ON DISCONTINUOUS FINITE ELEMENT APPROXIMATION FOR THE SOLUTION OF TRICOMI'S PROBLEM*

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§ 1. Introduction

In this paper we shall discuss a discontinuous finite element approximation for the solution of Tricomi's problem

$$\begin{cases} y\Phi_{,xx}-\Phi_{,yy}=f & \text{in } \Omega, \\ \Phi=0 & \text{on } \Gamma^2, \Gamma^3 \text{ and } \Gamma^4, \\ 2\Phi_{,x}+\Phi_{,y}=0 & \text{on } \Gamma^5, \\ \Phi & \text{unspecified on } \Gamma^1, \end{cases} \tag{1.1}$$

where Ω as shown in the figure of section 2 is a domain in the (x, y) plane bounded by the characteristics Γ^1 and Γ^2 passing through the points $(0, 1)$ and $(0, -1)$ respectively for $y\geqslant 0$ and bounded by the rectangle with sides Γ^3, Γ^4 and Γ^5 for $y<0$. This problem is a linear mixed type problem with equation hyperbolic in the part $y>0$ of Ω, elliptic in the other part $y<0$ of Ω and parabolic on the line $y=0$.

It is known that by transformation problem (1.1) can be reduced to a first order symmetric positive system. In [1] and [4] a finite element method for the solution of Tricomi's problem has been presented in the form of first order system, where the finite element space is a subspace of $H^1(\Omega)$ consisting of piecewise polynomials of degree $\leqslant r$. And an error bound in L_2 norm of $O(h^r)$ for this continuous finite element method was shown, based on the results of Lesaint in [2] for the finite element method for first order symmetric positive systems. This error estimate is not optimal compared with the approximation properties of the finite element space employed.

The discontinuous finite element approximation for Tricomi's problem to be discussed is constructed first by transforming (1.1) into a first order symmetric positive system and then applying a discontinuous finite element procedure presented in [3] to this reduced first order system. We aim at studying the stability and convergence properties of the finite element method for Tricomi's problem in a wider situation where the finite element space is only a subspace of $L_2(\Omega)$. We shall prove that the rate of convergence is $O(h^{r+\frac{1}{2}})$ provided the finite element space is chosen as the space of piecewise polynomials of degree $\leqslant r$. This result shows the effectiveness of the discontinuous finite element, and it also is an

* Received December 31, 1982.

improvement of the known result for the continuous finite element method.

§ 2. Reduction of Tricomi's Problem

We introduce the following transformation of the independent variables

$$u=\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad u_1=e^{-\lambda x}\Phi_{,x}, \quad u_2=e^{-\lambda x}\Phi_{,y}.$$

Then the equation in (1.1) can be written in form of first order system

$$\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_{,x}+\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_{,y}+\begin{pmatrix} \lambda y & 0 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}=\begin{pmatrix} e^{-\lambda x}f \\ 0 \end{pmatrix},$$

which is symmetric. Multiplying this system by matrix

$$T=\begin{pmatrix} a & by \\ b & a \end{pmatrix}$$

we obtain

$$Lu\equiv A_1u_{,x}+A_2u_{,y}+A_3u=F, \tag{2.1a}$$

where

$$A_1=\begin{pmatrix} ay & by \\ by & a \end{pmatrix}, \; A_2=\begin{pmatrix} -by & -a \\ -a & -b \end{pmatrix}, \; A_3=\begin{pmatrix} \lambda ay & \lambda by \\ \lambda by & \lambda a \end{pmatrix}, \; F=\begin{pmatrix} ae^{-\lambda x}\cdot f \\ be^{-\lambda x}\cdot f \end{pmatrix}.$$

It can be seen that with the choice of $a=2$, $b=1$ and $\lambda=0.1$ system (2.1a) is a symmetric positive system in Ω. In the sequel we shall fix this choice. Let L^* be the formal adjoint of L

$$L^*u\equiv-(A_1u)_{,x}-(A_2u)_{,y}+A_3^Tu.$$

We have

$$L+L^*=A_3+A_3^T-\frac{\partial A_1}{\partial x}-\frac{\partial A_2}{\partial y}=\begin{pmatrix} 0.4y+1 & 0.2y \\ 0.2y & 0.4 \end{pmatrix},$$

and $(L+L^*)$ is strictly positive, i.e. there exists a constant $\alpha>0$ such that

$$u^T(L+L^*)u\geqslant\alpha u^T\cdot u \quad \text{in } \Omega \text{ for } u\in R^2. \tag{2.2}$$

We now examine the corresponding boundary condition of system (2.1a). The boundary $\partial\Omega$ of Ω is shown in the figure. Let $n=(n_x, n_y)$ be the outer unit normal on $\partial\Omega$.

