\sigma\tau B_2(k\tau)=\frac{\sigma\nu\tau}{h}\{&\tilde{u}(1-h,\ k\tau)\tilde{u}_t(1-h,\ k\tau)+\tilde{g}_1(k\tau)\tilde{u}_t(1-h,\ k\tau)\\&-\tilde{g}_{1t}(k\tau)\tilde{u}(1-h,\ k\tau)-\tilde{g}_1(k\tau)\tilde{g}_{1t}(k\tau)+\tilde{u}(h,\ k\tau)\tilde{u}_t(h,\ k\tau)\\&+\tilde{g}_0(k\tau)\tilde{u}_t(h,\ k\tau)-\tilde{g}_{0t}(k\tau)\tilde{u}(h,\ k\tau)-\tilde{g}_0(k\tau)\tilde{g}_{0t}(k\tau)\}.\end{aligned}$$
Since
$$\tilde{u}(1-h,\ k\tau)\tilde{u}_t(1-h,\ k\tau)=\frac{1}{2}[\tilde{u}^2(1-h,\ k\tau)]_t-\frac{\tau}{2}[\tilde{u}_t^2(1-h,\ k\tau)],\ \text{etc.},$$
we have
$$\begin{aligned}\sigma\tau B_2(k\tau)\geqslant&\frac{\sigma\nu\tau}{2h}[\tilde{u}^2(1-h,\ k\tau)+\tilde{u}^2(h,\ k\tau)]_t\\&-\frac{\nu\tau^2}{2h}(\sigma+\varepsilon)[\tilde{u}_t^2(1-h,\ k\tau)+\tilde{u}_t^2(h,\ k\tau)]\\&-\frac{\nu\varepsilon}{h}[\tilde{u}^2(1-h,\ k\tau)+\tilde{u}^2(h,\ k\tau)]\\&-\frac{M}{\varepsilon h}(\tilde{g}_1^2(k\tau)+\tilde{g}_0^2(k\tau)+\tau^2\tilde{g}_{1t}(k\tau)+\tau^2 g_{0t}^2(k\tau)).\end{aligned} \tag{5.7}$$
We can estimate $\frac{m\tau}{2}$ $B_3(k\tau)$ in the same way as noted by (5.8). Similarly we have

$$\frac{\sigma m\tau^2}{2}B_4(k\tau)=\frac{\sigma m\nu\tau^2}{2h}[\tilde{u}_t^2(1-h,\ k\tau)+\tilde{u}_t^2(h,\ k\tau)-\tilde{g}_{1t}^2(k\tau)-\tilde{g}_{0t}^2(k\tau)].$$

Now we are going to estimate $|C_i(k\tau)|$, $i=1,\ 2,\ 3$. Firstly,

$$\begin{aligned}C_1(k\tau)=&\frac{1}{3}\tilde{g}_1(k\tau)\tilde{u}(1-h,\ k\tau)(\tilde{g}_1(k\tau)+\tilde{u}(1-h,\ k\tau))\\&-\frac{1}{3}\tilde{g}_0(k\tau)\tilde{u}(h,\ k\tau)(\tilde{g}_0(k\tau)+\tilde{u}(h,\ k\tau))\\&+\frac{1}{3}\tilde{g}_1(k\tau)\tilde{u}(1-h,\ k\tau)(u(1,\ k\tau)+u(1-h,\ k\tau))\\&-\frac{1}{3}\tilde{g}_0(k\tau)\tilde{u}(h,\ k\tau)(u(0,\ k\tau)+u(h,\ k\tau)).\end{aligned}\tag{5.8}$$

Since

$$\begin{aligned}&\tilde{g}_1^2(k\tau)\tilde{u}(1-h,\ k\tau)\leqslant\frac{\nu\varepsilon}{4h}\tilde{u}^2(1-h,\ k\tau)+\frac{h}{\varepsilon\nu}\tilde{g}_1^4(k\tau),\\&\tilde{g}_1(k\tau)\tilde{u}^2(1-h,\ k\tau)\leqslant\frac{\nu\varepsilon}{4h}\tilde{u}^2(1-h,\ k\tau)+\frac{h}{\varepsilon\nu}\tilde{g}_1^2(k\tau)\tilde{u}^2(1-h,\ k\tau),\\&\tilde{g}_1(k\tau)\tilde{u}(1-h,\ k\tau)\leqslant\frac{\nu\varepsilon}{4h}\tilde{u}^2(1-h,\ k\tau)+\frac{h}{\nu\varepsilon}\tilde{g}_1^2(k\tau),\ \text{etc.}\end{aligned}\tag{5.9}$$

We obtain

$$\begin{aligned}|C_1(k\tau)|\leqslant&\frac{\nu\varepsilon}{h}[\tilde{u}^2(1-h,\ k\tau)+\tilde{u}^2(h,\ k\tau)]\\&+\frac{M\nu h}{\varepsilon}[\tilde{g}_1^2(k\tau)\tilde{u}^2(1-h,\ k\tau)+\tilde{g}_0^2(k\tau)\tilde{u}^2(h,\ k\tau)]\\&+\frac{Mh}{\varepsilon}[\tilde{g}_1^2(k\tau)+\tilde{g}_0^2(k\tau)+\tilde{g}_1^4(k\tau)+\tilde{g}_0^4(k\tau)].\end{aligned}\tag{5.10}$$

Similarly we have

$$\begin{aligned}|C_2(k\tau)|\leqslant&\frac{\varepsilon\nu\tau^2}{h}[\tilde{u}_t^2(1-h,\ k\tau)+\tilde{u}_t^2(h,\ k\tau)]\\&+\frac{M\nu h\tau^2}{\varepsilon}[\tilde{g}_{1t}^2(k\tau)\tilde{u}^2(1-h,\ k\tau)+\tilde{g}_{0t}^2(k\tau)\tilde{u}^2(h,k\tau)]\\&+\frac{Mh\tau^2}{\varepsilon}[\tilde{g}_{1t}^2(k\tau)+\tilde{g}_{0t}^2(k\tau)+\tilde{g}_1^2(k\tau)\tilde{g}_{1t}^2(k\tau)+\tilde{g}_0^2(k\tau)\tilde{g}_{0t}^2(k\tau)],\end{aligned}\tag{5.11}$$

$$\begin{aligned}|C_3(k\tau)|\leqslant&\frac{\varepsilon\nu}{h}[\tilde{u}^2(1-h,\ k\tau)+\tilde{u}^2(h,\ k\tau)]\\&+\frac{\varepsilon\nu\tau^2}{h}[\tilde{u}_t^2(1-h,\ k\tau)+\tilde{u}_t^2(h,\ k\tau)]\\&+\frac{M\nu h}{\varepsilon}[\tilde{g}_1^2(k\tau)\tilde{u}^2(1-h,\ k\tau)+\tilde{g}_0^2(k\tau)\tilde{u}^2(h,\ k\tau)]\\&+\frac{M\nu h\tau^2}{\varepsilon}[\tilde{g}_{1t}^2(k\tau)\tilde{u}^2(1-h,\ k\tau)+\tilde{g}_{0t}^2(k\tau)\tilde{u}^2(h,\ k\tau)]\\&+\frac{Mh}{\varepsilon}[\tilde{g}_1^2(k\tau)+\tilde{g}_0^2(k\tau)+\tilde{g}_1^4(k\tau)+\tilde{g}_0^4(k\tau)+\tau^2\tilde{g}_1^2(k\tau)\tilde{g}_{1t}^2(k\tau)\\&+\tau^2\tilde{g}_0^2(k\tau)\tilde{g}_{0t}^2(k\tau)].\end{aligned}\tag{5.12}$$