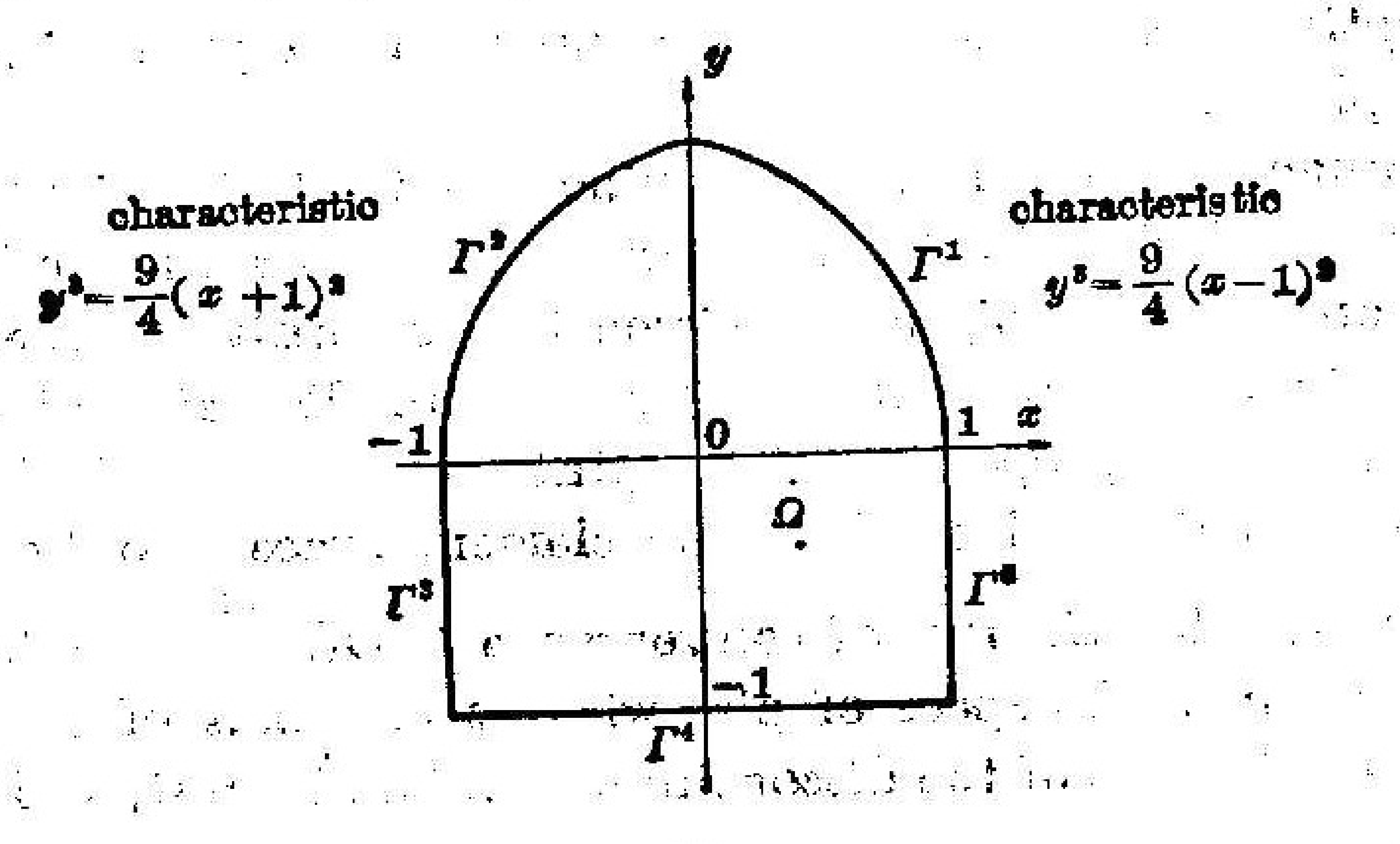

Fig. 1

The following matrix is called the boundary matrix

$$\beta=n_xA_1+n_yA_2=\begin{pmatrix}(2n_x-n_y)y & n_xy-2n_y\\ n_xy-2n_y & 2n_x-n_y\end{pmatrix}.$$

We thus have

$$u^T\beta u=\frac{(4-y)(n_yu_1-n_xu_2)^2-(n_y^2-yn_x^2)(2u_1+u_2)^2}{2n_x+n_y}. \tag{2.3}$$