We substitute the above estimations into (3.8); then

$$\|\tilde{u}(k\tau)\|_t^2+\tau(m-1-M\varepsilon)\|\tilde{u}_t(k\tau)\|^2+\nu|u(k\tau)|_1^2+\nu\tau\Big(\sigma+\frac{m}{2}\Big)[|\tilde{u}(k\tau)|_1^2]_t$$
$$+\nu\tau^2\Big(m\sigma-\frac{m}{2}-\sigma\Big)|\tilde{u}_t(k\tau)|_1^2$$
$$+\frac{\nu}{h}\Big[1-M\varepsilon-\frac{Mh}{\varepsilon}(\tilde{g}_1^2(k\tau)+\tilde{g}_0^2(k\tau)+\tau^2\tilde{g}_{1t}^2(k\tau)+\tau^2\tilde{g}_{0t}^2(k\tau))\Big](\tilde{u}^2(1-h,k\tau)$$
$$+\tilde{u}^2(h,k\tau))+\frac{\nu\tau}{2h}\Big(\sigma+\frac{m}{2}\Big)[\tilde{u}^2(1-h,k\tau)+\tilde{u}^2(h,k\tau)]_t$$
$$+\frac{\nu\tau^2}{2h}\Big(\sigma m-\frac{m}{2}-\sigma-M\varepsilon\Big)(\tilde{u}_t^2(1-h,k\tau)+\tilde{u}_t^2(h,k\tau))$$
$$\leqslant\tilde{R}_1(k\tau)+\tilde{R}_2(k\tau), \tag{5.13}$$

where

$$\tilde{R}_1(k\tau)=M\|\tilde{u}(k\tau)\|^2+M\sum_{l=1}^{q}\tilde{M}_l h^{-\frac{l}{2}}\|\tilde{u}(k\tau)\|^{l+2}+M\sum_{l=1}^{q}\tilde{N}_l h^{2-l}\|u(k\tau)\|^{2l+2}$$
$$+(-\nu+Mh(m-2\delta)^2\|u(k\tau)\|^2)|\tilde{u}(k\tau)|_1^2,$$
$$\tilde{R}_2(k\tau)=M\|\tilde{f}(k\tau)\|^2+\frac{M}{h}(\tilde{g}_1^2(k\tau)+\tilde{g}_0^2(k\tau)+\tau^2\tilde{g}_{1t}^2(k\tau)+\tau^2 g_{0t}^2(k\tau))$$
$$+Mh(\tilde{g}_1^4(k\tau)+\tilde{g}_0^4(k\tau)+\tau^2\tilde{g}_1^2(k\tau)\tilde{g}_{1t}^2(k\tau)+\tau^2\tilde{g}_0^2(k\tau)\tilde{g}_{0t}^2(k\tau)).$$

We suppose that N is a suitably small positive constant and

$$\tilde{g}_1^2\leqslant Nh,\quad \tilde{g}_0^2\leqslant Nh. \tag{5.14}$$

Now we are going to estimate the error. First we consider the case $\sigma>\frac{1}{2}$. By taking

$$m=m_1^*=\max\Big(1+M\varepsilon+p_0,\ \frac{2\sigma+M\varepsilon+Mh}{2\sigma-1}\Big),\quad p_0\geqslant 0,$$

we obtain from (5.13)

$$\|\tilde{u}(k\tau)\|_t^2+p_0\tau\|\tilde{u}_t(k\tau)\|^2+\nu|\tilde{u}(k\tau)|_1^2+\nu\tau\Big(\sigma+\frac{m}{2}\Big)[|\tilde{u}(k\tau)|_1^2]_t$$
$$+\frac{\nu\tau}{2h}\Big(\sigma+\frac{m}{2}\Big)[\tilde{u}^2(1-h,k\tau)+\tilde{u}^2(h,k\tau)]_t\leqslant\tilde{R}_1(k\tau)+\tilde{R}_2(k\tau),$$

so that

$$Q(\tilde{u}(k\tau),\nu,p_0)\leqslant\rho_k^{(1)}(\tilde{u}(0),\tilde{f},\tilde{g}_1,\tilde{g}_0)+\tau\sum_{j=0}^{k-1}\tilde{R}_1(j\tau), \tag{5.15}$$

where

$$\rho_k^{(1)}(\tilde{u}(0),\tilde{f},\tilde{g}_1,\tilde{g}_0)=\|\tilde{u}(0)\|^2+\tau\sum_{j=0}^{k-1}\|\tilde{f}(j\tau)\|^2$$
$$+\frac{M}{h}\sum_{j=0}^{k}(\tilde{g}_1^2(j\tau)+\tilde{g}_0^2(j\tau)+h^2\tilde{g}_1^4(j\tau)+h^2\tilde{g}_0^4(j\tau)).$$