The boundary conditions in (1.1) can be written in the form $Nu=0$ on $\partial\Omega$ with matrix N defined as follows:

$$N=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} \quad \text{on} \quad \Gamma^1=\left\{(x,y)\,|\,y^3=\frac{9}{4}(x-1)^2\right\},$$

$$N=\frac{2+\sqrt{y}}{\sqrt{1+y}}\begin{pmatrix}y & \sqrt{y}\\ \sqrt{y} & 1\end{pmatrix} \quad \text{on} \quad \Gamma^2=\left\{(x,y)\,|\,y^3=\frac{9}{4}(x+1)^2\right\},$$

$$N=\begin{pmatrix}0 & 0\\ 0 & 2+\frac{1}{2}|y|\end{pmatrix} \quad \text{on} \quad \Gamma^3=\{(x,y)\,|\,x=-1,\ -1\leqslant y\leqslant 0\},$$

$$N=\begin{pmatrix}5 & 0\\ 0 & 0\end{pmatrix} \quad \text{on} \quad \Gamma^4=\{(x,y)\,|\,y=-1,\ -1\leqslant x\leqslant 1\},$$

$$N=\begin{pmatrix}2|y| & |y|\\ |y| & \frac{1}{2}|y|\end{pmatrix} \quad \text{on} \quad \Gamma^5=\{(x,y)\,|\,x=1,\ -1\leqslant y\leqslant 0\}.$$

It can be seen that $Nu=0$ is an admissible boundary condition in the sense of Friedrichs (see [5]) by defining a matrix M on $\partial\Omega$ such that

$$u^TMu=\frac{(4-y)(n_yu_1-n_xu_2)^2+|n_y^2-yn_x^2|(2u_1+u_2)^2}{|2n_x+n_y|} \tag{2.4}$$

which satisfies the following conditions

(i) $N=\frac{1}{2}(M-\beta)$,

(ii) $M+M^*\geqslant 0$, here $M^*=M$,

(iii) $\operatorname{Ker}(\beta-M)+\operatorname{Ker}(\beta+M)=R^2$.

From (2.3) and (2.4) it is easy to see that

$$|u^T\beta u|\leqslant u^TMu \quad \text{on } \partial\Omega \quad \text{for } u\in R^2,$$

hence we have

$$0\leqslant\frac{1}{2}u^T(M-\beta)u\leqslant u^TMu \quad \text{on } \partial\Omega. \tag{2.5}$$

Now the reduction of problem (1.1) is completed. The reduced first order system, equation (2.1a), together with the boundary condition

$$Nu=0 \quad \text{on } \partial\Omega \tag{2.1b}$$

is a symmetric positive system of Friedrichs and satisfies conditions (2.2) and (2.5) which are a little stronger then what Friedrichs' system requires. We say that function v satisfies the adjoint boundary condition if

$$N^*u=0 \quad \text{for } (x, y)\in\partial\Omega,$$

where $N^*=N^T+\beta$. Notice that here M, β and N are all symmetric. We thus have

$$N+N^*=2N+\beta=M. \tag{2.6}$$

The method we shall discuss is based on the reduced system (2.1a), (2.1b) and we are looking for the solution of this system in $(H^1(\Omega))^2$. By means of the following identity

$$(Lu, v)_\Omega+\langle Nu, v\rangle_{\partial\Omega}=(u, L^*v)_\Omega+\langle u, N^*v\rangle_{\partial\Omega} \text{ for } u, v\in(H^1(\Omega))^2 \tag{2.7}$$

one of the variational formulation of (2.1a) and (2.1b) is: Find $u\in(H^1(\Omega))^2$ such that

$$(Lu, v)_\Omega+\langle Nu, v\rangle_{\partial\Omega}=(F, v)_\Omega \quad \forall v\in(H^1(\Omega))^2, \tag{2.8}$$

where $(\cdot, \cdot)_\Omega$ and $\langle\cdot, \cdot\rangle_{\partial\Omega}$ denote the inner product in $(L_2(\Omega))^2$ and in $(L_2(\partial\Omega))^2$ respectively.