Then we use Lemma 7 with

$$\eta(k\tau)=Q(\tilde{u}(k\tau),\nu,p_0),$$
$$\rho=\rho_k^{(1)}(\tilde{u}(0),\tilde{f},\tilde{g}_1,\tilde{g}_0),$$
$$V(k\tau)=|\tilde{u}(k\tau)|_1^2.$$

The error estimations for $\sigma=\frac{1}{2}$ or $\sigma<\frac{1}{2}$ can be derived similarly.

Theorem 3. *If the following conditions are satisfied*

(i) (5.1)—(5.3), *and* (5.14) *hold,*

(ii) $\sigma \geqslant \frac{1}{2}$ *or* $\lambda < \frac{4}{9\nu(1-2\sigma)}$,

(iii) $\rho_k^{(1)} \leqslant Nh^{2s}$, $k\tau \leqslant T(\rho_k^{(1)})$,

$$s = \begin{cases} s_1, & \text{if all } \widetilde{M}_i = 0,\ i \geqslant 1, \\ \frac{1}{2}, & \text{otherwise}, \end{cases}$$

$$s_1 = \max\left(-\frac{1}{2}\operatorname{sign}|m-2\delta|,\ \max_{\substack{1\leqslant i\leqslant q \\ \widetilde{N}_i \neq 0}}\left(\frac{i-2}{2i}\right)\right),$$

then

$$Q(\tilde{u}(k\tau),\ \nu,\ p_0) \leqslant Me^{Lk\tau}\rho_k^{(1)}(\tilde{u}(0),\ \tilde{f},\ \tilde{g}_1,\ \tilde{g}_0). \tag{5.16}$$

Especially, if $p = q = 0$, *and*

$$\delta > \begin{cases} \frac{1}{2}m_1^*, & \text{for } \sigma > \frac{1}{2}, \\ \frac{1}{2}m_2^*, & \text{for } \sigma = \frac{1}{2}, \\ \frac{1}{2}m_3^*, & \text{for } \sigma < \frac{1}{2}, \end{cases}$$

where

$$m_2^* = 1 + 4\varepsilon + p_0 + \frac{9\lambda\nu}{4},$$

$$m_3^* = \frac{4 + 16\varepsilon + 4p_0 + 18\lambda\nu\sigma}{4 + 18\lambda\nu\sigma - 9\lambda\nu}$$

then for all $\rho_k^{(1)}$ *and* k, *we have* (5.16).

Remark 2. The convergence can be obtained provided

$$L_h(u(jh,\ k\tau)) = \tilde{\tilde{f}}(jh,\ k\tau),\quad |\tilde{\tilde{f}}(jh,\ k\tau)| = O(h^{S_a}),\quad S_a \geqslant s,$$

$$|\tilde{g}_1(k\tau)| = O(h),\quad |\tilde{g}_0(k\tau)| = O(h).$$

Remark 3. If $\tilde{g}_1 = 0$, or $\tilde{g}_0 = 0$, then

$$\max_{0\leqslant j\leqslant N} |\tilde{u}^2(jh,\ k\tau)| \leqslant |\tilde{u}(k\tau)|_1^2.$$

Therefore

$$\sum_{j=1}^{N-1} h|\tilde{u}(jh,\ k\tau)|^{l+2} \leqslant h^{1-\frac{l}{2}}\|\tilde{u}(k\tau)\|^l|\tilde{u}(k\tau)|_1^2,$$

$$|\tau(\tilde{u}_t(k\tau),\ |\tilde{u}(k\tau)|^{l+1})| \leqslant \varepsilon\tau\|\tilde{u}_t(k\tau)\|^2 + \frac{M}{\varepsilon}h^{3-l}\|\tilde{u}(k\tau)\|^{2l}|\tilde{u}(k\tau)|_1^2;$$

so (5.15) becomes

$$Q(\tilde{u}(k\tau),\ \nu,\ p_0) \leqslant \rho_k^{(1)}(\tilde{u}(0),\ \tilde{f},\ \tilde{g}_1,\ \tilde{g}_0)$$
$$+ M\sum_{j=0}^{k-1}\Big\{\|\tilde{u}(j\tau)\|^2 + (-\nu + Mh(m-2\delta)^2\|\tilde{u}(j\tau)\|^2$$
$$+ \sum_{l=1}^{p}\widetilde{M}_l h^{1-\frac{l}{2}}\|\tilde{u}(j\tau)\|^l + \sum_{l=1}^{q}\widetilde{N}_l h^{3-l}\|u(j\tau)\|^{2l}|\tilde{u}(j\tau)|_1^2\Big\}.$$

Finally we get the same result as in Theorem 3, but

$$s = \max\left(-\frac{1}{2}\operatorname{sign}|m-2\delta|,\ \max_{\substack{1\leqslant i\leqslant p \\ \widetilde{M}_i \neq 0}}\left(\frac{i-2}{2i}\right),\ \max_{\substack{1\leqslant i\leqslant q \\ \widetilde{N}_i \neq 0}}\left(\frac{i-3}{2i}\right)\right).$$

Clearly if $p\leq 2$, $q\leq 3$, then $s\leq 0$.

§ 6. The Second Boundary Value Problem

For simplicity, we suppose that (5.1), (5.3) hold, and

(i) $\delta=\sigma=0$, (6.1)

(ii) $\tilde{u}_{\bar{x}}(1, k\tau)=\tilde{g}_1(k\tau)$, $\tilde{u}_x(0, k\tau)=\tilde{g}_0(k\tau)$. (6.2)

Then (3.7) holds with $B_0(k\tau)=A_3(k\tau)=0$. From (5.6) and Lemma 6, we have

$$|B_1(k\tau)|\leq\nu\varepsilon|\tilde{u}(k\tau)|_1^2+M(\|\tilde{u}(k\tau)\|^2+\tilde{g}_1^2(k\tau)+\tilde{g}_0^2(k\tau)).$$

Similarly

$$\left|\frac{m\tau}{2}B_3(k\tau)\right|\leq\varepsilon\tau^2|\tilde{u}_t(k\tau)|_1^2+M\tau^2\|\tilde{u}_t(k\tau)\|^2+\tilde{g}_1^2(k\tau)+\tilde{g}_0^2(k\tau).$$

We can prove that

$$|(2\tilde{u}(k\tau), J(\tilde{u}(k\tau), u(k\tau)+\tilde{u}(k\tau)))|$$
$$\leq\varepsilon|\tilde{u}(k\tau)|_1^2+M\|\tilde{u}(k\tau)\|^2+Mh^{-1}\|\tilde{u}(k\tau)\|\,|\tilde{u}(k\tau)|_1^2,$$
$$|m\tau(\tilde{u}_t(k\tau), J(\tilde{u}(k\tau), u(k\tau)+\tilde{u}(k\tau)))|$$
$$\leq\varepsilon\tau\|\tilde{u}_t(k\tau)\|^2+Mh(\|\tilde{u}(k\tau)\|^2+\|\tilde{u}(k\tau)\|^2|\tilde{u}(k\tau)|_1^2).$$