Remark. The matrix M defined by (2.4) has the following explicit form:

$$M=\beta=\frac{2-\sqrt{y}}{\sqrt{1+y}}\begin{pmatrix} y & \sqrt{y} \\ \sqrt{y} & 1\end{pmatrix} \quad \text{on } \Gamma^1,$$

$$M=-\beta=\frac{2+\sqrt{y}}{\sqrt{1+y}}\begin{pmatrix} y & \sqrt{y} \\ \sqrt{y} & 1\end{pmatrix} \quad \text{on } \Gamma^2,$$

$$M=\begin{pmatrix} 2|y| & |y| \\ |y| & 2+|y|\end{pmatrix} \quad \text{on } \Gamma^3 \text{ and } \Gamma^5,$$

$$M=\begin{pmatrix} 9 & 2 \\ 2 & 1\end{pmatrix} \quad \text{on } \Gamma^4.$$

§ 3. Discontinuous Finite Element Approximation

Let $\{\mathscr{L}_h, h>0\}$ be a family of triangulations of Ω. Each $\mathscr{L}_h$ divides Ω into a finite collection of nonoverlapping elements: $\bar\Omega=\bigcup_{K\in\mathscr{L}_h} K$, where h is the maximum diameter of the elements in $\mathscr{L}_h$. We assume that $\{\mathscr{L}_h\}$ is nondegenerate, i.e. the ratio of the radii r_1 and r_2 of the circumscribed and inscribed circles of each element is bounded:

$$r_1/r_2\leqslant C \quad \text{for all elements and all } h.$$

Let $P_r(K)$ be the set of all polynomials of degree $\leqslant r$ on K. We define

$$V_h=(\prod_{K\in\mathscr{L}_h} P_r(K))^2.$$

Thus V_h is a finite dimensional subspace of $(L_2(\Omega))^2$.

Let $n_K=(n_{kx}, n_{ky})$ be the outer unit normal on ∂K. We now extend the matrices β and M in section 2, originally defined on $\partial\Omega$, to ∂K for all $K\in\mathscr{L}_h$ by using n_{kx}, n_{ky} to replace n_x, n_y in (2.3) and (2.4), and we denote the extensions by β_k and M_k respectively. Similarly to (2.5) we have

$$0\leqslant\frac{1}{2}u^T(M_k-\beta_k)u\leqslant u^TM_ku \quad \text{on } \partial K,\ K\in\mathscr{L}_h. \tag{3.1}$$

Definition. *u_h is said to be a discontinuous finite element approximation of the* *solution of* (2.1a), (2.1b) *if for any* $v_h \in V_h$

$$(Lu_h - F,\ v_h)_\Omega + \langle Nu_h,\ v_h\rangle_{\partial\Omega} + \frac{1}{2}\sum_{K\in\mathscr{L}_h}\int_{\partial K^0}((M_k-\beta_k)[u_h],\ v_h)ds = 0, \tag{3.2}$$

where $\partial K^0 = \partial K - (\partial K \cap \partial\Omega)$, $[u_h] = u_h - \tilde{u}_h$ *and* $\tilde{u}_h$ *derotes the trace of* u_h *on* ∂K *from the exterior of element* K.

This definition coincides with the one given by Lesaint in [3]. Introduce the following bilinear form for piecewise smooth functions φ and ψ

$$B(\varphi,\ \psi) = \sum_{K\in\mathscr{L}_h}\left\{(L\varphi,\ \psi)_K + \frac{1}{2}\int_{\partial K}((M_k-\beta_k)(\varphi-\xi_h),\ \psi)ds,\right\} \tag{3.3}$$

where

$$\xi_h = \begin{cases} 0 & \text{on } \partial K \cap \partial\Omega \\ \tilde{\varphi} & \text{the external trace of } \varphi \text{ on } \partial K - (\partial K \cap \partial\Omega). \end{cases}$$

Then (3.2) is equivalent to

$$B(u_h,\ v_h) = (F,\ v_h) \quad \forall v_h \in V_h. \tag{3.4}$$

Lemma 1. *For any piecewise smooth function* φ

$$B(\varphi,\ \varphi) = \frac{1}{2}(\varphi,\ (L+L^*)\varphi)_\Omega + \frac{1}{2}\langle\varphi,\ M\varphi\rangle_{\partial\Omega} + \frac{1}{2}\sum_{S\bar{\in}\partial\Omega}\int_S([\varphi],\ M_k[\varphi])ds,$$

where the summation $\sum_{S\bar{\in}\partial\Omega}$ *is taken over all side faces of the elements of* $\mathscr{L}_h$ *which do not belong to* $\partial\Omega$.

Proof. Notice that by Green's formula

$$(L\varphi,\ \psi)_K = (\varphi,\ L^*\psi)_K + \int_{\partial K}(\beta_k\varphi,\ \psi)ds.$$

We have

$$\begin{aligned} B(\varphi,\ \varphi) = \sum_{K\in\mathscr{L}_h}&\left\{\frac{1}{2}(L\varphi,\ \varphi)_K + \frac{1}{2}(\varphi,\ L^*\varphi)_K + \frac{1}{2}\int_{\partial K}(\beta_k\varphi,\ \varphi)ds\right. \\ &\left.+\frac{1}{2}\int_{\partial K}((M_k-\beta_k)(\varphi-\xi_h),\ \varphi)\,ds\right\} \\ =\frac{1}{2}&(\varphi,\ (L+L^*)\varphi)_\Omega + \frac{1}{2}\sum_{K\in\mathscr{L}_h}\left\{\int_{\partial K}(\beta_k\xi_h,\ \varphi)ds\right. \\ &\left.+\int_{\partial K}(M_k(\varphi-\xi_h),\ \varphi)\,ds\right\}. \end{aligned}$$

Since $\beta_k|_{S^+} = -\beta_k|_{S^-}$ and $M_k|_{S^+} = M_k|_{S^-}$, it is easy to see that

$$\sum_{K\in\mathscr{L}_h}\int_{\partial K}(\beta_k\xi_h,\ \varphi)\,ds = 0$$

and

$$\sum_{K\in\mathscr{L}_h}\int_{\partial K}(M_k(\varphi-\xi_h),\ \varphi)\,ds = \langle\varphi,\ M\varphi\rangle_{\partial\Omega} + \sum_{S\bar{\in}\partial\Omega}\int_S([\varphi],\ M_k[\varphi])ds.$$

These together show the desired identity.

Since $B(u_h,\ u_h) = (F,\ u_h)$, we have by (2.2), (3.1) and Lemma 1 with $\varphi = u_h$

$$\|u_h\|^2 \leqslant C(u_h,\ (L+L^*)u_h) \leqslant C(F,\ u_h) \leqslant C\|F\|\cdot\|u_h\|,$$

hence u_h is bounded in L_2 norm, i.e.

$$\|u_h\| \leqslant C\|F\|. \tag{3.5}$$

This estimate shows that the discrete problem (3.2) has a unique solution for any given $F\in(L_2(\Omega))^2$

We shall prove a better stability estimate than (3.5). To do this we introduce the following norm for the piecewise smooth function v_h

$$|\!|\!|v_h|\!|\!|=\left\{\|v_h\|^2+h\|Lv_h\|^2+\langle v_h,\ Mv_h\rangle_{\partial\Omega}+\sum_{S\in\partial\Omega}\int_S([v_h],\ M_k[v_h])ds\right\}^{\frac{1}{2}}.$$

Lemma 2. *There exist positive constants C and $\varkappa_0$ independent of h such that for* $0<\varkappa\leqslant\varkappa_0$

$$\varkappa\cdot|\!|\!|v|\!|\!|^2\leqslant B(v,\ v+\varkappa hLv)+C\varkappa^2h^2\sum_{K\in\mathscr{K}_h}\|Lv\|^2_{L(\partial K)},$$

where v is any piecewise smooth function.

Proof. By Lemma 1 and (2.2), there exists some constant C_0 such that

$$B(v,\ u)\geqslant C_0\left\{\|v\|^2+\langle v,\ Mv\rangle_{\partial\Omega}+\sum_{S\in\partial\Omega}\int_S([v],\ M_k[v])\,ds\right\}.$$

From the definition of $B(\cdot,\ \cdot)$,

$$B(v,\ Lv)=(Lv,\ Lv)_\Omega+\frac{1}{2}\int_{\partial\Omega}((M-\beta)v,\ Lv)\,ds$$
$$+\frac{1}{2}\sum_{K\in\mathscr{K}_h}\int_{\partial K^*}((M_k-\beta_k)(v-\xi_h),\ Lv)\,ds.$$

Hence by applying Schwarz' inequality, (2.5) and (3.1) we have

$$\varkappa hB(v,\ Lv)\geqslant\varkappa h\|Lv\|^2-\frac{C_0}{2}\langle v,\ Mv\rangle_{\partial\Omega}-C_1\varkappa^2h^2\langle Lv,\ Lv\rangle_{\partial\Omega}$$
$$-\frac{C_0}{2}\sum_{S\in\partial\Omega}\int_S([v],\ M_k[v])ds-C_2\varkappa^2h^2\sum_{K\in\mathscr{K}_h}\int_{\partial K^*}(Lv,\ Lv)\,ds$$
$$\geqslant\varkappa h\|Lv\|^2-\frac{C_0}{2}\langle v,\ Mv\rangle_{\partial\Omega}-\frac{C_0}{2}\sum_{S\in\partial\Omega}\int_S([v],\ M_k[v])\,ds$$
$$-C^*\varkappa^2h^2\sum_{K\in\mathscr{K}_h}\|Lv\|^2_{L_2(\partial K)},$$