By substituting the above estimations into (3.7), we obtain

$$\|\tilde{u}(k\tau)\|_t^2+\tau(m-1-M\varepsilon)\|\tilde{u}_t(k\tau)\|^2+\nu|\tilde{u}(k\tau)|_1^2+\frac{m\nu\tau}{2}[|\tilde{u}(k\tau)|_1^2]_t$$
$$-\frac{m\nu\tau^2}{2}|\tilde{u}_t(k\tau)|_1^2-\varepsilon\tau^2|\tilde{u}_t(k\tau)|_1^2$$
$$\leq M\|\tilde{u}(k\tau)\|^2+M\sum_{l=1}^{p}\bar{M}_l h^{-\frac{l}{2}}\|\tilde{u}(k\tau)\|^{l+2}+M\sum_{l=1}^{q}\bar{N}_l h^{2-l}\|\tilde{u}(k\tau)\|^{2l+2}$$
$$+\nu(1-M\varepsilon-Mh^{-1}\|\tilde{u}(k\tau)\|^2)|\tilde{u}(k\tau)|_1^2+M(\|\tilde{f}(k\tau)\|^2+\tilde{g}_1^2(k\tau)+\tilde{g}_0^2(k\tau)).$$

Theorem 4. *If the following conditions are satisfied*

(1) *conditions* (5.1), (5.3), (6.1) *and* (6.2) *hold,*

(2) $\lambda<\dfrac{1}{2\nu}$,

(3) $\rho_k^{(2)}(\tilde{u}(0), \tilde{f}, \tilde{g}_1, \tilde{g}_0)=\|\tilde{u}(0)\|^2+\tau\sum_{j=0}^{k}(\|\tilde{f}(j\tau)\|^2+\tilde{g}_1^2(j\tau)+\tilde{g}_0^2(j\tau))\leq Nh$,

then for all $k\tau\leq T(\rho_k^{(2)})$, *we have*

$$Q(\tilde{u}(k\tau), \nu, p_0)\leq Me^{Lk\tau}\rho_k^{(2)}(\tilde{u}(0), \tilde{f}, \tilde{g}_1, \tilde{g}_0).$$

We can get the convergence as in Remark 2.

§ 7. The Case $\nu(x, t, U)=\nu(U)$

For simplicity we suppose that

(i) $u(x, t)=u(x+1, t)$, $Nh=1+h$, (7.1)

(ii) $\nu(x, t, u)=\nu(u)$, (7.2)

(iii) for all $u\in(a, b)$, we have

$$0<\nu_0<\nu(x,\,t,\,u)<\nu_1, \quad \left|\frac{\partial\nu(u)}{\partial u}\right|\leqslant\nu_2, \tag{7.3}$$

(iv) $\delta=\sigma=0$, (7.4)

(v) $F=0$. (7.5)

We can obtain the following error equality

$$\begin{aligned}&\|\tilde u(k\tau)\|_t^2+\tau(m-1)\|\tilde u_t(k\tau)\|^2+2|u(k\tau)|^2_{1,\nu(u(k\tau)+\tilde u(k\tau))}\\ &\quad-m\tau(\tilde u_t(k\tau),\,\Delta_h^{\nu(u(k\tau)+\tilde u(k\tau))}\tilde u(k\tau))+m\tau(\tilde u_t(k\tau),\,J(\tilde u(k\tau),\,u(k\tau)+\tilde u(k\tau)))\\ &\quad+(2\tilde u(k\tau)+m\tau\tilde u_t(k\tau),\,J(u(k\tau)+\delta\tau u_t(k\tau),\,\tilde u(k\tau)))\\ &\quad=(2\tilde u(k\tau)+m\tau\tilde u_t(k\tau),\,\tilde f(k\tau)+\Delta_h^{\tilde\nu(k\tau)}u(k\tau)),\end{aligned} \tag{7.6}$$

Assume $u(jh,\,k\tau)\in(a,\,b)$, $\widetilde{\widetilde{M}}_l(z)$ is such a nonnegative function that if $z\leqslant Nh$, then $\widetilde{\widetilde{M}}_l(z)\leqslant\alpha_l$; especially, $\widetilde{\widetilde{M}}_1(z)\leqslant\frac{\nu_1^2}{\nu_0}$. We have

$$\begin{aligned}&|m\tau(\tilde u_t(k\tau),\,\Delta_h^{\nu(u(k\tau)+\tilde u(k\tau))}\tilde u(k\tau))|\\ &\quad\leqslant\frac{\tau}{4}\|\tilde u_t(k\tau)\|^2+4\lambda m^2\widetilde{\widetilde{M}}_1(\|\tilde u(k\tau\|^2)\,|\tilde u(k\tau)|^2_{1,\nu(u(k\tau)+\tilde u(k\tau))},\\ &|m\tau(\tilde u_t(k\tau),\,J(\tilde u(k\tau),\,\tilde u(k\tau)))|\\ &\quad\leqslant\frac{\tau}{4}\|\tilde u_t(k\tau)\|^2+h\widetilde{\widetilde{M}}_2(\|\tilde u(k\tau)\|^2)\|\tilde u(k\tau)\|^2|\tilde u(k\tau)|^2_{1,\nu(u(k\tau)+\tilde u(k\tau))},\\ &|(2\tilde u(k\tau),\,\Delta_h^{\tilde\nu(k\tau)}u(k\tau))|\\ &\quad\leqslant\widetilde{\widetilde{M}}_3(\|\tilde u(k\tau)\|^2)\Big(\frac{1}{\varepsilon}\|\tilde u(k\tau)\|^2+\varepsilon|\tilde u(k\tau)|^2_{1,\nu(u)k\tau)+\tilde u(k\tau))}\Big).\end{aligned}$$

We can estimate other terms in (7.6) and get

$$\begin{aligned}&\|\tilde u(k\tau)\|_t^2+\tau(m-2)\|\tilde u_t(k\tau)\|^2\\ &\quad\leqslant\widetilde{\widetilde{M}}_4(\|\tilde u(k\tau)\|^2)\|\tilde u(k\tau)\|^2+(-2+(\varepsilon+h)\widetilde{\widetilde{M}}_5(\|\tilde u(k\tau)\|^2)\\ &\quad\quad+4\lambda m^2\widetilde{\widetilde{M}}_1(\|\tilde u(k\tau)\|^2)\,|u(k\tau)|^2_{1,\nu(u(k\tau)+\tilde u(k\tau))}+M\|\tilde f(k\tau)\|^2.\end{aligned} \tag{7.7}$$

Let h, ε be sufficiently small and $m=2$; then we have from (7.7)

$$\begin{aligned}\|\tilde u(k\tau)\|^2&\leqslant\rho_k(\tilde v(0),\,\tilde f)+2\tau\sum_{j=0}^{k-1}\widetilde{\widetilde{M}}_6(\|\tilde u(j\tau)\|^2)\|u(j\tau)\|^2\\ &\quad+\tau\sum_{j=0}^{k-1}[-2+(\varepsilon+h)\widetilde{\widetilde{M}}_7(\|\tilde u(j\tau)\|^2)+16\lambda\widetilde{\widetilde{M}}_1(\|\tilde u(j\tau)\|^2)]\\ &\quad\times|\tilde u(j\tau)|^2_{1,\nu(u(j\tau)+\tilde u(j\tau))}.\end{aligned}$$

Finally we use Lemma 7 with

$$\eta(k\tau)=\|\tilde u(k\tau)\|^2,$$
$$V(k\tau)=|u(k\tau)|^2_{1,\nu(u(k\tau)+\tilde u(k\tau))}.$$

Theorem 5. *If $u(jh,\,k\tau)\in(a,\,b)$, conditions* (7.1)—(7.6) *hold, and*

$$\lambda<\frac{\nu_0}{8\nu_1^2}, \quad \rho_k(\tilde u(0),\,\tilde f)\leqslant Nh, \quad k\tau\leqslant T(\rho_k),$$

then

$$\|\tilde u(k\tau)\|^2\leqslant Me^{Lk\tau}\rho_k(\tilde u(0),\,\tilde f).$$

The convergence can be derived by using the same technique as in Theorem 2.

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