where constant C^* only depends on C_0. So

$$B(v,\ v+\varkappa hLv)=B(v,\ v)+\varkappa hB(v,\ Lv)$$
$$\geqslant\frac{C_0}{2}\left\{\|v\|^2+\langle v,\ Mv\rangle_{\partial\Omega}+\sum_{S\in\partial\Omega}\int_S([v],M_k[v])ds\right\}$$
$$+\varkappa h\|Lv\|^2-C^*\varkappa^2h^2\sum_{K\in\mathscr{K}_h}\|Lv\|^2_{L_2(\partial K)}$$
$$\geqslant\min\left(\frac{C_0}{2},\ \varkappa\right)|\!|\!|v|\!|\!|^2-C^*\varkappa^2h^2\sum_{K\in\mathscr{K}_h}\|Lv\|^2_{L_2(\partial K)},$$

which proves the lemma by choosing $\varkappa_0=\dfrac{C_0}{2}$ and $C=C^*$.

§ 4. Rate of the Convergence

Let $u\in H^1(\Omega)$ be the solution of the continuous problem (2.1a), (2.1b). Then it is easy to see that

$$B(u,\ v_h)=(F,\ v_h)\quad\forall v_h\in V_h.$$

Thus by (3.4) the error $e_h=u-u_h$ satisfies equation

$$B(e_h, v_h) = 0 \quad \forall v_h \in V_h. \tag{4.1}$$

Introduce an approximation L_h of L as follows:

$$L_h = A_{1h}\partial_x + A_{2h}\partial_y + A_{3h},$$

where

$$A_{ih} = A_i(y_K^*) \quad \text{in } K \text{ for } K \in \mathscr{L}_h,\ i=1, 2, 3,$$

y_k^* is the y-coordinate of the centroid of K.

Then we have

$$L = L_h + \widetilde{C}hL', \quad L' = \frac{\partial A_1}{\partial y}\partial_x + \frac{\partial A_2}{\partial y}\partial_y + \frac{\partial A_3}{\partial y}.$$

Since $u_h + hL_hu_h \in V_h$, by (4.1) we have with $\eta = u - v_h$ for any $v_h \in V_h$

$$B(e_h, e_h + \varkappa hL_he_h) = B(e_h, \eta) + \varkappa hB(e_h, L_h\eta).$$

Applying Lemma 2 with $v = e_h$ we obtain

$$\begin{aligned}\varkappa\|e_h\|^2 &\leqslant B(e_h, e_h + \varkappa hLe_h) + C^*\varkappa^2h^2 \sum_{K\in\mathscr{L}_h} \|Le_h\|^2_{L_2(\partial K)} \\ &\leqslant B(e_h, \eta) + \varkappa hB(e_h, L_h\eta) + \widetilde{C}\varkappa h^2B(e_h, L'e_h) \\ &\quad + C^*\varkappa^2h^2 \sum_{K\in\mathscr{L}_h} \|Le_h\|^2_{L_2(\partial K)}.\end{aligned} \tag{4.2}$$

We now estimate each term on the right hand side of (4.2). First, by Schwarz' inequality, (2.5) and (3.1) we have for $\varepsilon_1 > 0$

$$\begin{aligned}B(e_h, \eta) &= \sum_{K\in\mathscr{L}_h} (Le_h, \eta)_K + \frac{1}{2}\int_{\partial\Omega}((M-\beta)e_h, \eta)\,ds \\ &\quad + \frac{1}{2}\sum_{K\in\mathscr{L}_h}\int_{\partial K^*}((M_k-\beta_k)(e_h-\xi_h), \eta)\,ds \\ &\leqslant \varepsilon_1\Big\{h\sum_{K\in\mathscr{L}_h}\|Le_h\|^2_{L_2(K)} + \langle e_h, Me_h\rangle_{\partial\Omega} + \sum_{S\in\partial\Omega}\int_S([e_h], M_k[e_h])\,ds\Big\} \\ &\quad + C_{\varepsilon_1}\{h^{-1}\sum_{K\in\mathscr{L}_h}\|\eta\|^2_{L_2(K)} + \sum_{K\in\mathscr{L}_h}\|\eta\|^2_{L_2(\partial K)}\}.\end{aligned}$$

Similarly, for $\varepsilon_2 > 0$

$$\begin{aligned}hB(e_h, L_h\eta) &= \sum_{K\in\mathscr{L}_h} h(Le_h, L_h\eta)_K + \frac{1}{2}h\int_{\partial\Omega}((M-\beta)e_h, L_h\eta)\,ds \\ &\quad + \frac{1}{2}h\sum_{K\in\mathscr{L}_h}\int_{\partial K^*}((M_k-\beta_k)(e_h-\xi_h), L_h\eta)\,ds \\ &\leqslant \varepsilon_2\Big\{h\sum_{K\in\mathscr{L}_h}\|Le_h\|^2_{L_2(K)} + \langle e_h, Me_h\rangle_{\partial\Omega} + \sum_{S\in\partial\Omega}\int_S([e_h], M_k[e_h])\,ds\Big\} \\ &\quad + C_{\varepsilon_2}\{h\sum_{K\in\mathscr{L}_h}\|\eta\|^2_{H^1(K)} + h^2\sum_{K\in\mathscr{L}_h}\|L_h\eta\|_{L_2(\partial K)}\},\end{aligned}$$

and for $\varepsilon_3 > 0$

$$\begin{aligned}h^2B(e_h, L'e_h) &= \sum_{K\in\mathscr{L}_h} h^2(Le_h, L'e_h)_K + \frac{1}{2}h^2\int_{\partial\Omega}((M-\beta)e_h, L'e_h)\,ds \\ &\quad + \frac{1}{2}h^2\sum_{K\in\mathscr{L}_h}\int_{\partial K^*}((M_k-\beta_k)(e_h-\xi_h), L'e_h)\,ds \\ &\leqslant \varepsilon_3\Big\{h\sum_{K\in\mathscr{L}_h}\|Le_h\|^2_{L(K)} + \langle e_h, Me_h\rangle_{\partial\Omega} + \sum_{S\in\partial\Omega}\int_S([e_h], M_k[e_h])\,ds \\ &\quad + C_{\varepsilon_3}\{h^3\sum_{K\in\mathscr{L}_h}\|e_h\|^2_{H^1(K)} + h^4\sum_{K\in\mathscr{L}_h}\|L'e_h\|^2_{L_2(\partial K)}\}.\end{aligned}$$

Since the triangulations $\{\mathscr{L}_h\}$ are nondegenerate, the following inverse inequality holds

$$\sum_{K\in\mathscr{L}_h}\|v_h\|^2_{H^1(K)}\leqslant C\cdot h^{-2}\sum_{K\in\mathscr{L}_h}\|v_h\|^2_{L_2(K)}\quad \text{for } v_h\in V_h.$$

Thus by using the triangle inequality and noting that $v_h-u_h\in V_h$ we have

$$\begin{aligned}h^3\sum_{K\in\mathscr{L}_h}\|e_h\|^2_{H^1(K)}&\leqslant Ch^3\sum_{K\in\mathscr{L}_h}\|\eta\|^2_{H^1(K)}+Ch^3\sum_{K\in\mathscr{L}_h}\|v_h-u_h\|^2_{H^1(K)}\\&\leqslant C\{h^3\sum_{K\in\mathscr{L}_h}\|\eta\|^2_{H^1(K)}+h\sum_{K\in\mathscr{L}_h}\|v_h-u_h\|^2_{L_2(K)}\}\\&\leqslant C\{h\sum_{K\in\mathscr{L}_h}\|\eta\|^2_{H^1(K)}+h\|e_h\|^2\}.\end{aligned}$$

Similarly, since $L'(v_h-u_h)$ and $L(v_h-u_h)$ are piecewise polynomials we have

$$\begin{aligned}h^4\sum_{K\in\mathscr{L}_h}\|L'e_h\|^2_{L_2(\partial K)}&\leqslant Ch^4\sum_{K\in\mathscr{L}_h}\|L'\eta\|^2_{L_2(\partial K)}+Ch^4\sum_{K\in\mathscr{L}_h}\|L'(v_h-u_h)\|^2_{L_2(\partial K)}\\&\leqslant Ch^4\sum_{K\in\mathscr{L}_h}\|L'\eta\|^2_{L_2(\partial K)}+Ch^3\sum_{K\in\mathscr{L}_h}\|v_h-u_h\|^2_{H^1(K)}\\&\leqslant C\{h^4\sum_{K\in\mathscr{L}_h}\|L'\eta\|^2_{L_2(\partial K)}+h\sum_{K\in\mathscr{L}_h}\|v_h-u_h\|^2_{L_2(K)}\}\\&\leqslant C\{h^4\sum_{K\in\mathscr{L}_h}\|L'\eta\|^2_{L_2(\partial K)}+h\sum_{K\in\mathscr{L}_h}\|\eta\|^2_{L_2(K)}+h\|e_h\|^2\}\end{aligned}$$

and

$$\begin{aligned}h^2\sum_{K\in\mathscr{L}_h}\|Le_h\|^2_{L_2(\partial K)}&\leqslant Ch^2\sum_{K\in\mathscr{L}_h}\|L\eta\|^2_{L_2(\partial K)}+Ch^2\sum_{K\in\mathscr{L}_h}\|L(v_h-u_h)\|^2_{L_2(\partial K)}\\&\leqslant C\{h^2\sum_{K\in\mathscr{L}_h}\|L\eta\|^2_{L_2(\partial K)}+h\sum_{K\in\mathscr{L}_h}\|L(v_h-u_h)\|^2_{L_2(K)}\}\\&\leqslant C\{h^2\sum_{K\in\mathscr{L}_h}\|L\eta\|^2_{L_2(\partial K)}+h\sum_{K\in\mathscr{L}_h}\|Le_h\|^2_{L_2(K)}+h\sum_{K\in\mathscr{L}_h}\|\eta\|^2_{H^1(K)}\}.\end{aligned}$$

Substituting these bounds into (4.2) yields

$$\begin{aligned}\varkappa\|e_h\|^2\leqslant&(\varepsilon_1+\varkappa\varepsilon_2+\widetilde{C}\varkappa\varepsilon_3+C\varkappa^2+C(\varepsilon_3)h)\|e_h\|^2+C\{h^{-1}\sum_{K\in\mathscr{L}_h}\|\eta\|^2_{L_2(K)}\\&+h\sum_{K\in\mathscr{L}_h}\|\eta\|^2_{H^1(K)}+\sum_{K\in\mathscr{L}_h}\|\eta\|^2_{L_2(\partial K)}+h^2\sum_{K\in\mathscr{L}_h}\|L\eta\|^2_{L_2(\partial K)}\\&+h^3\sum_{K\in\mathscr{L}_h}\|L'\eta\|^2_{L_2(\partial K)}\}.\end{aligned}$$

So by suitably choosing ε_i, $i=1, 2, 3$, and $\varkappa$ we obtain for h small enough

$$\begin{aligned}\|e_h\|^2\leqslant C\{&h^{-1}\sum_{K\in\mathscr{L}_h}\|\eta\|^2_{L_2(K)}+h\sum_{K\in\mathscr{L}_h}\|\eta\|^2_{H^1(K)}+\sum_{K\in\mathscr{L}_h}\|\eta\|^2_{L_2(\partial K)}\\&+h^2\sum_{K\in\mathscr{L}_h}(\|\eta_{,x}\|^2_{L_2(\partial K)}+\|\eta_{,y}\|^2_{L_2(\partial K)})\}.\end{aligned}\tag{4.3}$$

It is well known that the subspaces $\{V_h\}$ defined in section 2 has the following approximation properties: if $u\in(H^{r+1}(\Omega))^2$, $r\geqslant1$, then

$$\inf_{v_h\in V_h}\{\sum_{K\in\mathscr{L}_h}(\|u-v_h\|^2_{L_2(K)}+h^2\|u-v_h\|^2_{H^1(K)})\}^{\frac12}\leqslant Ch^{r+1}\|u\|_{H^{r+1}(\Omega)},$$

$$\inf_{v_h\in V_h}\{\sum_{K\in\mathscr{L}_h}(\|u-v_h\|^2_{L_2(\partial K)}+h^2\|D(u-v_h)\|^2_{L_2(\partial K)})\}^{\frac12}\leqslant Ch^{r+\frac12}\|u\|_{H^{r+1}(\Omega)}.$$

Therefore, by (4.3) we finally have the following result for the discontinuous finite element method.

Theorem. *The discrete problem (3.2) has a unique solution u_h for any $F\in(L_2(\Omega))^2$. And if the exact solution $u\in H^{r+1}(\Omega)$ $(r\geqslant1)$, then for small h*

$$\|u-u_h\|+(\sum_{K\in\mathscr{L}_h}h\|L(u-u_h)\|^2_{L_2(K)})^{\frac12}\leqslant Ch^{r+\frac12}\|u\|_{H^{r+1}(\Omega)}.$$

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